Optimal Bank Regulation and Fiscal Capacity

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First Draft: August 2012
This Draft: November 2013

Key Words: Optimal Bank Regulation, Derivative Regulation, Fiscal Capacity, Moral Hazard, Pecuniary Externalities

Abstract

Financial regulation is synchronized across countries despite the fact that countries vary widely in their ability to bail out their banking sector in the event of a financial crisis. This paper addresses the question of whether countries with different fiscal capacity should optimally have different ex-ante minimum bank capital requirements. In an environment with endogenously incomplete markets and overinvestment because of moral hazard and pecuniary externalities, I show that countries with larger fiscal capacity should have lower ex-ante minimum bank capital requirements. This result is the opposite of what one may expect given that countries with larger fiscal capacity often have stronger moral hazard. I also show that, in addition to a minimum bank capital requirement, regulators in countries with strong moral hazard (which are countries with a concentrated financial sector and large fiscal capacity) should impose a limit on the net liabilities pledged by financial institutions in a crisis state. This would entail restricting the amount of put options/CDS contracts sold by financial institutions, among other measures.

1 Introduction

Countries differ significantly in their ability to bail out their financial system during a crisis. At the same time, there is a substantial synchronization of the type and the level of ex-ante financial sector regulation across countries due to the Basel Accords. This paper addresses the question of whether governments with different abilities to bail-out their banking system during a financial crisis (different fiscal capacity) should have more or less stringent ex-ante minimum bank capital requirements.

1Contact information: vania.stavrakeva@gmail.com. This paper is a version of my JMP and was written while I was a graduate student in economics at Harvard University. I am very grateful to my advisors Emmanuel Farhi, Gita Gopinath, Kenneth Rogoff and Jeremy Stein for their invaluable guidance. I would also like to thank Philippe Aghion, Mitra Akhtari, Javier Bianchi, Oliver Hart, Rohan Kekre, Pablo Kurlat, Frederic Malherbe, Andrei Shleifer, Alp Simsek, Adriano Rampini, Alexandra Roulet, Dorothee Rouzet and Jenny Tang for very helpful comments and discussions.
requirements. It also examines the issue of what other types of regulatory instruments are required in addition to minimum bank capital requirements, if any.

In a model with both pecuniary externalities and moral hazard, I show that countries with larger fiscal capacity should have lower ex-ante minimum bank capital requirements. This result is the opposite of what one may expect given that countries with larger fiscal capacity often have stronger moral hazard. Furthermore, a second interesting and novel result emerges; countries with large fiscal capacity and a concentrated banking sector (which are the countries with strong moral hazard) should also impose a limit on the liabilities of the financial sector in a future crisis state, when a government bail-out is anticipated. In particular, in addition to imposing minimum bank capital requirements, the regulators of countries with strong moral hazard should limit the sale of put option contracts by financial institutions, such as CDS contracts, among other measures.

I build a three period model in the spirit of Lorenzoni (2008) where markets are endogenously incomplete. Bankers are modelled as entrepreneurs that have access to a linear production technology. Bankers invest and borrow every period using short term state contingent collateralized contracts. The project has to be refinanced in the middle period in order to remain productive. If a crisis state occurs, in order to refinance the project, bankers are forced to sell part of their capital stock to the less productive outside sector (consumers), which generates fire-sales. One can interpret the assumption that bankers are more productive than the outside sector at channeling savings into investment as bankers being better at monitoring loans and screening, which I do not model explicitly. The government can intervene during a crisis and provide a bail-out to the bankers by taxing the consumers. However, taxing is costly, and the cost depends on the fiscal capacity of the country.

The combination of future fire-sales and the assumption that the banking sector is more productive than the outside sector generates pecuniary externalities in the spirit of Lorenzoni (2008), which lead to ex-ante overinvestment relative to the constrained Central Planner’s allocation. Bankers do not internalize the fact that the more they invest ex-ante, the larger the fire-sale of financial assets during a future crisis is, which tightens the budget constraints of the other bankers. This is welfare reducing because it increases the inefficient transfer of resources from the bankers — the more productive agents — to the consumers — the less productive agents. Moral hazard is a second source of ex-ante inefficiency in the model. When banks are not infinitesimally small and they anticipate a bail-out in the future, they internalize the fact that the more they invest ex-ante, the larger the fire-sale is during a crisis, which will lead to a larger bail-out ex-post. The moral hazard also leads to ex-ante overinvestment and is stronger, the more concentrated the banking sector is and the larger the fiscal capacity of the country is.

In this framework, conditional on assuming that the policy maker has a sufficient number of instruments to replicate the constrained Central Planner’s allocation, one can prove the first

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2 The minimum bank capital requirement constrains banks to finance at least a fraction of their risky assets using equity. This fraction is referred to as the minimum bank capital ratio.

3 Capital stock/capital will be sometimes used as a synonym to investment throughout the paper and should not be confused with the capital in the "minimum bank capital requirement" expression which refers to bank equity.
key result of this paper — that countries with smaller fiscal capacity should have higher ex-ante minimum bank capital requirements. This result will be present even if the model had no moral hazard and is driven by the two key assumptions that drive the pecuniary externalities — the presence of fire-sales combined with the assumption that the banking sector is more productive than the outside sector. The intuition is the following. For a given investment level, countries that can afford a smaller bail-out will have a larger fire-sale in a crisis, which will lead to a larger transfer of resources from the more productive to the less productive sector. Therefore, the constrained Central Planner in more fiscally constrained countries perceives ex-ante investment as less attractive, and he optimally chooses to invest less relative to the constrained Central Planner of a country with a larger fiscal capacity. Since the ex-ante investment chosen by the constrained Central Planner and the optimal minimum bank capital ratio are inversely related, smaller fiscal capacity implies optimally higher ex-ante minimum bank capital ratio. In summary, countries with larger fiscal capacity can prop up asset prices more during a crisis and can alleviate any inefficiencies arising from fire-sales. As a result, they can "afford" to have larger investment booms ex-ante.

Since a larger bail-out will lead to a stronger moral hazard if the banks are not infinitesimally small, intuitively, one would expect that larger fiscal capacity would imply a higher (not lower) ex-ante minimum bank capital requirement. The reason why this intuition proves to be incorrect relies crucially on the fact that the ex-ante regulatory instrument considered — a minimum bank capital requirement — is a "quantity" policy instrument and on the assumption that the policy maker has a sufficient number of instruments to replicate the constrained Central Planner's allocation. By setting the minimum bank capital ratio, the policy maker can directly set the quantity invested by bankers in order to replicate the constrained Central Planner's allocation. The moral hazard does not play a role since it does not affect the investment chosen by the constrained Central Planner. However, the strength of the moral hazard affects the first order conditions of the banker in the decentralized equilibrium, who internalizes the benefit but not the cost of the bail-out. Therefore, if a "price" instrument was used instead, such as a tax on ex-ante investment, the optimal ex-ante regulation would have been a function of the strength of the moral hazard.\textsuperscript{4} With such a "price" instrument, the policy maker can affect only the marginal cost of investment as perceived by the banker, but cannot directly set the amount invested by the banker.

The second key result of this paper is that countries with strong moral hazard should impose a second ex-ante regulatory instrument that limits bank liabilities during a crisis when a bail-out is expected. The moral hazard in this model works through two different channels. The first one is the standard channel where bankers overinvest ex-ante. The second moral hazard channel is that if the moral hazard is strong enough and the bankers already face a minimum bank capital requirement, they optimally choose to pledge too high of a payment in the crisis state relative to what the constrained Central Planner would optimally pledge. The intuition for the second channel is that a larger payment promised in the crisis state leads to a more severe fire-sale and, therefore, to a larger expected bail-out.

\textsuperscript{4}"Price" instruments, similar to a tax on period zero investment, have been considered in the literature (see Stein (2012), Bianchi (2011) and Jeanne and Korinek (2012)).
More than one hundred countries worldwide currently have some form of a minimum liquidity requirement and also Basel III promotes liquidity regulation in the form of minimum amount of safe asset holdings. Also derivative regulations is at the forefront of the current policy debate. However, there is little theoretical understanding as to why such instruments will be required in addition to the minimum bank capital requirement. According to the results of this paper, the "sufficient" statistics, which regulators should target, is not the ex-ante bank holdings of safe assets, but the net liabilities of the financial sector in a potential future crisis when a bail out will be required. Knowledge of derivative positions will be crucial for forecasting future financial sector liabilities. In this paper I argue that countries with strong moral hazard should regulate derivative contracts (for example regulate the quantity of CDS contacts sold) even if there is no counterparty risk and even if the trades are fully collateralized. Furthermore, such a regulatory instrument will be required in addition to the minimum bank capital requirement.

The cross country heterogeneity of fiscal capacity (the government’s ability to provide a bail-out during a financial crisis) plays a crucial role in this model. I model the fiscal capacity of a country by introducing an exogenous parameter that directly affects the marginal cost of taxing and, as a result, the cost of an extra dollar of bank bail-out. In reality, a bank bail-out can be financed either by taxing, borrowing or printing money (if an independent monetary policy is available). The access to and the cost of these policy tools jointly determine the fiscal capacity of a country. However, a country will optimally use all of these instruments up to the point where the marginal costs of each one of them are equalized. The marginal cost of each of these instruments, in equilibrium, will be equal to the cost of an extra dollar of bail-out. Therefore, one can think of the marginal cost of the bail-out as a sufficient statistic, which is why I abstract from modelling sovereign borrowing and printing money explicitly. Such an approach simplifies the model significantly and, at the same time, preserves the generality of the results.

The last crisis provided plenty of evidence that countries vary widely in their abilities to provide a financial sector bail out. For a given size of the banking sector and for a given bail-out, the marginal cost of the bail-out will be larger if a country has a smaller tax base, higher cost of sovereign borrowing during a crisis or does not have access to an independent monetary policy. For example, a country like Switzerland, which has a large banking sector relative to the GDP of the country is relatively more fiscally constrained than the United States and would optimally choose to bail out a smaller fraction of its banking sector during a crisis. In response to the 2007-2008

5The World Bank survey on bank regulation shows that in 2010, 103 out of 127 countries had some form of a minimum ratio on liquid assets, such as a regulatory minimum ratio on liquid assets as a percentage of total balance sheet or deposit base.

6In order to be able to forecast the net liabilities of financial institutions in a future crisis, regulators will need information on the entire bank balance sheet of every bank and also on how the various exposures are correlated. They will also have to rely on comprehensive Value at Risk models.

7In the presence of no counterparty risk and well collateralized positions, according to Basel III, financial institutions need not hold equity against their derivative positions.

8Borrowing from the future can be thought of as future taxation while printing money is an inflationary tax, hence, the title "Optimal Bank Regulation and Fiscal Capacity".

9Using BankScope data, in 2007, in the US, the assets of the banking sector were 120% of the GDP of the country.
financial crisis, in 2011, Swiss regulators deviated from the Basel I norm of 8 percent minimum bank capital ratio by significantly increasing the minimum bank capital requirement to 19 percent, which according to the conclusions of this paper was a change in the right direction. During the 2011-2013 European sovereign debt crisis, Greece and Spain are two prime examples of countries that were fiscally constrained since they faced a high cost of sovereign borrowing and were limited in the amount of bail-outs they could provide to their financial sectors. Finally, all else equal, countries with independent monetary policy such as the United States are less fiscally constraint than the countries in the European Monetary Union since in order to reduce the marginal cost of the bail-out, they can print money during a financial crisis, in addition to borrowing and taxing.

Related Literature

There are three key assumptions that lead to the main result of this paper and they are fairly standard in the literature. The first assumption is that bankers face a borrowing constraint. It is in the spirit of Kiyotaki and Moore (1997) and is based on the seminal paper by Hart and Moore (1994). The second assumption is that the banking sector is more productive than the outside sector (the consumers), and it has been used in many papers such as Kiyotaki and Moore (1997), Lorenzoni (2008), Brunnermeier and Sannikov (2013), among others. It can be further justified by the fact that financial institutions are considered more efficient than savers at providing credit to firms, because of their ability to monitor the borrower at a lower cost (for example see Holmstrom and Tirole (1997)). Hence, the value of loans (investment) is higher in the hands of the bankers than in the hands of the consumers. On the empirical side, the importance of the banking sector and the value lost by terminating the relationship between banks and firms are studied by the literature on relationship banking (see Slovin, Sushka, and Polonchek (1993), Peek and Rosengren (1997), Gan (2007), Boot (2000) and Freixas and Rochet (2008) for a literature review). The third key assumption is that the government can circumvent the banker’s borrowing constraint via its power to tax and it was used, besides others, by Holmstrom and Tirole (1998) and Gorton and Huang (2004). Any paper with a government bank bail-out has a similar assumption in the background.

Starting with the seminal work of Bagehot (1873), moral hazard has been proposed as one of the main reasons for bank regulation. However, a growing literature on financial sector regulation has emerged, which emphasizes the role of fire sales and pecuniary externalities —


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Even counting the shadow banking sector in the US, the total assets of the banking sector (as defined above) plus the shadow banking system becomes 300% of the US GDP in 2007 (see the Financial Stability Board estimates, 2007). In 2007, in Switzerland, the assets of just UBS and Credit Suisse were 700% of the GDP of the country.

10In 2011, the 10 year sovereign debt interest rate was 5.4% in Spain and 35% in Greece.

11Hart and Moore (1994) show that if entrepreneurs can run away with the cash flow and can threaten to withdraw their human capital, they can borrow only against collateral.
externalities can emerge in a microfounded environment. The key difference between this paper and Lorenzoni (2008) is that he does not allow for an ex-post bail-out and, as a result, for heterogeneous fiscal capacity. Also Lorenzoni’s paper does not study concentrated banking sectors and optimal policy plays a minor role. I also simplify Lorenzoni (2008)’s framework. Lorenzoni (2008) assumes that in addition to the borrowing constraint on the side of the bankers, there is a limited commitment friction on the side of the consumers. I show that as long as the banks are owned by the consumers (an implicit ex-post lump sum transfer), limited commitment of the consumers will not be necessary to generate pecuniary externalities.

The only other paper, besides this one, that studies the optimal mix of ex-ante regulation and ex-post bail-out, which has both pecuniary externalities and moral hazard, is the one by Jeanne and Korinek (2012). In contrast to this paper, in Jeanne and Korinek (2012) markets are exogenously incomplete. Endogenous market incompleteness is important to understand the key sources of inefficiency. It is also crucial for some of the key results of the paper such as the result that, for countries with stronger moral hazard, regulators need to control the liabilities of the banking sector in a crisis state (impose derivative regulation). Most importantly, Jeanne and Korinek (2012) do not ask one of the key questions raised by this paper: How the optimal mix of ex-ante and ex-post bank regulation should vary with the fiscal capacity of the country?

Some of the recent papers which also find that there is a role for minimum liquidity regulations due to moral hazard are Farhi and Tirole (2012), Acharya, Shin, and Yorulmazer (2011), Repullo (2005) and Keister (2012). In an environment with a non-targeted bail-out in the form of lowering the borrowing rate of banks, Farhi and Tirole (2012) show that there are complementarities in the actions of bankers and, as a result, multiple equilibria. If banks expect low interest rates during crises, they might end up holding too little of the safe asset, which, in equilibrium, will force the policy maker to keep interest rates low in a crisis. As a result, minimum liquidity requirements can improve welfare. In a Diamond and Dybvig (1983) environment with multiple equilibria, Keister (2012) shows that an expected government bail-out leads to moral hazard and to bankers choosing lower liquidity ex-ante relative to what the Central Planner would choose. In a lot of those papers though there is a role either only for a minimum bank capital requirement or a liquidity regulation and the type of liquidity regulation is quite different from the one I propose here.

The Basel Accord recommendation of synchronized regulation is often justified by the idea of creating a "level-playing" field for banks. For a summary of the bank regulation literature see Santos (2001). Acharya (2002) studies whether minimum capital requirements should be synchronized across countries given the presence of different bank closure policies and he argues in favor of heterogeneous cross country bank regulation. Bengui (2011) builds a two country model and discusses optimal bank regulation in an environment with pecuniary externalities where he emphasizes the importance of international coordination.

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12There is a large literature on pecuniary externalities where the inefficiency comes from binding borrowing constraints where prices enter the borrowing constraint (for example Stein (2012), Bianchi (2011)). In this paper, as in Lorenzoni (2008), the source of the pecuniary externality will be that bankers do not internalize the fact that their actions are tightening the budget constraints of the other bankers, not the borrowing constraints.
Finally this paper also relates to the literature about different types of regulatory instruments pioneered by Weitzman (1974). According to Weitzman (1974), if the policy maker has an access to state contingent policy instruments, she can replicate the constrained Central Planner’s allocation using either a "price" or a "quantity" instrument. However, the comparative statics can be very different depending on which type of instruments is used, as I show in this paper.

The structure of the paper is the following. In Section 2, I present the set up of the model. Section 3 provides the solution to the decentralized problem and the constrained Central Planner’s problem. Section 4 proves that there is overinvestment in this economy while Section 5 shows how the constrained Central Planner’s allocation can be decentralized. Section 6 discusses optimal policy as a function of fiscal capacity and proves the key results of this paper. Section 7 compares "quantity" regulatory instruments to "price" regulatory instruments. Section 8 concludes and provides further discussion.

2 Model Set-Up

The model is a three period model where \( t = 0, 1, 2 \). The discount factor between periods is one. There is aggregate uncertainty only in the middle period, \( t = 1 \), and there is no idiosyncratic uncertainty. In \( t = 1 \), there are two states of nature — a high state and a low state. The period zero probabilities of the high and the low states occurring are \( \pi_h \) and \( \pi_l \), where \( \pi_l + \pi_h = 1 \). I will consider parametrization where the fire-sale occurs only in the low state in period one, which is what I will refer to as the crisis state. There are two goods — a capital good and a consumption good where the consumption good is the numeraire good. Each period and state of nature, the consumption good can be transformed into a capital good one-to-one. The capital good has to be employed in a production technology in every period and the consumption good is perishable.

There are three agents in the economy — consumers, bankers (modelled as entrepreneurs) and the government. The banks are owned by the consumers, who are risk neutral.\(^{13}\)\(^{14}\) The implicit assumptions that I make are that the banks cannot borrow from their equity owners and that the objective function of the bankers is to maximize dividend payments to equity holders.\(^{15}\) The

\(^{13}\)Alternatively, one can think of this set up as there being a representative family that splits into bankers and consumers in the beginning of period zero and the bankers are given an exogenous starting capital. The bankers and consumers reunite in \( t = 2 \) and consume jointly. The bankers can only borrow from consumers that are not members of their family and can only optimize the profits from operating their production technology. This interpretation is in the spirit of Gertler and Kiyotaki (2010).

\(^{14}\)The model can be easily changed so that both consumers and bankers are treated as agents that consume separately and I can impose an ex-ante welfare Pareto improvement criterion as in Lorenzoni (2008). Most of the qualitative results would remain unchanged as long as the government has an access to last period lump sum transfers from the bankers to the consumers.

\(^{15}\)The assumption that bankers optimize dividend payments is equivalent to the bankers optimizing the welfare of the equity holders (consumers) in the case where there is infinite number of banks. If banks are large (finite number), in order for those two objective functions to coincide, one would need the additional assumption that the bail-out of bank \( i \) is not financed by taxing the equity owners of bank \( i \) and also in addition to maximizing the dividend
government is benevolent and optimizes the welfare of the consumers. It chooses the optimal regulatory instruments and provides a bail-out to the banking sector in a crisis.

2.1 Bankers

Production Technology

Assume that there are \( N \) identical bankers/entrepreneurs. Every banker has an access to a bank specific linear production technology.\(^{16}\) In \( t = 0 \), banker \( i \) starts with no pre-existing capital stock and has to choose the amount he invests given by \( k_0^i \), where the price of capital is \( q_0 \). In \( t = 1 \) and state \( s \), the project produces \( a_{1s}k_0^i \) units of the consumption good where \( s \in \{l, h\} \) and \( a_{1l} < a_{1h} \). In \( t = 1 \), in order for the capital stock, \( k_0^i \), to remain productive, it has to be refinanced. Banker \( i \) has to invest an additional amount of \( \gamma < 1 \) per every unit of period zero capital. Otherwise, the capital depreciates one hundred percent.\(^{17}\) One can think of the refinancing cost as a long term project that requires refinancing in order to remain productive (for example, workers have to be paid and more equipment has to be purchased). The total required re-investment in order for all capital to remain productive is \( \gamma k_0^i \) units of the consumption good. I will consider functional forms such that it will be always optimal to refinance all of the capital.\(^{18}\) In \( t = 1 \), banker \( i \) can also adjust the scale of the project by choosing \( k_1^i \), where the price of period one capital is \( q_1 \). The capital sold by banker \( i \) in state \( s \) and period one is \( k_1^{i,F} = \min \{0, k_0^i - k_1^i \} \). I will refer to \( k_1^{i,F} \) as the fire-sold capital and the aggregate amount of fire-sold capital is defined as an average \( k_1^{F} = \frac{1}{N} \sum_{i=1}^{N} k_1^{i,F} \). Using averages, instead of simple sums, is standard in macro models and implies that when one takes the limit of \( N \to \infty \), the equilibrium allocation converges to the case of continuum of banks distributed uniformly on \([0, 1]\).\(^{19}\) Finally, in \( t = 2 \), there is no further uncertainty and the amount invested in \( t = 1 \) produces \( A k_1^i \), where \( A > 0 \). Also in \( t = 2 \), banker \( i \) can sell the capital to the consumers for the price of \( q_2 \), after he pays the refinancing cost, \( \gamma \).

State Contingent Debt Contracts Subject to Borrowing Constraints

In \( t = 0 \), each banker is endowed with \( n \) units of the consumption good. \( n \) is an exogenous parameter and in this model represents the equity of the banker. Banker \( i \) can also borrow from the consumers that are not the equity owners of bank \( i \). However, credit markets are imperfect due to an agency friction in the spirit of Hart and Moore (1994) and Kiyotaki and Moore (1997). I assume that the state of nature is verifiable but banker \( i \) can always run away with the cash flow \((a_{1s}k_0^i \text{ and } A k_1^i)\) and can also withdraw her human capital.

\(^{16}\)Implicitly, I shut off any frictions between the firm and the bank, which, while relevant for the real world, are not the focus of this paper because they will not affect the key qualitative results.

\(^{17}\)Alternatively, I could have set \( \gamma \) to zero and assumed that \( a_{1l} < 0 \) which would be another way to generate a fire sale.

\(^{18}\)This assumption is just for simplicity and can be relaxed.

\(^{19}\)The results do not change if simple sums are used instead of averages.
Given these assumptions, the contract that emerges in equilibrium is a short-term, state contingent debt contract subject to a collateral constraint on the part of the banker. In $t = 0$, banker $i$ can sell a promise to pay $d_{1s}^i$ units of the consumption good in $t = 1$, state $s$, at the price $\pi_s p_{1s}$ per unit. This implies that the total period zero borrowing of banker $i$ is $\sum_s \pi_s p_{1s} d_{1s}^i$. Also in $t = 1$ state $s$, banker $i$ can sell a new promise to pay $d_{2s}^i$ units of the consumption good in $t = 2$ state $s$ at the price $p_{2s}$ per unit. As a result, his period one, state $s$ borrowing is $d_{2s}^i p_{2s}$. The prices, $p_{1s}$ and $p_{2s}$, will be determined in equilibrium.

I assume that if banker $i$ borrows in period zero, in period one he can give an enforceable "take it or leave it" offer to the lender — either take $\theta (q_{1s} - \gamma) k_{0s}^i$ or the bank will be closed (the banker will withdraw his human capital) and no output will be produced in period two. $1 - \theta$ is the fraction of the value of the collateral that has to be paid in legal fees if the consumer has to seize the collateral. (One can set $\theta$ equal to one and all the results remain.) If the bank is closed, the consumer will withhold the capital, pay the legal fees and the refinancing cost and resell the capital, which will generate $\theta (q_{1s} - \gamma) k_{0s}^i$ units of the consumption good. In equilibrium, the consumer will always accept the "take it or leave it" offer. Anticipating that, in $t = 0$, the consumer will write only a short-term, state contingent debt contract with the banker subject to the collateral constraint, $d_{1s}^i \leq \theta (q_{1s} - \gamma) k_{0s}^i$. Once the banker repays his old debt in $t = 1$, he can enter a new collateralized contract with the consumers. Since in period two he will never pay more than the value of the collateral after legal fees, the banker can borrow only against the collateral constraint, $d_{2s}^i \leq \theta (q_{2s} - \gamma) k_{1s}^i$.

Finally, in the low state in period one (the crisis state), banker $i$ will receive a bail-out equal to $B_i^1$ as a transfer from the government, where $B_i^1$ is endogenously determined. Banker $i$ can pay dividends every period and state of nature but he will optimally do so only in the last period, $t = 2$, when he gives all of the profits to the consumers.

### 2.2 Consumers

There is a continuum of risk neutral consumers distributed uniformly over the unit interval. They are the only agents that consume in this economy. In every period and state of nature, every consumer receives an endowment $e$. He can enter a state contingent debt contract with each banker both in periods zero and in period one, as described in the previous section. The preferences of the representative consumer are given by

$$U_0^c = c_0 + \sum_s \pi_s (c_{1s} + c_{2s})$$

Consumers also have an access to a production technology which uses capital as an input good and transforms it into the consumption good within the same period. Once the production

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20 In fact, for the simulations presented in the paper, I will set $\theta$ equal to 1.
technology produces the consumption good, the capital depreciates one hundred percent. When modelling the production technology of the consumers, in order to generate a downward sloping demand for capital, I use an approach similar to many papers in the literature on financial frictions that follow the seminal paper of Kiyotaki and Moore (1997). In \( t = 0, 1 \), the production technology of the consumers is given by \( F(k_{ts}^T) \) where \( k_{ts}^T \) is the amount of capital employed. In \( t = 2 \), I assume that the production technology is such that one unit of capital is transformed into one unit of consumption. The assumption that the production technology is different across periods is a simplification and can be relaxed. However, given that the model is a finite period model, relaxing it would imply that there will be always a fire-sale in \( t = 2 \) both in the high and the low state. This will be an unappealing feature of the model given that in reality crises are followed by normal times.

The production technology satisfies the following assumptions. \( F(k_{ts}^T) \) is at least three times differentiable on \((0, \infty)\). \( F(k_{ts}^T) \) and \( F'(k_{ts}^T) \) are continuous on \([0, \infty)\). Also \( F(0) = 0, F'(0) = 1, F''(0) = F'''(0) = 0, F''(k_{ts}^T) < 0 \) if \( k_{ts}^T \in (0, \infty) \) and \( \lim_{k_{ts}^T \to \infty} F'(k_{ts}^T) \geq \gamma \). The assumptions made on \( F(\cdot) \) guarantee that if there is no fire-sale, it will be never profitable for the consumers to transform consumption into capital one-to-one in order to use the production technology since the marginal product of capital is less than one. Therefore, from market clearing, it will be the case that \( k_{ts}^T = k_{ts}^F \). The assumption \( F''(\cdot) < 0 \) guarantees that the larger the fire-sale is, the lower the price of capital will be, which is a proxy for a downward sloping demand for capital. Finally, \( \lim_{k_{ts}^T \to \infty} F'(k_{ts}^T) \geq \gamma \) ensures that the resale price of capital, \( q_{ts} \), is always greater than the refinancing cost and, hence, all the capital is refinanced. Notice that the bankers’ production technology is more productive than the consumers’, which is a key assumption.

If the production technology of the consumer is employed in \( t = 0, 1 \), which will be the case only when there is a fire-sale, the price of capital will be pinned down by the marginal product of capital of the consumers’ production technology, \( q_{ts} = F'(k_{ts}^T) \). If it is not employed, then \( q_{ts} = 1 \). In \( t = 2 \), the price of capital will be always equal to one, \( q_{2s} = 1 \), given the assumption that capital can be transformed into consumption one-to-one. In equilibrium, in \( t = 0 \), \( q_0 = 1 \), because bankers enter period zero with no capital and, therefore, there will never be a fire-sale. I will consider parametrization where in \( t = 1 \), there will be a fire-sale only in the low state, \( q_{1h} < 1 \), and not in the high state, \( q_{1h} = 1 \). Since consumers are risk neutral, in equilibrium, the prices of the state contingent debt contracts will be \( p_{1s} = p_{2s} = 1 \). A detailed solution to the consumer’s problem is provided in the Appendix, Section A.1.

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\(^{21}\) The assumption that 100% of the capital depreciates after the production technology of the consumers is employed can be relaxed and is only for simplicity.

\(^{22}\) The fact that crises are followed by normal times will be naturally captured if the model is extended to an infinite period model and the production technology is given by \( F'(k_{ts}^T) \) in all periods.

\(^{23}\) More precisely, the assumptions that guarantee this result are that \( F'(k_{ts}^T) \) is continuous on \([0, \infty)\), \( F'(0) = 1 \), and \( F''(k_{ts}^T) < 0 \) if \( k_{ts}^T \in (0, \infty) \).
2.3 Policy Maker

The policy maker optimizes the welfare of the consumers who are also the owners of the banks. He has an access to ex-ante and ex-post policy instruments. The ex-ante policy instruments are a minimum bank capital requirement and a limit on the payment promised in the crisis state by bankers. The minimum bank capital requirement is defined as \( \rho^i \leq \frac{n}{k_0} \) where \( \rho^i \) is the minimum bank capital ratio of bank \( i \).\(^{24}\) \( \rho^i \) is the minimum fraction of period zero investment that has to be financed using equity. I chose to focus throughout most of the paper on the minimum bank capital requirement as the ex-ante policy instrument used to address overinvestment, because it is the regulatory policy currently implemented by almost all countries worldwide. I allow the policy maker also explicitly to restrict the amount of payments a banker pledges in the crisis state by imposing the constraint \( d_{il}^i \leq \nu^i \). While such an instrument is currently not used in practice, in this paper I will show that the presence of such an instrument will allow policy makers of countries with strong moral hazard to improve upon aggregate welfare. From a regulatory perspective, in order to map this regulation to contracts used in the real world, policy makers would want to limit the liabilities bankers face in a crisis state due to standard debt contracts as well as derivative contracts such as CDS contracts.

The ex-post regulatory instrument is an optimal government bail-out during crises. I implicitly assume that the government can circumvent the collateral constraint of bankers during crises via its power to tax the consumers and transfer resources to the bankers. For simplicity, I assume that bail-outs are prohibitively costly when there is no crisis due to high political costs of transferring money from taxpayers to the financial sector in normal times.\(^{25}\) Denote the levied tax as \( T_l \). Market clearing implies that \( T_l = B_l \). I assume that taxation is costly, and this cost can vary across countries. The size of the deadweight loss from taxing is given by \( \delta (B_l, \chi) \) where \( 0 \leq \chi < \infty \). \( B_l = \sum_{i=1}^{N} \frac{1}{N} B_l^i \) is the aggregate bail-out and \( B_l^i \) is the targeted bail-out given to bank \( i \).\(^{26}\)

The parameter \( \chi \) captures the ability of a country to provide an extra dollar of bail-out, which I call fiscal capacity in this paper. The smaller \( \chi \) is, the more fiscally constrained the country is (lower fiscal capacity). Instead of introducing the exogenous parameter \( \chi \), I could have chosen to model the cost of the bail-out simply as the deadweight loss from the taxes, \( T_l \), scaled by the tax base, \( e \), which would have a closer mapping to the public policy literature. In that case one can interpret the fiscal capacity parameter \( \chi \) as \( e \) — the tax base of the country. However, in reality, bail-outs are financed in three different ways — by taxing the residents of the country during the financial crisis, by sovereign borrowing or by printing money if the government has an access to an

\(^{24}\) In order to link the model to the Basel III minimum capital requirement, one can interpret \( k \) as loans, fixed income and investment in stocks and the state contingent borrowing, \( d \), as derivative positions. The reason why in this model I assume that no equity provisions are required for \( d \) (and only for \( k_0 \)) is because in Basel III derivative positions require equity holdings only if there is counterparty risk and in this model there is no counterparty risk. Also it will become clear that regulators should target the liquidity of the banking sector in a crisis. Therefore, requiring equity holdings for all derivative positions will not be the optimal policy.

\(^{25}\) One can relax the assumption that there are bail-outs only in the crisis state. The qualitative results remain.

\(^{26}\) This assumption captures the fact that it is the aggregate tax size that affects the deadweight loss from taxing.
independent monetary policy. The costs associated with sovereign borrowing, taxing and printing money can be realistically proxied using convex cost functions. In equilibrium, a country which has access to all three instruments will use all of them up to the point where the marginal cost of taxing is equal to the marginal cost of printing money and to the marginal cost of sovereign borrowing, which will, in turn, equal the marginal cost of the bail-out. As a result, even if I were to write a full blown model with sovereign borrowing and printing of money, in addition to taxing, the marginal cost of the bail-out would be a sufficient statistic. Therefore, I choose to model the marginal cost of taxing and, hence, the bail-out, \( \frac{\partial \delta(B_t, \chi)}{\partial B_t} \), in a reduced form way in order to obtain analytical results. The interpretation of \( \chi \) is flexible and should be linked not only to the size of the tax base of the country but also to variables such as the size of sovereign debt, the cost of printing money and also the tax collection ability of a country.

I assume that \( \delta(B_t, \chi) \) is a convex and increasing function with respect to the aggregate bail-out, which implies \( \frac{\partial \delta(B_t, \chi)}{\partial B_t} > 0 \) and \( \frac{\partial^2 \delta(B_t, \chi)}{\partial^2 B_t} > 0 \) if \( \chi < \infty \). The larger the total size of the bail-out is, the larger the deadweight loss is and also the marginal cost of the bail out increases with the total size of the bail out. The convexity of the deadweight loss is a standard assumption in the public finance literature which captures the distortionary cost of labor taxation in a reduced form way. A few additional assumptions are required. For a given \( B_t \), I assume that \( \frac{\partial \delta(B_t, \chi)}{\partial \chi} < 0 \) and \( \frac{\partial^2 \delta(B_t, \chi)}{\partial B_t \partial \chi} < 0 \), which implies that the larger the fiscal capacity of a country is, the smaller the deadweight loss from the bail-out is and the smaller the marginal cost of the bail-out is. The final set of assumptions are that \( \frac{\partial \delta(B_t, \chi)}{\partial^2 B_t} < 0 \) and the deadweight loss due to the bail-out, \( \delta(B_t, \chi) \), the marginal cost of the bailout, \( \frac{\partial \delta(B_t, \chi)}{\partial B_t} \) and \( \frac{\partial^2 \delta(B_t, \chi)}{\partial^2 B_t} \) are all equal to zero when \( \chi \to \infty \) (infinite fiscal capacity) and approach infinity when \( \chi = 0 \) (no fiscal capacity). For example, a functional form that satisfies all of these conditions and will be used in the simulations is \( \delta(B_t, \chi) = \frac{1}{\chi} B_t^{\eta} \) where \( \eta > 1 \).

### 2.4 Assumptions

In addition to the assumptions made so far on the functional forms of \( F(\cdot) \) and \( \delta(\cdot) \), in order to have a well defined and non-trivial problem, I also assume that the following inequalities are satisfied. The first assumption is that the expected return on the period zero investment is less than one, when the return in the low state is zero

\[
\pi_h (1 + a_{1h} - \gamma) < 1 \quad \text{Assumption 1}
\]

If Assumption 1 is violated, it will always be optimal for the banker to lever to the maximum in period zero and invest as much as possible, which will make the problem trivial. The following assumption ensures that if there is no fire-sale, the expected return on period zero investment is greater than the cost

\[
1 < \sum_s \pi_s [1 - \gamma + a_{1s}] \quad \text{Assumption 2}
\]
If Assumption 2 was violated, period zero investment would be always zero. In order to have a fire-sale in the model, it has to be the case that the fraction of the capital value that can be pledged, \( \theta (1 - \gamma) \), plus the return on period zero capital in the crisis state, \( a_{1l} \), is less than the refinancing cost of capital, \( \gamma \). I also assume that the refinancing cost is less than one, which is the highest possible price of capital.

\[ a_{1l} + \theta (1 - \gamma) < \gamma < 1 \]

Assumption 3

To ensure uniqueness, I also assume\(^{27}\)

\[
F' (k_{1l}^T) - \theta (1 - \gamma) + F'' (k_{1l}^T) k_{1l}^T > 0
\]

Assumption 4

\[
\frac{\partial (F' (k_{1l}^T) - \theta (1 - \gamma) + F'' (k_{1l}^T) k_{1l}^T)}{\partial k_{1l}^T} = F''' (k_{1l}^T) k_{1l}^T F'' (k_{1l}^T) < 0
\]

Assumption 5 guarantees that in the high state there is never a fire-sale (i.e. the output per unit of \( k_0 \) is higher than the refinancing cost of \( k_0 \))

\[ a_{1h} > \gamma \]

Assumption 5

Assumption 6 is necessary in order to generate fire sales in the model. It guarantees that the optimal bail-out and the amount of resources that can be transferred to the crisis state using state contingent debt contracts are not sufficient to cover the refinancing cost of period zero capital during the crisis state.

\[
\frac{(a_{1l} - \gamma)^n}{1 - (1 - \gamma)} + (\delta')^{-1} \left( \frac{A - \gamma}{1 - \theta (1 - \gamma)} \right) < 0
\]

Assumption 6

Assumption 7 ensures that the return to period zero and period one investment (after the refinancing cost is paid) is non-negative

\[ A - \gamma > 0, \quad a_{1s} \geq 0 \]

Assumption 7

### 3 Solving the Model

#### 3.1 Decentralized Equilibrium

This section develops the optimization problem of banker \( i \), assuming no commitment and that the policy maker provides an optimal ex-post bail-out during a crisis. For now, I do not introduce any ex-ante policy instruments in order to compare the decentralized equilibrium with no ex-ante regulation to the constrained Central Planner’s allocation and to prove that there is a role for ex-ante regulation. Throughout the paper, I will do the analysis for any number of banks \( N \), and

\(^{27}\)In models with binding borrowing constraints there might be multiple equilibria because both the demand for and the supply of fire-sold capital is downward sloping. For details, see Lorenzoni (2008). The possibility of multiple equilibria in this class of models is interesting but is not the focus of this paper.
that, if the low state is realized, the policy maker chooses

\[ B \text{ in } t \]

He also takes into account that his choice of payment in the last period by choosing the profits as dividends to the consumers. At the end of that if \( N < \infty \), banker \( i \) internalizes the fact that his actions affect the price of capital in the middle period, because they affect the aggregate fire-sale. Furthermore, in equilibrium, it will be never optimal to pay dividends in \( t = 0 \) and \( t = 1 \) and, hence, I omit those choice variables from the set-up without loss of generality. Since I assume that banker \( i \) does not have an access to a commitment technology, I solve the model backwards.\(^\text{28}\)

The actions in reverse order are the following. In \( t = 2 \), all bankers produce and pay out all the profits as dividends to the consumers. At the end of \( t = 1 \), banker \( i \) maximizes the dividend payment in the last period by choosing \( \{ k_{1s}^i, d_{2s}^i \} \) and taking as given the endogenous state variables \( \{ B_s^i, k_0^i, d_{1s}^i \} \). Banker \( i \) maximizes

\[
\max_{k_{1s}^i, d_{2s}^i} (A + 1 - \gamma) k_{1s}^i - d_{2s}^i
\]

subject to the collateral constraint in \( t = 1 \)

\[
d_{2s}^i \leq \theta (1 - \gamma) k_{1s}^i \quad \quad [\lambda_{2s}^i]
\]

where the Lagrangians are given in square brackets. Banker \( i \) also takes into account the period one budget constraint

\[
k_{1s}^i F'(k_{1s}^i) + d_{1s}^i \leq (F'(k_{1s}^i) + a_{1s} - \gamma) k_0^i + B_s^i + d_{2s}^i \quad \quad [z_{1s}^i]
\]

where \( B_h^i = 0 \) since I assumed bail-outs are prohibitively costly if there is no crisis.

At the beginning of \( t = 1 \), first, banker \( i \) repays the promised debt \( d_{1s}^i \) to the consumers. After that, if the low state is realized, the policy maker chooses \( B_s^i \) given the state variables \( \{ k_{1s}^i, d_{1s}^i \} \). He also takes into account that his choice of \( B_s^i \) affects the choices banker \( i \) will make at the end of \( t = 1 \). Given that consumers are risk neutral, the objective function of a benevolent policy maker in \( t = 1 \) in the low state is to maximize last period’s total output.

\[
\max_{k_{11}^i, B_s^i} 2e + F(k_{11}^i) - F'(k_{11}^i) k_{11}^i - \delta(B_i; \chi) + d_{1l}
\]

\[
+ \sum_{i=1}^{N} \frac{1}{N} \left[ z_{11}^{i,1,F} \left((A + 1 - \gamma) k_{11}^i - d_{2l}^i - B_s^i + (A + 1 - \gamma) k_0^i + B_s^i + d_{2l}^i - k_{11}^i F'(k_{11}^i) - d_{1l}^i)\right) \right]
\]

\(^{28}\)Potentially, there can be time inconsistency in the decentralized problem. When banks are large, they internalize the fact that their period one investment decision will affect the tightness of their period zero borrowing constraint against the low state, if it is binding, through the fire sale price. There will be no time inconsistency in the case with a continuum of banks. For more details see Davila (2011).
Proposition 1

Given Assumptions 1-7 and the Assumptions made on the functional forms of $F(\cdot)$ and $\delta(\cdot)$, considering a symmetric equilibrium with an ex-post optimal bail-out and no ex-ante regulation, one can prove the following

1. There is no fire-sale in the high state, $q_{1H} = 1$, and there is a fire-sale in the low state, $q_{1L} < 1$.
2. Given the additional Assumption 8 provided in the Appendix, (required only for the $N < \infty$ case), the equilibrium is unique and exists and is one of the following types:

   Type 1) $z_0 = z_{1L} > z_{1H}$; ($\lambda_{1L} = 0$, $\lambda_{1H} > 0$) (interior equilibrium)

   Type 2) $z_0 > z_{1H}$; ($\lambda_{1H} > 0$) (corner equilibrium where the banker borrows to the maximum in $t = 0$) where

\[
Z_0 = \sum_s \pi_s \left[ z_{1s} \left( F^T(k_{1s}^T) + \frac{1}{N} F''(k_{1s}^T) k_{1s}^T + a_{1s} - \gamma + \frac{\partial B_l}{\partial k_0} \right) + \lambda_{1s} \theta \left( F'(k_{1s}^T) - \gamma + \frac{1}{N} F''(k_{1s}^T) k_{1s}^T \right) \right] \tag{5a}
\]

\[
Z_{1s} = \frac{A + (1 - \theta) (1 - \gamma) - \lambda_{1s} \theta A}{\frac{1}{N} F''(k_{1s}^T) k_{1s}^T - \frac{\partial B_l}{\partial k_{1s}}} + F'(k_{1s}^T) - \theta (1 - \gamma) \tag{5b}
\]

I also assume that the endowment of the consumer, $e$, is large enough so that $B_t < e + d_{1L} - p_2 d_{2L}$. 

At the end of $t = 0$, banker $i$ optimizes expected dividends paid to the consumers in $t = 2$, taking into account his future optimal actions and the first order condition of the policy maker with respect to the bail-out in the crisis state. The period zero objective function of banker $i$ is given by

\[
\max_{k_0^i, d_{1s}^i} \sum_s \pi_s \left[ (A + 1 - \gamma) k_{1s}^i - d_{2s}^i \right]
\]

subject to his period one budget constraints given by equation 2 (with a period zero Lagrangian $\pi_s z_{1s}^i$). Also banker $i$ takes into account the period zero budget constraint

\[
k_0^i \leq \sum_s \pi_s d_{1s}^i + n \quad [z_0^i]
\]

The optimization problem is also subject to the period zero collateral constraints.

\[
d_{1s}^i \leq \theta (F^T(k_{1s}^T) - \gamma) k_0^i \quad [\pi_s \lambda_{1s}^i]
\]

For detailed derivations see Appendix, Section A.2.
The optimal bail-out is pinned down by

\[ 1 + \frac{\partial \delta (B_t, \chi)}{\partial B_t} = z_1^{1P} = \frac{F''(k_{1h}^T) k_{1}^T + A + (1-\theta)(1-\gamma)}{F''(k_{1l}^T) k_{1l}^T + F' (k_{1l}^T) - \theta(1-\gamma)} \]  

(5c)

where

\[ \frac{\partial B^i_j}{\partial k_{1l}^{i,F}} = \frac{\partial B^i_j}{\partial k_{0}^{i,F}} = \frac{-\partial B^i_j}{\partial k_{1l}^{i,F}} = \frac{1}{\partial^2 \delta(B_t, \chi)} N \frac{\partial z_1^{1P}}{\partial k_{1l}^{i,F}} \geq 0 \]  

(5d)

Also \( B_{1h}^i = 0 \) and \( \frac{\partial z_1^{1P}}{\partial k_{1l}^{i,F}} \) is given by equation 24 in the Appendix. The first order conditions with respect to \( d_{2s} \) and \( d_{1s} \) imply \( \lambda_{2s} > 0 \) and \( \lambda_{1s} = z_0 - z_{1s} \geq 0 \).

**Proof of Proposition 1.** See Appendix, Section A.4.1. □

The key variables that characterize the equilibrium type are \( z_{1s} \) and \( z_0 \). \( z_0 \) is the period zero marginal value of one unit of the consumption good (which, from now on, I will refer to as the marginal value of one dollar) in the hands of the banker as perceived by the banker. \( z_{1s} \) is the scaled marginal value of a dollar in \( t = 1 \), state \( s \), in the hands of the banker as perceived by the banker. When the banker decides optimally whether to keep an extra dollar in \( t = 0 \) or to "transfer" it to \( t = 1 \), state \( s \), using a state contingent contract, the relevant variables to compare are \( z_0 \) and \( z_{1s} \). If the banker saves/"transfers" an extra dollar from \( t = 0 \) to period \( 1 \), state \( s \), he will get \( \frac{1}{\pi_s} \) units of the consumption good in \( t = 1 \), state \( s \), and his ex-ante welfare will increase by \( (\pi_s z_{1s}) \frac{1}{\pi_s} = z_{1s} \). If the banker keeps the dollar in \( t = 0 \), his ex-ante welfare will improve by \( z_0 \). Throughout the rest of the paper, I will refer to \( z_{1s} \) and \( z_0 \) as the marginal value of wealth in the hands of the banker, as perceived by the banker, in \( t = 1 \), state \( s \), and in \( t = 0 \) respectively.

Let us analyze the first order conditions in more detail. One can prove that given the assumptions made \( z_{1s} > 1 \) and \( z_0 > 1 \). \( z_{1s} > 1 \) and \( z_0 > 1 \) imply that in \( t = 0 \) and in \( t = 1 \), the banker optimally does not pay dividends to the consumer whose marginal value of wealth is one.\(^{30}\) Also \( z_{1s} > 1 \) implies that the banker wants to transfer the maximum amount of resources from period two to periods zero and one since in the last period the marginal value of wealth is equal to one — the marginal utility of consumption. As a result, the period one borrowing constraints will always bind, implying that \( \lambda_{2s} > 0 \) and \( d_{2s} = \theta(1-\gamma) k_{1s} \). If the equilibrium is interior, then \( z_0 = z_{1l} > z_{1h} \), which implies that the banker values wealth more in period zero and in the crisis state than he values wealth in the high state in \( t = 1 \). Therefore, in period zero, the banker borrows to the maximum against the high state which implies \( \lambda_{1h} > 0 \) and \( d_{1h} = \theta(1-\gamma) k_0 \).

The economic intuition behind the first order conditions is the following. The first order condition of the policy maker with respect to the bail-out at the beginning of \( t = 1 \), equation 5c, determines the optimal bail-out, \( B_t \), by equating the marginal cost to the marginal benefit of the

\(^{30}\) The proof that \( z_{1s} > 1 \) can be found in Section A.2 of the Appendix and \( z_0 > 1 \) follows from the proof of Proposition 1 which shows that in both types of feasible equilibria types, it is the case that \( z_0 \geq z_{1s} \). This confirms the assumption made when solving the problem that dividends will be optimally paid out only in \( t = 2 \).
bail-out. The marginal cost of the bail-out is simply the direct transfer of one dollar from the consumer to the banker plus the marginal increase of the deadweight loss from the bail-out, $\frac{\partial \delta(B_t, \chi)}{\partial B_t}$. The marginal benefit of the bail-out is the marginal value of an extra dollar in the hands of the bankers in the crisis state as perceived by the policy maker, $z_{1i}^{1P} (k_{1i}^F)$. From equation 5c, it is clear that only the aggregate bail-out is pinned down, $B_t$, and not the bank specific bail-out, $B^i_t$. In order to solve the model, I assume that the equilibrium is symmetric where the government gives the same bail-out to each bank, $B_t = B^i_t$, and every banker internalizes that when making decisions in period $t = 0$.

By totally differentiating equation 5c with respect to $k_{1i}^{iF}$, one can derive how banker $i$ perceives his individual fire-sale to affect the targeted bail-out he receives, $\frac{\partial B^i_t}{\partial k_{1i}^{iF}}$, which will be an important term that enters the first order conditions with respect to $k^i_0$ and $k^{iF}_{1i}$. The formula for $\frac{\partial B^i_t}{\partial k_{1i}^{iF}}$ is given by equation 5d. When banks are not infinitesimally small and the country has some fiscal capacity, $N < \infty$ and $\chi > 0$, banker $i$ partially internalizes the fact that the larger his individual fire-sale is, the larger the optimal bail-out is, $\frac{\partial B^i_t}{\partial k_{1i}^{iF}} > 0$. Since the banker is large he realizes that his actions affect the fire-sale and, therefore, the optimal amount of bail-out received. A larger fire-sale implies a larger transfer of resources from the more productive sector — bankers — to the less productive sector — consumers, leading to a larger optimal bail-out (captured by the term $\frac{\partial z_{1i}^{1P}(k_{1i}^{iF})}{\partial k_{1i}^{iF}} > 0$). For countries with a large number of banks, for a given level of the fire-sale, the perceived impact of the individual fire-sale on the aggregate fire-sale is smaller (captured by the term $\frac{1}{N}$). In the limiting case with a continuum of banks, $N \rightarrow \infty$, $\frac{\partial B^i_t}{\partial k_{1i}^{iF}} = 0$. This result will be crucial in order to prove later on that in the case with a continuum of banks there is no moral hazard while there will be moral hazard if banks are large. Finally, the bail-out received is smaller, for a given level of the fire-sale, if the country is more fiscally constrained, (captured by the fact that $\frac{\partial \delta(B_t, \chi)}{\partial B_t}$ is positive and increases when $\chi$ decreases). When $\chi = 0$, the bail-out will be zero and, hence, $\frac{\partial B^i_t}{\partial k_{1i}^{iF}} = 0$.

Equation 5b is the first order condition with respect to $k_{1s}$ (from the period zero optimization problem), which pins down the marginal value of wealth in the hands of the bankers in period one as perceived by the banker, $z_{1s}$. In equilibrium, $z_{1s}$ is equal to the marginal benefit of $k_{1s}$ over the "effective" marginal cost of purchasing an extra unit of $k_{1s}$. The marginal benefit of an extra dollar invested in $t = 1$ is the cash flow received in $t = 2$, $A$, plus the resale value of capital of

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\[^{31}\text{In this model, with linear production technology of the bankers, the policy maker is indifferent whether to give the bail-out money to Bank of America which takes over Merill Lynch (the bank that needs the bail-out) or he gives the money directly to Merill Lynch. Optimally, the policy maker wants to achieve an aggregate fire sale of } k_{1i}^F = k_{1i}^T \text{ which is determined by equation 5c. For a proof of this result, see Appendix, Section A.2. Of course, the ex-post bail-out design affects the ex-ante incentives of banks. Acharya, Shin, and Yorulmazer (2011) and Nosal and Ordonez (2013) are two interesting papers that address the question of what is the optimal ex-post bail-out design that minimizes moral hazard ex-ante.}

In this model I focus on an environment where the policy maker has a sufficient number of ex-ante instruments to correct for the moral hazard (he can replicate the constrained Central Planner’s allocation). Therefore, the ex-post bail-out design is not crucial for the comparative static of the optimal minimum bank capital requirement with respect to fiscal capacity.
one minus the refinancing cost minus the debt payment \((1 - \gamma) - \theta (1 - \gamma)\). If the Lagrangian on the period zero borrowing constraint is binding, \( \lambda_{1s} > 0 \), and there is a fire-sale, the extra unit of capital purchased in period one will increase the resale value of period zero capital and relax the period zero borrowing constraint, which is captured by the \(-\lambda_{1s} \theta \frac{1}{N} F''(k_{1s}^T) k_0 > 0\) term. This will increase the marginal benefit of \(k_{1s}\). Next, let us consider the denominator of equation 5b, which is the "effective" marginal cost of an extra \(k_{1s}\). The direct marginal cost of investment is the price of an extra unit of investment \(q_{1s} = F'(k_{1s}^T)\), which is reduced by the fact that the banker can lever against the capital — captured by the term \(-\theta (1 - \gamma)\). The indirect marginal cost is given by the term \(\frac{1}{N} F''(k_{1s}^T) k_{1s}^T - \frac{\partial B_1^i}{\partial k_{1s}}\) and is relevant only if the bank is not infinitesimally small, \(N < \infty\), and there is a fire-sale. The indirect cost includes a monopolistic effect, \(\frac{1}{N} F''(k_{1s}^T) k_{1s}^T < 0\), which makes the "effective" cost of period one capital lower because the banker realizes that an extra unit of period one capital will increase the per unit price of the fire-sold capital for a given \(k_0\). In addition, if the bank is large and there is a fire-sale, the bank also realizes that an extra \(k_{1s}\) will decrease the marginal bail-out received because the fire-sale will be smaller, which is captured by the term \(-\frac{\partial B_1^i}{\partial k_{1s}} > 0\). This last effect will increase the "effective" marginal cost of period one capital.

The first order condition with respect to \(k_0\), given by equation 5a, pins down \(z_0\) — the marginal value of wealth in period zero as perceived by the banker. Since the price of period zero investment is one, an extra dollar in \(t = 0\) implies an extra unit of investment purchased in \(t = 0\). Therefore, the marginal value of wealth in period zero, \(z_0\), in equilibrium, is equal to the marginal benefit of \(k_0\). In \(t = 1\), state \(s\), the return from an extra dollar \(k_0\) is the cash flow, \(a_{1s}\), plus the resale price, \(q_{1s} = F'(k_{1s}^T)\), minus the refinancing cost, \(\gamma\). In addition, if the banker is not infinitesimally small, he internalizes the fact that the "effective" marginal return to \(k_0\) is smaller, because an extra \(k_0\) (for a given \(k_{1s}\)) leads to a larger fire-sale and a lower price of the fire-sold capital. Hence the return will decrease by \(\frac{1}{N} F''(k_{1s}^T) k_{1s}^T < 0\) (a monopolistic effect). However, higher \(k_0\) will also increase the perceived bail-out received by increasing the fire-sale, which is captured by the term \(\frac{\partial B_1^i}{\partial k_0} > 0\). This will increase the "effective" return to an extra dollar invested in period zero. Notice that the banker internalizes only the benefit but not the cost of the bail-out. The "effective" period one return on \(k_0\) is re-invested, which is why it is multiplied by the marginal value of wealth in \(t = 1\) state \(s\), \(z_{1s}\). If the period zero collateral constraint is binding, an extra \(k_0\) has the additional benefit of relaxing the borrowing constraint, which is why the marginal benefit of \(k_0\) also includes the term \(\lambda_{1s} \theta (F'(k_{1s}^T) - \gamma + \frac{1}{N} F''(k_{1s}^T) k_0)\).

**Graphical Proof of Existence and Uniqueness**

In order to solve for the optimal allocation, first, I solve for the optimal amount of period zero investment, \(k_0\). I will provide intuition for the proof of existence and uniqueness using a graphical approach of how \(k_0\) is determined. In subsequent sections, I will build on Figure 1 to prove that there is overinvestment if the policy maker does not have an access to any ex-ante regulatory instruments.

Define the following function of \(k_0\)

\[
\psi(k_0) = z_{1s}(k_0) - z_0(k_0) \text{ where } k_0 \in \left[ k_0, t_{0}^{\text{max}} \right]
\]
where in Proposition 1 of the Appendix I derive the relevant range \([k_0, k_0^{\text{max}}]\). Also \(z_{1l}(k_0)\) and \(z_0(k_0)\) are the marginal values of wealth in period zero and in the low state in period one if the equilibrium is of Type 1. If the equilibrium is of Type 1 (interior equilibrium), then \(k_0^*\) will be determined by \(\psi(k_0^*) = 0\), where the star denotes equilibrium values. If the equilibrium is of Type 2 (corner equilibrium) then \(k_0^* = k_0^{\text{max}}\) and the bank will borrow to the maximum in \(t = 0\).^32

I focus on the interior equilibrium of Type 1 which will be the more interesting case. \(k_0^*\) is pinned down using the first order condition with respect to \(d_i\), which is similar to an Euler equation and given by \(z_{1l}(k_0^*) = z_0(k_0^*)\). In equilibrium, the banker is indifferent between investing an extra dollar in period zero and saving the extra dollar towards the crisis state using a state contingent contract. ^33

Figure 1: Existence and Uniqueness of the Decentralized Equilibrium

First, notice that \(\frac{\partial z_{1l}}{\partial k_0} > 0\) because the marginal value of wealth in the crisis state, as perceived by the banker increases with period zero investment, while the marginal value of wealth in period zero decreases with period zero investment, \(\frac{\partial z_0}{\partial k_0} < 0\). Let us start with \(\frac{\partial z_{1l}}{\partial k_0} > 0\). In the case with a continuum of banks, \(N \to \infty\), \(\frac{\partial z_{1l}}{\partial k_0} > 0\) because a larger \(k_0\) increases the fire-sale which lowers the price of capital during a crisis. As a result, an extra dollar is more valuable in the crisis state since it can purchase more units of capital. If banks are large, \(N < \infty\), in order for \(\frac{\partial z_{1l}}{\partial k_0} > 0\), Assumption 8 is required as well. Assumption 8 guarantees that the "effective" cost of \(k_{1l}\) decreases as \(k_0\) increases, because the direct effect of the price decrease as \(k_0\) increases is not offset by the fact that the perceived marginal bail-out received increases with \(k_0\), \(\frac{\partial^2 B_{1l}}{\partial k_{1l}^2} \frac{\partial k_{1l}}{\partial k_0} > 0\).

Next let us consider why \(\frac{\partial z_0}{\partial k_0} < 0\). In equilibrium \(z_0\) is equal to the marginal benefit of an extra \(k_0\). An extra \(k_0\) will lead to a larger fire-sale and to a lower re-sale price of \(k_0\) in the crisis state,

^32The equilibrium is of Type 2 if \(\psi(k_0) < 0\) for all \(k_0 \in \left[k_0, k_0^{\text{max}}\right]\).

^33If the banker invests the extra dollar in period zero, his ex-ante welfare increases by the benefit of investing an extra unit of \(k_0\), which is given by \(z_0(k_0)\). If he saves the extra dollar towards the crisis state and invests it then, his ex-ante welfare increases by \(z_{1l}(k_0)\). Figure 1 depicts \(\psi(k_0)\).

^34The functional forms used for \(\delta(\cdot)\) and \(F(\cdot)\) in the graph are given in the beginning of the Appendix. The parameters used to produce Figure 1 are \(\gamma = 0.7, \alpha = 0.8, A = 1, a_{1l} = 1.5, a_{1l} = 0, \pi_h = .55, n = .5, \theta = 1, \eta = 1.5, N = 10\).
thereby lowering the marginal benefit of an extra dollar invested in $t = 0$. In the case of a finite number of banks, $N < \infty$, Assumption 8 guarantees that this direct effect dominates the indirect effect, where larger $k_0$ leads to a larger fire-sale in the crisis state and larger perceived marginal bail-out, $\frac{\partial^2 B^*_i}{\partial k_i^T \partial k_0} > 0$, which increases the marginal benefit of an extra dollar invested in $t = 0$.

Therefore, as $k_0$ increases, $z_1$ increases and $z_0$ decreases, leading to $\psi'(k_0) > 0$. As a result, $\psi(k_0)$ will cross the zero line at most once.

3.2 Constrained Central Planner’s Problem Without Commitment

In this section I solve for the constrained Central Planner’s problem without commitment. The constrained Central Planner optimizes the welfare of the consumers who are also the owners of the banks. The Central Planner faces exactly the same constraints as the banker in the decentralized equilibrium — the borrowing constraints plus the first order conditions of the consumer. Setting up the constrained Central Planner’s problem in such a way implies that he cannot directly transfer resources from the consumer to the banker unless he uses the bail-out instrument, which is costly.

Essentially, the Central Planner chooses the investment and borrowing decision of every banker, where he takes into account any externalities this decision imposes on the rest of the bankers and on consumers. Given that the equilibrium considered is symmetric, this maps into a problem where the Central Planner chooses aggregate variables, taking into account that his actions affect prices and, in particular, the price of capital in $t = 1$, $q_{1s} = F' \left( k_{1,s}^T \right)$. The Central Planner also takes into account the rest of the equilibrium prices given by $q_0 = q_{2s} = 1$, $p_{1s} = p_{2s} = 1$. In this section, I preserve the assumption that the bail-out will occur only if there is a fire-sale.

Solving the problem backwards, in $t = 1$, the Central Planner maximizes the welfare of the consumers

$$\max_{B_{1s}, k_{1s}, d_{1s}} 2c + F \left( k_{1s}^T \right) - F' \left( k_{1s}^T \right) k_{1s}^T + d_{1s} - \left( B_{s} + \delta \left( B_{s} \right) \right) + \left( A + 1 - \gamma \right) k_{1s} - d_{2s}$$

subject to the collateral constraint in period one, equation 1, with a Lagrangian given by $\lambda_{1s}^{CP}$, and subject to the period one budget constraint, equation 2, with a Lagrangian given by $z_{1s}^{1CP}$. Notice that the Central Planner, unlike the banker in the decentralized equilibrium, also internalizes the cost of the bail-out, $- \left( B_{s} + \delta \left( B_{s} \right) \right)$. Another difference from the optimization problem of the banker is that the Central Planner also takes into account that a larger aggregate fire-sale improves the welfare of the consumer via the profits from operating the consumer’s production technology given by $F \left( k_{1s}^T \right) - F' \left( k_{1s}^T \right) k_{1s}^T$. Last but not least, the Central Planner also internalizes the fact that the actions of a single banker impose an externality on consumers and other bankers by affecting the price of fire sold capital, $F' \left( k_{1s}^T \right)$. This mechanism will be at the heart of the pecuniary externality.

In $t = 0$, the Central Planner chooses $\{k_0, d_{1s}\}$, taking into account his future optimal actions, in order to optimize the following ex-ante welfare function
The optimal bail-out is determined by the following types:

(ii) The equilibrium of the constrained Central Planner’s problem exists and is unique and is one of state, \( q \)  with a Lagrangian given by \( \pi_s z_{1s}^{CP} \) and the budget constraint in \( t = 0 \), equation 3, where the Lagrangian is \( z_0^{CP} \). The Central Planner also takes into account the period zero collateral constraints given by equation 4 with a Lagrangian \( \pi_s \lambda_{1s}^{CP} \). For details on the set-up and the solution see Appendix, Section A.3.

**Proposition 2**

(i) Given Assumptions 1-7 and the assumptions made on the functional forms of \( F (\cdot) \) and \( \delta (\cdot) \), there is never a fire-sale in the high state, \( q_{1h} = 1 \) and there is a fire-sale in the low state, \( q_{1l} < 1 \).

(ii) The equilibrium of the constrained Central Planner’s problem exists and is unique and is one of the following types:

- Type 1) \( z_0^{CP} = z_{1l}^{CP} > z_{1h}^{CP} \) (interior equilibrium);
- Type 2) \( z_0^{CP} > z_{1s}^{CP} \) (corner equilibrium where the banker borrows to the maximum in \( t = 0 \)).

The optimal bail-out is determined by

\[
1 + \frac{\partial \delta (B_l, \chi)}{\partial B_l} = z_{1l}^{1,CP} \tag{6a}
\]

(iii) If also Assumption 9 is satisfied (provided in the Appendix, Section A.4.2), the only possible equilibrium is the interior equilibrium of Type 1 where

\[
\sum \pi_s (-F'' (k_{1s}^T) k_{1s}^T + z_{1s}^{CP} (F' (k_{1s}^T) + a_{1s} - \gamma + F'' (k_{1s}^T) k_{1s}^T)) + \pi_h \lambda_{1h}^{CP} \theta (1 - \gamma) = z_0^{CP} \tag{6b}
\]

\[
z_{1h}^{CP} = \frac{A + (1 - \theta) (1 - \gamma)}{1 - \theta (1 - \gamma)} \tag{6c}
\]

\[
z_{1l}^{1,CP} = z_{1l}^{CP} = \frac{F'' (k_{1l}^T) k_{1l}^T + A + (1 - \theta) (1 - \gamma)}{F' (k_{1l}^T) - \theta (1 - \gamma) + F'' (k_{1l}^T) k_{1l}^T} \tag{6d}
\]

and the first order conditions with respect to \( d_{2s} \) and \( d_{1s} \) imply \( \lambda_{2s}^{CP} > 0 \), \( \lambda_{1l}^{CP} = 0 \), \( \lambda_{1h}^{CP} = z_0^{CP} - z_{1h}^{CP} > 0 \).

**Proof of Proposition A.4.2.** See Appendix, Section A.4.2. ■

Similarly to the decentralized equilibrium, one can prove that \( z_{1s}^{CP} > 1 \) and \( z_0^{CP} > 1 \). The equilibria types, which are classified based on the borrowing contract between the banker and the consumer, are the same as the decentralized equilibria types. The Central Planner borrows to the maximum against the value of last period capital, \( \lambda_{2s}^{CP} > 0 \). Furthermore, if the equilibrium is interior, in \( t = 0 \), he borrows first against the high state and only then against the low state in \( t = 0 \), i.e. \( \lambda_{1l}^{CP} = 0 \) and \( \lambda_{1h}^{CP} > 0 \). The Central Planner perceives an extra dollar in the hands of the bankers to be more valuable in the crisis state rather than in the non-crisis state, \( z_{1l}^{CP} > z_{1h}^{CP} \), due to the presence of a fire-sale which increases the inefficient transfer of resources from the more
productive sector — bankers — to the less productive sector — consumers. Finally, since the Central Planner internalizes the cost and the benefit of the bail-out and in equilibrium he chooses a bail-out such that the marginal cost and benefit of the bail-out are equated, the bail-out does not enter the first order conditions of the Central Planner.

Regarding proving existence and uniqueness of the constrained Central Planner’s allocation, one can use a similar graphical approach as in the case of the decentralized equilibrium. Define the following function

$$\psi^{CP}(k_0) = z_{1l}^{CP}(k_0) - z_0^{CP}(k_0)$$

where $z_0^{CP}(k_0)$ and $z_{1l}^{CP}(k_0)$ are the marginal values of wealth in the hands of the bankers as perceived by the Central Planner in period zero and in the low state in period one if the equilibrium is of Type 1. If the equilibrium is of Type 1, $k_0^{CP}$ will be determined by $\psi^{CP}(k_0^{CP}) = 0$ and if the equilibrium is of Type 2, $k_0^{CP} = k_0^{\text{max}}$ (the bank will borrow to the maximum in $t = 0$). In order to prove existence and uniqueness it will be sufficient to prove that $\psi^{CPT}(k_0) > 0$. As before, larger $k_0$ implies larger fire-sale. As a result, more wealth is transferred from the more productive to the less productive sector and, hence, the marginal value of wealth in the crisis state as perceived by the Central Planner, $z_{1l}^{CP}$, is larger. At the same time, $z_0^{CP}$, which in equilibrium equals the marginal benefit of an extra $k_0$ as perceived by the Central Planner, decreases with $k_0$. The reason is that a larger $k_0$ implies a larger fire-sale and larger inefficient transfer of resources. Hence the marginal benefit of $k_0$, $z_0^{CP}$, decreases with $k_0$.

4 Overinvestment

In this sub-section I compare the constrained Central Planner’s allocation to the decentralized equilibrium with no ex-ante regulation. I prove that in period $t = 0$, when the economy is in normal times (no fire-sale), bankers overinvest relative to the constrained Central Planner. If $N \to \infty$, the overinvestment is due to future pecuniary externalities and, if $N < \infty$, it is due to both future pecuniary externalities and moral hazard.

Before proving overinvestment, first I prove Corollary 1, which states that the Central Planner perceives the marginal value of wealth in the hands of the banker in the crisis state to be larger than the banker in the decentralized equilibrium does, $z_{1l}^{CP} > z_{1l}$. This result will be the main component to the proof that there is ex-ante overinvestment in this model.

35 For details on the derivations, see the proof of Proposition 2.

36 In an extension in an older version of the paper, I showed that if the economy starts in a crisis state with a fire sale, contemporaneous pecuniary externalities can potentially lead to underinvestment and not overinvestment. The reason why this is the case is similar to the result in He and Kondor (2012).
Corollary 1  Conditional on Assumptions 1-7, 9 and the assumptions made on the functional forms of $F(\cdot)$ and $\delta(\cdot)$ and given Assumption 10 (required only for the $N < \infty$ case),

$$
\frac{1}{\delta''(B_1)N} \frac{\partial z^1_{1L}}{\partial k^T_{1l}} z_{1L}^{CP} > \left( \left(1 - \frac{1}{N}\right) z_{1L}^{CP} - 1 \right) F''(k^T_{1l}) k^T_{1l}
$$

Assumption 10

the Central Planner values an extra dollar in the hands of the banker in the crisis state by more than the banker does in the decentralized equilibrium, for a given $k_0$.

$$
z_{1L}^{CP} > z_{1L}
$$

Proof of Corollary 1.  See Appendix, Section A.4.3.

Having proved Corollary 1, I proceed to prove overinvestment.

Proposition 3  Conditional on Assumptions 1-8 and 10 and the Assumptions made on the functional forms of $F(\cdot)$ and $\delta(\cdot)$, comparing the constrained Central Planner’s allocation and the decentralized equilibrium with no ex-ante regulation, there is always overinvestment, $k_{0,CP}^* < k_0^*$, if the equilibrium is of Type 1 for the Central Planner (interior equilibrium) and there is no overinvestment, $k_{0,CP}^* = k_0^*$, if the equilibrium is of Type 2 for the Central Planner (corner equilibrium).

Proof of Proposition 3.  Since I already proved in the previous sections that given the assumptions made, $\psi^{CP}(k_0) > 0$ and $\psi'(k_0) > 0$, in order to prove that there is overinvestment, it is sufficient to show that as long as both the Central Planner’s and the decentralized equilibria are of Type 1, $\psi^{CP}(k_0) > \psi(k_0)$ (see Figure 2 below).  One can re-write $\psi^{CP}(k_0) - \psi(k_0)$ as

$$
\psi^{CP}(k_0) - \psi(k_0) = \frac{(z_{1L}^{CP}(k_0) - z_{1L}(k_0)) (1 - \theta (1 - \gamma) + \pi (\gamma - a_{1L}))}{1 - \pi b \theta (1 - \gamma)} > 0
$$

where the inequality follows from Assumption 3 and Corollary 1 and the expressions for $\psi^{CP}(k_0)$ and $\psi(k_0)$ are given by equations 52 and 46 in the Appendix.  It is clear that if the equilibrium is of Type 1 for the Central Planner (Assumption 9 is satisfied) and Type 2 for the banker, then there is overinvestment since Type 2 equilibrium implies that the banker will borrow and invest to the maximum in period zero.  Also if the equilibrium of the Central Planner is of Type 2, there will be no overinvestment and one can easily show that the equilibrium will be of Type 2 for the banker as well.

Figure 2 depicts $\psi(k_0)$ and $\psi^{CP}(k_0)$, using the same parametrization as in Figure 1.

From equation 8 one can see that proving overinvestment is equivalent to proving that $z_{1L}^{CP}(k_0) > z_{1L}(k_0)$, where the equilibrium is of Type 1 for the Central Planner and for the banker in the decentralized equilibrium.  In what follows, I present the intuition for the proof of 1 and the formal proof is presented in the Appendix, Section A.4.3.

There are two reasons why there is overinvestment in this model — future pecuniary externalities and moral hazard.
Future Pecuniary Externality

In order to isolate the pecuniary externality channel, let us first consider the case of a country with no fiscal capacity, $\chi = 0$, and hence, no bail-out $B_l = 0$. The lack of bail-out implies that the difference between $z_{1l}^{CP}(k_0)$ and $z_{1l}(k_0)$ is only due to the pecuniary externalities. If $\chi = 0$, the main reason why $z_{1l}^{CP}(k_0)$ differs from $z_{1l}(k_0)$ is that the Central Planner internalizes the fact that an extra dollar in the crisis state in the hands of a banker will decrease the fire-sale, which will relax the budget constraints of the other bankers and will ameliorate the inefficient transfer of capital from the more productive to the less productive sector.\(^{37}\) This effect pushes $z_{1l}^{CP}(k_0)$ to be larger than $z_{1l}(k_0)$.\(^{38}\) Since the Central Planner values wealth in the crisis state more than the banker does and higher $k_0$ leads to a lower bank net worth in the crisis state, the Central Planner optimally chooses a lower $k_0$ than the banker in the decentralized equilibrium (as can be seen on Figure 2).\(^{39}\) If banks are infinitesimally small ($N \to \infty$), they take the price of capital as given,\(^{37}\) This effect is captured by the $F''(k_{1l})k_{1l}^T < 0$ term in the denominator of $z_{1l}^{CP}(k_0)$. In contrast, if $\infty > N > 1$, the banker only partially internalizes his impact on prices (captured by the $\frac{1}{N}F''(k_{1l})k_{1l}^T < 0$ term in the denominator of $z_{1l}(k_0)$).

\(^{38}\) The Central Planner, unlike the banker, also internalizes the fact that an extra dollar in the hands of the banker in the crisis state implies a smaller fire-sale and lower profits for the consumer (which is captured by the $F''(k_{1l})k_{1l}^T$ term in the numerator of $z_{1l}^{CP}(k_0)$) which pushes $z_{1l}^{CP}(k_0)$ to be lower than $z_{1l}(k_0)$. One can think of this force as a monopolistic underinvestment force. Assumption 10 is a sufficient and necessary condition for $z_{1l}^{CP}(k_0)$ to be higher than $z_{1l}(k_0)$. For example, if $N \to \infty$, Assumption 10 is always satisfied while if $N = 1$, it will not be satisfied if $\chi = 0$. If $N = 1$, there could still be overinvestment due to the moral hazard channel if $\chi > 0$. Assumption 10 will not be required if the banker maximized consumer’s welfare rather than dividends even for the $N < \infty$ case.

\(^{39}\) It is a well known fact that in a standard Arrow-Debreu economy with no frictions, where agents are small and take prices as given, there are pecuniary externalities as well but there is no inefficiency (unlike in this model). The reason is that, in a standard Arrow-Debreu economy, the change in the price is just a wealth transfer from one agent to another and, in equilibrium, the marginal utility of wealth across agents is equalized, implying that the net effect on welfare is zero. This is why the assumption that bankers are more productive than consumers (i.e. they have different marginal valuations of wealth) is crucial for the pecuniary externalities in this model to lead to an inefficient decentralized allocation.
and the pecuniary externality is the strongest.  

**Moral Hazard**

First, consider the case of a continuum of banks. In that case, there is no moral hazard, despite the presence of a targeted bail-out. The proof is intuitive. If a country has a continuum of banks \( N \to \infty \), one can show that \( k_0 - k_0^{CP} \) does not vary with the size of the fiscal capacity, \( \chi \geq 0 \), and, therefore, with the size of the bail-out, implying that there is no moral hazard in the continuum of banks case. As a result, if \( N \to \infty \), the reason for the overinvestment is only due to the pecuniary externalities and not due to the moral hazard.  

The intuition is that when banks are small, they do not internalize the fact that their actions affect the bail-out they receive since the bail-out, even when targeted, depends only on the aggregate fire-sale (i.e. \( \frac{\partial B_i}{\partial k_{1t}^{BP}} = 0 \)).  

Next consider the case of \( N < \infty \) and some fiscal capacity \( \chi > 0 \), which will be the only case where there will be moral hazard. The moral hazard is captured by the term \( \frac{\partial B_i}{\partial k_{1t}^{BP}} = \frac{1}{\frac{\partial z_{1t}^{BP}(k_0)}{\partial B_i} N} \frac{\partial z_{1t}^{BP}}{\partial k_{1t}^{BP}} \) in the denominator of \( z_{1t}(k_0) \). When banks are large and the country has some fiscal capacity, they internalize the fact that an extra dollar in the crisis state will decrease the fire-sale and hence the marginal bail-out they receive. As a result, since larger investment in \( t = 0 \) will lead to a lower bank net worth and a larger fire-sale in the crisis state, in order to maximize the bail-out received, bankers choose a larger \( k_0 \) than what the Central Planner would want them to choose.  

In general, as we increase \( N \), it is unclear whether the overinvestment, which is given by \( k_0^* - k_0^{CP} \), will be smaller or larger since the pecuniary externality is stronger but the moral hazard is weaker. As it turns out, the size of the overinvestment will not be relevant when determining the optimal minimum bank capital requirement, conditional on the policy maker having sufficient number of instruments to replicate the constrained Central Planner’s allocation.

## 5 Decentralize the Constrained Central Planner’s Allocation

In the previous section I proved that given the assumptions made, in \( t = 0 \), the banker overinvests relative to the constrained Central Planner and, hence, there is a role for ex-ante regulation. One way to correct for the overinvestment is to use a minimum bank capital requirement, which is the policy instrument currently employed by almost all countries. Banker \( i \) is required to finance at

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40 One can see that clearly in the \( \chi = 0 \) case. The smaller \( N \) is, the difference between \( z_{1t}^{BP}(k_0) \) and \( z_{1t}(k_0) \), which proxies the extent of the overinvestment, shrinks. As a result smaller \( N \) implies weaker pecuniary externality.

41 If a country has a continuum of banks and no fiscal capacity (\( \chi = 0 \) and \( N \to \infty \)), which implies no bail-out and no moral hazard, the amount of the overinvestment \( z_{1t}^{BP}(k_0) - z_{1t}(k_0) \) coincides with the \( z_{1t}^{BP}(k_0) - z_{1t}(k_0) \) of a country with a continuum of banks and any level of fiscal capacity (\( N \to \infty \) for any \( \chi \)).

42 The linearity of the production technology of the bankers is one of the key reason why the moral hazard is not present in the case of \( N \to \infty \). Assuming a concave production technology will change the result, but even then the moral hazard will be stronger if \( N \) is smaller for the same reasons as the ones discussed in the remaining part of this sub-section.
least a fraction $\rho^t$ of his risky investment using internal equity, which, in this model, is equal to the period zero net worth of banker $i$, $n$. The minimum bank capital requirement constraint is given by $\rho^t k_0^i \leq n$ and will be an additional constraint that banker $i$ will have to take into account when choosing his optimal allocation. The following proposition presents the results from the banker’s optimization problem given the minimum bank capital constraint.\footnote{For details regarding the set up see Appendix, Section A.2.}

**Proposition 4** Given Assumptions 1-8, Assumption 10 and the assumptions made on the functional forms of $F(\cdot)$ and $\delta(\cdot)$, for a given exogenous minimum bank capital ratio such that $\rho > \frac{n}{k_0^i(\rho=0)}$ and considering a symmetric equilibrium, the decentralized equilibrium can be one of the following four types:

1. Type 1) $z_{1l}(k_{1l}^T(\rho)) = z_0(k_{1l}^T(\rho)) > z_{1h}$ if $k_{1l}^T(\rho) \in (\tilde{k}_{1l}^T, \tilde{k}_{1l}^{T,\max})$
2. Type 2) $z_0(k_{1l}^T(\rho)) > z_{1s}(k_{1l}^T(\rho))$ if $k_{1l}^T(\rho) = \tilde{k}_{1l}^{T,\max}$
3. Type 3) $z_{1l}(k_{1l}^T(\rho)) = z_0(k_{1l}^T(\rho)) = z_{1h}$ if $k_{1l}^T(\rho) = \tilde{k}_{1l}^T$
4. Type 4) $z_{1h} = z_0(k_{1l}^T(\rho)) > z_{1l}(k_{1l}^T(\rho))$ if $k_{1l}^T(\rho) \in [0, \tilde{k}_{1l}^T]$ \(\tilde{k}_{1l}^{T,\max}\) is determined in Section A.4.1 in the Appendix. $\tilde{k}_{1l}^T$ is unique and exists if $0 < \tilde{k}_{1l}^T < \tilde{k}_{1l}^{T,\max}$; $\tilde{k}_{1l}^T$ is determined by

$$M(k_{1l}^T) = \frac{1}{N}F''(k_{1l}^T)k_{1l}^T + \frac{1}{\delta''(B_1)N}\frac{\partial z_{1l}^T}{\partial k_{1l}^T} + F'(k_{1l}^T) - 1.$$ 

**Proof of Proposition 4.** See Appendix, Section A.4.4. \(\blacksquare\)

The condition that the exogenous $\rho$ is such that $\rho > \frac{n}{k_0^i(\rho=0)}$ guarantees that the minimum bank capital requirement constraint will be always binding. The most interesting case to consider is to set the minimum bank capital ratio in such a way as to replicate the optimal period zero investment chosen by the Central Planner, i.e. $k_0^* = k_0^{CP}$. Therefore, from now on let us consider the case $\rho^* = \frac{n}{k_0^{CP}}$. Notice that the minimum bank capital ratio is a "quantity" regulatory instrument since it directly determines the quantity of period zero investment chosen by the banker.

Given the presence of a binding ex-ante minimum bank capital requirement, Proposition 4 states that the decentralized equilibrium can be one of four types. What differentiates the equilibria is how much the banker values wealth in the high state in $t=1$ relative to the low state in $t=1$ (the crisis state). In Proposition 2, I already proved that the only two possible borrowing contracts for the Central Planner are of Type 1 and 2. If the equilibrium is interior (of Type 1), the Central Planner always values wealth more in the crisis state than in the high state in $t=1$ and, hence, borrows to the maximum against the high state and only then borrows against the crisis state. From Proposition 4 it is clear that a single instrument in the form of a minimum bank capital requirement might not be sufficient to replicate the constrained Central Planner’s allocation. In addition to choosing $k_0$ in $t=0$, the banker has another degree of freedom, which is to choose how to transfer resources across states of nature and time. More precisely, he can choose how much to
borrow/save towards the crisis state, \( d_{1l} \).\(^{44}\) For example, if the decentralized equilibrium is of Type 4, even though \( k_{0}^{*} = k_{0}^{CP} \), the banker optimally chooses to borrow first to the maximum against the low state and only then to borrow against the high state, which implies \( d_{1l}^{*} > d_{1l}^{CP} \). This result is represented graphically below.

Consider parametrization where the equilibrium is of Type 1 for the Central Planner (Assumption 9 is satisfied) and set \( \rho^{*} = \frac{n}{k_{0}^{P}}. \)\(^{45}\)

When the banking sector is fairly concentrated (in the Figure above \( N = 3 \)) and when the country has a large fiscal capacity (which is when the moral hazard will be strong), the banker optimally chooses to borrow too much (save too little) towards the crisis state relative to the Central Planner. If the dashed line in Figure 3 is above the solid line, the borrowing contract for the banker is either of Type 3 or of Type 4 while for the Central Planner the borrowing contract is always of Type 1. The intuition for the result is the following.

The moral hazard presents itself in two different dimensions. On the one hand, the banker is tempted to invest too much in \( t = 0 \) relative to the Central Planner and, on the other hand, the banker might be also tempted to pledge too high of a payment in the crisis state if the moral hazard is strong enough. Both too much investment in period zero, \( k_{0} \), and too high of a payment pledged in the crisis state, \( d_{1l} \), will lead to a larger fire-sale which will maximize the bail-out received. This result is stated formally in the following Corollary 2.

**Corollary 2** If \( N < \infty \) and \( \chi > 0 \), bankers realize that they affect the bail-out received both via \( k_{0}^{i} \) and \( d_{1i}^{i} \), i.e. \( \frac{\partial B_{i}}{\partial k_{0}^{i}} = \frac{\partial B_{i}}{\partial k_{0}^{CP}} \frac{\partial k_{0}^{CP}}{\partial k_{0}^{i}} > 0 \) and \( \frac{\partial B_{1}}{\partial d_{1i}} = \frac{\partial B_{1}}{\partial d_{1i}^{CP}} \frac{\partial d_{1i}^{CP}}{\partial d_{1i}} > 0 \) where \( \frac{\partial B_{i}}{\partial k_{0}^{i}} \) and \( \frac{\partial B_{1}}{\partial d_{1i}} \) are total derivatives.

\(^{44}\) The amount of period zero borrowing is given by \( \sum \sigma_{i} d_{1i} = k_{0}^{CP} - n \) and also \( d_{2i} \) is determined by the borrowing constraint in \( t = 2 \) which is binding. Hence, what is left for the banker to choose is \( d_{1i} \) since \( d_{1h} = \frac{k_{0}^{CP} - \sigma_{h} d_{1i} - n}{\sigma_{h}} \) and \( k_{0} = k_{0}^{CP} \).

\(^{45}\) The parameters used are the same as in Figures 1 and 2 with the exception that \( N = 3 \) and I vary the fiscal capacity, \( \chi \).
derivatives. Also for a given $k^T_{II}$, the fewer the banks are and the larger the fiscal capacity is, the stronger the moral hazard is; $\frac{\partial^2 B^i_l}{\partial k^T_{II} \partial N} < 0$, $\frac{\partial^2 B^i_l}{\partial k^T_{II} \partial \lambda} > 0$ and $\frac{\partial^2 B^i_l}{\partial d^i_{II} \partial N} < 0$, $\frac{\partial^2 B^i_l}{\partial d^i_{II} \partial \lambda} > 0$.

**Proof of Corollary 2.** See Appendix, Section A.4.5. 

Corollary 2 states that as long as the country has some fiscal capacity, $\chi < \infty$, and the number of banks is finite, $N < \infty$, the banker internalizes the fact that his period zero actions affect the size of the bail-out, which is how the moral hazard enters the model. The moral hazard is captured by $\frac{\partial B^l_i}{\partial k^T_{II} \partial N} > 0$ and $\frac{\partial B^l_i}{\partial d^i_{II} \partial N} > 0$. A large fiscal capacity and a more concentrated banking sector exacerbate the moral hazard problem. The intuition is that when the banks are large (small number of banks), they know that their marginal impact on the fire-sale is large and, as a result, they affect the optimal bail-out by more, leading to a stronger moral hazard. Similarly, if a country has a larger fiscal capacity, it can afford to provide a larger bail-out, which implies that the moral hazard is stronger.

As a result, a second ex-ante instrument in the form of a limit on the payment promised in the crisis state $d^i_{II} \leq \nu^i$, in addition to the minimum bank capital requirement, might be necessary to replicate the constrained Central Planner’s allocation for countries with strong moral hazard. The exact conditions are specified in Proposition 5.

**Proposition 5** Consider parametrization where the equilibrium is of Type 1 for the constrained Central Planner.

Part 1) A single instrument, given by $\rho^* = \frac{n}{k^T_{II}}$, is sufficient to replicate the constrained Central Planner’s allocation if

(i) There is no moral hazard — $N \rightarrow \infty$ or $\chi = 0$

(ii) Small fiscal capacity — $\chi \leq \chi^* (N)$ for any $N$

(iii) A large number of banks — $N \geq N^* (\chi)$ for any $\chi$

Part 2) Two instruments, given by $\nu^* = d^i_{II}^{CP}$ and $\rho^* = \frac{n}{k^T_{II}}$, are required to replicate the constrained Central Planner’s allocation if

(i) Large fiscal capacity — $\chi > \chi^* (N)$ for any $N < \infty$

(ii) A small number of banks — $N < N^* (\chi)$ for any $\chi > 0$

If $N^* (\chi)$ are $\chi^* (N)$ are interior solutions, they are determined by the systems of equations $M \left(k^T_{II}, N^* (\chi)\right) = 0$, $BC_{II} \left(k^T_{II}\right) = 0$ and $M \left(k^T_{II}, \chi^* (N)\right) = 0$, $BC_{II} \left(k^T_{II}, \chi^* (N)\right) = 0$, respectively where

$$M = \frac{1}{N} F'' \left(k^T_{II}\right) k^T_{II} + \frac{1}{\partial^2 \beta \left(B^j_{II}\chi\right)} \frac{\partial^2 \lambda \left(B^j_{II}\chi\right)}{\partial k^T_{II} \partial N} + F' \left(k^T_{II}\right) - 1$$

$$BC_{II} = \pi_i \left(k^T_{II} \left(F' \left(k^T_{II}\right) - \theta (1 - \gamma)\right) + B^j_{II} \left(k^T_{II}, \chi\right)\right) - \left((1 - \theta (1 - \gamma) + \pi_i (1 - a_{II})) \frac{n}{\rho^* - n}\right)$$
Proof of Proposition 5. See Appendix, Section A.4.6.

Proposition 5 states that a second instrument, in addition to the minimum bank capital requirement, is required only if the banking sector is fairly concentrated and a country has a large fiscal capacity. The intuition why the policy maker should optimally limit the pledged payments by the bankers in the crisis state only if the moral hazard is strong enough is the following. There are two sets of forces which determine whether the banker values wealth more in the crisis state or in the high state in $t = 1$, and they push in different directions. The first set of forces, which pushes towards higher valuation of wealth in the crisis state, are that capital is cheaper during a crisis and also an extra dollar in the crisis state will lead to a lower fire-sale, which implies higher resale value of the fire-sold capital. However, the countervailing force is the benefit from maximizing the bail-out by pledging too high of a payment in the crisis state. Only once the perceived benefit of the bail-out becomes large enough (which is the case when the fiscal capacity is large and banks are large), the banker starts to value wealth in the crisis state less relative to the high state. That is why a second regulatory instrument would be required only for countries with a strong moral hazard.

The way to interpret the limit on the payment pledged in the crisis state by banker $i$, $v^i$, more broadly is as a limit on the net liabilities of banker $i$ in a future crisis during which a bail-out is anticipated. In order to be able to forecast the net liabilities in a crisis state, the regulator will need detailed bank balance sheet data and he will have to rely on projections from value-at-risk models. For example, the sale of put options and CDS contracts, in particular, can leave banks with significant liabilities during a systemic banking crisis. Therefore, imposing limits on some derivative positions (even if there is no counterparty risk, which is the case in this model) will be crucial for countries with strong moral hazard.

6 Comparative Statics of Optimal Bank Regulation With Respect to Fiscal Capacity

The question arises how should the optimal minimum bank capital requirement, $\rho^*$, and also the optimal limit on the payment pledged in the crisis state by bankers, $v^*$, vary across countries with different fiscal capacity. In this subsection, I prove one of the key results of this paper — that smaller fiscal capacity implies a larger optimal ex-ante minimum bank capital requirement. I also build on the result from the previous section, which showed that countries with larger fiscal capacity are more likely to need a second regulatory instrument in the form of a limit on the liabilities pledged by the financial sector in a crisis. In this section I show that conditional on a country requiring

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\[46\text{See equation 5b.}\]
Proposition 6  Conditional on Assumptions 1-8 and 10 and the assumptions made on the functional forms of $F(\cdot)$ and $\delta(\cdot)$, if the policy maker has a sufficient number of instruments to replicate the constrained Central Planner’s allocation and the parametrization is such that the Central Planner’s equilibrium is of Type 1 (Assumption 9 is satisfied), the optimal minimum bank capital ratio is smaller for countries with larger fiscal capacity, $\frac{\partial \rho^*}{\partial \chi} < 0$.

Proof of Proposition 6. See Appendix, Section A.4.7.

In this section I assume that the policy maker has a sufficient number of instruments to replicate the constrained Central Planner’s allocation. The optimal minimum bank capital ratio, $\rho^*$, and $k_{0,CP}^*$ are inversely related, $\rho^* = \frac{n}{k_{0,CP}^*}$. As a result, in order to prove Proposition 6, it is sufficient to prove that the Central Planner of a country with a larger fiscal capacity will optimally choose to invest more ex-ante relative to the Central Planner of a country with a smaller fiscal capacity, i.e. $\frac{\partial k_{0,CP}^*}{\partial \chi} > 0$. I present the intuition of the proof graphically.\footnote{With the exception of $\chi$, the rest of the parameters are the same as in Figure 1.}

Figure 4: Larger Fiscal Capacity Implies Larger Ex-Ante Investment

Figure 4 plots $\psi_{CP}(k_0, \chi)$ for two countries with different fiscal capacity. On the graph, the country with $\chi = 0.6$ has a smaller fiscal capacity and a larger marginal cost of an extra dollar of bail-out relative to the country with $\chi = 1$. Given that $\frac{\partial \psi_{CP}(k_0; \chi)}{\partial k_0} > 0$, in order to prove that the Central Planner of a country with a larger fiscal capacity would optimally choose a larger period zero investment relative to the Central Planner of a country with a smaller fiscal capacity, it is sufficient to prove that, for a given $k_0$, the dashed line is below the solid line, i.e. $\frac{\partial \psi_{CP}(\chi; k_0)}{\partial \chi} < 0$. Larger fiscal capacity for a given $k_0$ implies a larger bail-out and a smaller fire-sale during a crisis leading to a smaller inefficient transfer of resources from the bankers to the consumers. Therefore, the policy maker who can optimally afford a larger bail-out will value a dollar in the hands of the
bankers in a crisis by less, \( z_{CP}^{CP}(\chi = 0.6) > z_{CP}^{CP}(\chi = 1) \). Similarly, given that the policy maker of a less fiscally constrained country can contain the downside of a crisis, which would imply that the return on \( k_0 \) in a crisis is higher, from an ex-ante perspective, the marginal benefit of \( k_0 \) is higher, \( z_{CP}^{CP}(\chi = 0.6) < z_{CP}^{CP}(\chi = 1) \). Intuitively, the policy maker in the country with the larger fiscal capacity can prop up prices by more ex-post during a crisis and control the downside, for a given level of bank assets, which implies that ex-ante he optimally chooses to have a larger investment boom \( (k_{CP}^{CP}(\chi = 1) > k_{CP}^{CP}(\chi = 0.6)) \). As it turns out, the assumptions that generate the pecuniary externality are the key driving force behind the result that less fiscally constrained countries should have lower ex-ante minimum bank capital requirements. The result remains even if there was no ex-post bail-out and, as a result, no moral hazard in this model.

Next, let us consider how \( v^* \) varies with the fiscal capacity of the country where the assumption that the policy maker has a sufficient number of instruments to replicate the constrained Central Planner’s allocation is still in place. Larger fiscal capacity implies larger \( k_0^{CP} \). Given that it is always the case that the period zero borrowing constraint binds in the high state for the Central Planner, the larger period zero investment is financed with larger ex-ante borrowing against the crisis state (or less money transferred to the crisis state). This implies that larger fiscal capacity results in larger \( v^* (\frac{\partial d_{CP}^{CP}}{\partial \chi} = \frac{\partial v^*}{\partial \chi} > 0) \). The minimum liquidity requirement will be binding only for countries with fiscal capacity, \( \chi \), for which the dashed line is above the solid line in Figure 3. In that sense, no regulation will be required for countries with small fiscal capacity. However, conditional on the constraint binding, \( v^* \) increases as the fiscal capacity increases.

According to the results of this model, if one compares the US (a country with large fiscal capacity) and Switzerland (a country with small fiscal capacity), the US should optimally have a lower ex-ante minimum bank capital requirement than Switzerland which is consistent with current regulation. However, since the moral hazard might be potentially stronger in the US due to the larger fiscal capacity, US regulators might need to use a second instrument which limits the amount of CDS contracts and put options that US financial institutions can sell (among other measures). Such an instrument might not be required in Switzerland. However, if both the US and Switzerland were to need a second regulatory instrument, Swiss banks will face a tighter limit on the net amount of put options they can sell, for example.

\[48\] One can also calculate the optimal \( \rho \), conditional on no minimum liquidity requirement. For countries with large fiscal capacity and a few banks (strong moral hazard), the constrained Central Planner’s allocation can no longer replicated. However, simulations for various parameter combinations (not reported here) show that it is still the case that larger fiscal capacity implies lower \( \rho \) (or even \( \rho = 0 \)). If the policy maker cannot prevent the banker from borrowing first against the crisis state, the optimal allocation from the problem with only an ex-ante minimum bank capital requirement is to borrow to the maximum, which is what the banker from the decentralized equilibrium with no ex-ante regulation would optimally choose himself.
7 "Price" Versus "Quantity" Instrument

The result that countries with larger fiscal capacity should optimally have lower ex-ante minimum bank capital requirements might appear counter-intuitive at first; the reason being that large fiscal capacity implies stronger moral hazard, while the optimal policy recommendation is to regulate less ex-ante. In the spirit of Weitzman (1974), one can show that the constrained Central Planner’s allocation can be achieved using either a tax on period zero investment or a minimum bank capital requirement (conditional on imposing also a limit on the payment promised in the crisis state if necessary). However, in this section, I show that whether larger fiscal capacity implies more or less ex-ante regulation depends critically on the instrument used. More precisely, if the policy maker has an access to a "price" instrument such as a tax on period zero investment, the result is the opposite from the case where the instrument used is a "quantity" instrument such as a minimum bank capital requirement. If the regulatory instrument was a tax on period zero investment, larger fiscal capacity implies an optimally higher tax on period zero investment if the moral hazard is present.

I solve the problem of the banker using a tax on period zero investment instead of a minimum bank capital requirement. The only change in the set up is that the period zero budget constraint becomes

\[ k_0^i (1 + \tau_{k_0}^i) - n + T_{k_0}^i \leq \sum_s \pi_s d_{1s}^i \quad [z_0^i] \]

where \( \tau_{k_0}^i \) is the bank specific tax on period zero capital. The revenues from the proportional tax are distributed equally back to the bankers using a lump sum tax, \( T_{k_0}^i \), which is negative and given by \( T_{k_0}^i = -\sum_{i=1}^{N} k_0^i \tau_{k_0}^i \). Equation 10 makes it clear that by setting \( \tau_{k_0}^i \), the policy maker can no longer set directly the amount of \( k_0^i \), which was true in the case of a minimum bank capital requirement. \( \tau_{k_0}^i \) affects the marginal cost of \( k_0^i \) (or the "price" of \( k_0^i \)), as perceived by the banker, which is why a tax instrument can be thought of as a "price" instrument.

The following proposition states that a larger fiscal capacity implies a larger tax on period zero investment as long as \( 1 < N < \infty \), and the tax on period zero investment is constant if \( N \to \infty \) (the case with a continuum of banks and no moral hazard).

**Proposition 7** Conditional on Assumptions 1-10 and on the functional forms assumed for \( F(\cdot) \) and \( \delta(\cdot) \), if the policy maker has an access to two ex-ante instruments — an ex-ante tax on period zero investment ("price" instrument), \( \tau_{k_0} \), and a limit on the payment promised in the crisis state, \( v \), one can show that \( \tau_{k_0}^* > 0 \). If \( N \to \infty \) (no moral hazard) then \( \frac{\partial \tau_{k_0}^*}{\partial x} = 0 \). If \( 1 < N < \infty \) then \( \frac{\partial \tau_{k_0}^*}{\partial x} > 0 \). \( \tau_{k_0}^* \) and \( \frac{\partial \tau_{k_0}^*}{\partial x} \) are given by

\[
\tau_{k_0}^* = \left[ \frac{z_{CP}^i (k_{11}^T, CP)}{z_{11}^i (k_{11}^T, CP, \chi)} - 1 \right] \Phi > 0
\]
\[
\frac{\partial \tau_{k_0}^*}{\partial \chi} = - \frac{\partial z_{ll}(k_{ll}^{T,CP}, \chi)}{\partial \chi} \frac{z_{ll}^{CP}(k_{ll}^{T,CP})}{z_{ll}(k_{ll}^{T,CP}, \chi)^2} \Phi \geq 0
\]

where \(\Phi = \frac{1-\theta(1-\gamma)+\pi_2(\gamma-\alpha_2)}{(1-\theta)} > 0\).

**Proof of Proposition 7.** See Appendix, Section A.4.8. ■

If the instrument of choice is a tax on period zero investment, equation 11a shows that the optimal tax is positive since the banker wants to overinvest relative to the Central Planner due to both pecuniary externality and moral hazard. This is captured by the fact that \(z_{ll}^{CP}(k_{ll}^{T}) > z_{ll}(k_{ll}^{T}, \chi)\) which I proved in Corollary 1. The optimal tax, \(\tau_{k_0}^*\), approximately equals the size of the overinvestment which is given by the scaled difference between the marginal benefit of \(k_0\), as perceived by the banker, minus the marginal benefit of \(k_0\) as perceived by the constrained Central Planner. This difference is approximately equal to \(\frac{z_{ll}^{CP}(k_{ll}^{T,CP}) - z_{ll}(k_{ll}^{T,CP}, \chi)}{z_{ll}(k_{ll}^{T,CP}, \chi)}\). The larger the difference in the perceived marginal benefit of \(k_0\) is, the larger the tax on capital has to be, in order for the policy maker to be able to replicate the constrained Central Planner’s allocation. \(\chi\) does not enter directly or indirectly the marginal benefit of \(k_0\) as perceived by the Central Planner, since the Central Planner internalizes both the marginal cost and the marginal benefit of the bail-out and, in equilibrium, they cancel out. Also due to the linearity assumption, the equilibrium fire-sale from the Central Planner’s problem, \(k_{ll}^{T,CP}\), does not vary with the fiscal capacity of the country (see equation 6b). In contrast, \(\chi\) enters directly the marginal benefit of \(k_0\) as perceived by the banker. Conditional on a finite number of banks, \(N < \infty\), the banker perceives the bail-out to be larger for countries with a larger fiscal capacity.\(^{49}\) Therefore, in order to achieve a given level of \(k_0\), the policy maker will have to increase \(\tau_{k_0}\) by more for countries with larger fiscal capacity, since the perceived bail-out and, hence, the moral hazard in those countries are stronger. If one considers the case of \(N \rightarrow \infty\), \(\frac{\partial z_{ll}(k_{ll}^{T,CP}, \chi)}{\partial \chi} = 0\), because there is no moral hazard. In that case, \(\tau_{k_0}\) is still positive but it is no longer a function of the fiscal capacity.\(^{50}\)

In summary, the key reason why the size of the moral hazard affects the "price" instrument and not the "quantity" instrument is the following. The moral hazard enters into the model through the first order condition of the banker since the banker internalizes the benefit of the bail-out, but not the cost. If the ex-ante regulatory instrument is a tax on period zero investment, \(\tau_{k_0}^*\) is determined by combining the first order condition of the banker with respect to \(k_0\), and the first order condition

\(^{49}\)Mathematically, this implies that \(\frac{\partial z_{ll}(k_{ll}^{T,CP}, \chi)}{\partial \chi} > 0\) and \(\frac{\partial z_{ll}^{CP}(k_{ll}^{T,CP})}{\partial \chi} = 0\).

\(^{50}\)The reason why the pecuniary externality does not affect the size of the ex-ante tax, even though the externality enters the first order condition of the banker in the decentralized equilibrium, is because \(k_{ll}^{T,CP}\) does not vary with the fiscal capacity due to the linearity assumption. If one were to introduce concavity in the production technology of the bankers, then there will be two opposing forces. Larger fiscal capacity will imply stronger moral hazard which will push \(\tau_{k_0}\) higher. It will also imply smaller \(k_{ll}^{T,CP}\) in equilibrium (unlike in the linear case) and also the pecuniary externalities will be smaller which will push the optimal ex-ante tax in the opposite direction (make it lower). In contrast, if a minimum capital requirement was used instead, the comparative static of \(\rho^*\) with respect to \(\chi\) will be still that larger fiscal capacity implies smaller ex-ante minimum bank capital requirement, as long as the decreasing returns to scale are not too strong during a crisis, which is a reasonable assumption.
of the Central Planner with respect to \( k_0 \). In contrast, when the instrument is a minimum bank capital requirement, the first order condition of the banker with respect to \( k_0 \) will no longer play a role and, hence, the strength of the externalities does not affect the optimal minimum bank capital requirement. The strength of the moral hazard will affect only the Lagragian of the minimum bank capital requirement which one can think of as a shadow tax. This is why, as long as the policy maker can replicate the constrained Central Planner’s allocation, the size of the moral hazard per se does not affect the optimal minimum bank capital ratio — a "quantity" instrument, but it affects the ex-ante tax on period zero investment — a "price" instrument.

8 Conclusion and Further Discussion

This paper derives a normative result which is that there should be differential cross country bank regulation given that countries vary in their ability to bail-out their financial sector during a crisis. More precisely, countries with larger fiscal capacity should have lower ex-ante minimum bank capital requirements relative to countries with smaller fiscal capacity. In order to know whether one country should have lower ex-ante minimum bank capital requirement than another, a policy maker has to measure the fiscal capacity of a country. This can be done by considering variables such as the size of the banking sector relative to GDP, the availability of independent monetary policy and a forecast of the cost of sovereign borrowing in a crisis. Therefore, the result is fairly information insensitive assuming that there are no large differences across countries regarding the other parameters of the model and that the marginal benefit of the bail-out is similar across countries. The second key result of the paper is that countries with large fiscal capacity and concentrated banking sectors should also impose a limit on the amount of net liabilities banks have during a crisis that requires a government bail-out. This includes regulating derivative contracts, which will leave the financial sector with high net liability during a systemic crisis.

One can make the argument that one of the main reasons why countries followed the Basel Accords and synchronized their bank regulation was to introduce a "level playing" field for their banks. This model is a legacy model and does not consider the dynamics of what might happen if I were to introduce heterogeneous regulation and banks were allowed to relocate across countries. However, usually there are large fixed costs to banks relocating either because of fixed investment in human capital and buildings or due the fact that markets are naturally segmented. The segmentation is due to the fact that monitoring costs are lower if the banks are closer to the borrowers. Given the natural market segmentation, governments would have some leeway in terms of having differential regulation up to a certain limit.

References


A Appendix

The functional forms used for the simulations are the following. The consumers’ production technology in $t = 0, 1$ is given by

$$F\left(k_{ts}^T\right) = \begin{cases} \frac{(1-\gamma)}{\alpha} - \frac{(1-\gamma)e^{-\alpha k_{ts}^T}}{\alpha} + \gamma k_{ts}^T > 0 \text{ if } k_{ts}^T > 0 \\ k_{ts}^T \text{ if } k_{ts}^T = 0 \end{cases}$$

where $\alpha > 0$ is a parameter that controls the concavity of the production technology and $0 < \gamma < 1$ is the refinancing cost. The larger $\alpha$ is, the smaller $q_{ts}$ is, for a given $k_{ts}^T$. This functional form guarantees that the assumptions made regarding $F\left(k_{ts}^T\right)$ are satisfied. If $k_{ts}^T > 0$

$$q_{ts} = F'\left(k_{ts}^T\right) = (1-\gamma)e^{-\alpha k_{ts}^T} + \gamma > \gamma \text{ where } 1 > \gamma > 0$$
and the price will be lower than one) and keep it until in period 2 in order to get a return of one. If that were the case, there would be no fire sales in the first place.

\[ F''(k_{ts}^T) = -\alpha (1 - \gamma) e^{-\alpha k_{ts}^T} < 0 \]

\[ F'''(k_{ts}^T) = \alpha^2 (1 - \gamma) e^{-\alpha k_{ts}^T} > 0 \]

\( F(k_{ts}^T) \) and \( F''(k_{ts}^T) \) are continuous on \( t \in [0, \infty) \), \( F(k_{ts}^T) \) is at least three times differentiable on \( k_{ts}^T \in (0, \infty) \); \( F(0) = 0, F'(0) = 1, F''(0) = F'''(0) = 0, F''(k_{ts}^T) < 0 \) on \( k_{ts}^T \in (0, \infty) \) and \( \lim_{k_{ts}^T \to \infty} F'(k_{ts}^T) \geq \gamma \). The functional form for the deadweight loss due to taxing is \( \delta (B_t, \chi) = \frac{1}{\chi} B_t^n \) where \( \eta > 1 \).

A.1 The Problem of the Representative Consumer

I solve the problem of the representative consumer backwards. Consumers are infinitesimally small which implies that they take prices as given. In period 2, the representative consumer maximizes

\[ \max_{k_{1s}^T, d_{2s}} \left[ 2e + d_{1s} - p_{2s}d_{2s} + d_{2s} + F(k_{1s}^T) - q_{1s}k_{1s}^T \right] \]

The first order condition with respect to \( k_{1s}^T \) pins down the equilibrium price of capital in \( t = 1 \) as a function of the amount of equilibrium fire-sold capital, \( q_{1s} = F'(k_{1s}^T) \). The first order condition with respect to \( d_{2s} \) is an Euler equation which implies \( p_{2s} = 1 \). In \( t = 0 \), the representative consumer optimizes (taking into account his period 1 and 2 first order conditions)

\[ \max_{d_{1s}, k_{0}^T} \left[ 3e - \sum_s \pi_s p_{1s}d_{1s} + \sum_s \pi_s (d_{1s} + F(k_{1s}^T) - q_{1s}k_{1s}^T) \right] \]

The first order conditions with respect to \( d_{1s} \) and \( k_{0}^T \) imply \( p_{1s} = 1 \) and \( q_0 = F'(k_{0}^T) \). However, since in period zero the bankers enter with no capital stock, then \( q_0 = 1 \). Given that \( F'(k_{0}^T) \leq 1 \), in \( t = 0 \), consumers will not use their production technology and in equilibrium \( k_{0}^T = 0 \). Since the representative consumer is risk neutral, his demand for state contingent debt at the equilibrium price of one is infinitely elastic up to the point where he runs out of money. I consider only parametrization where the consumer never runs out of money.\(^{51}\)

A.2 Decentralized Equilibrium – No Commitment

In this section of the Appendix I solve the optimization problem of banker \( i \) where the policy maker provides an optimal bail-out in \( t = 1 \). Also the policy maker has an access to two ex-ante (period

\(^{51}\)The assumption that capital cannot be stored and has to be used for production every period implies that consumers cannot simply purchase capital in \( t = 1 \) in the low state (which is the state where there will be a fire-sale and the price will be lower than one) and keep it until \( t = 2 \) in order to get a return of one. If that were the case, there would be no fire sales in the first place.
zero) policy instruments $\rho^i k_0^i \leq n$ and $d_1^i \leq \nu^i$. I assume that the policy maker cannot commit. As a result, the optimal targeted bail-out, $B_i^t$, will be determined in $t = 1$ and banker $i$ will internalize the fact that his actions in period $t = 0$ will affect $B_i^t$. In contrast, $\rho^i$ and $\nu^i$ are predetermined in the beginning of period $t = 0$, before banker $i$ makes any decisions. Also since I solve the model for an economy with $N$ banks, banker $i$ takes into account the first order conditions of the consumers, which pin down prices $p_{1s} = p_{2s} = 1$, $q_0 = q_{2s} = 1$ and $q_{1s} = F'(k_{1s}^t)$ where from market clearing $k_{1s}^t = k_{1s}^F = \sum_{i=1}^N \frac{1}{N} k_{1s}^i = \sum_{i=1}^N \frac{1}{N} \max\{0, k_0^i - k_{1s}^i\}$.

I solve the model backwards. In $t = 2$, all bankers produce and pay out all the profits as dividends to the consumers. At the end of $t = 1$, banker $i$ maximizes the dividend payment in the last period by choosing $\{k_{1s}^i, d_{2s}^i\}$ and taking as given the endogenous state variables $\{k_0^i, d_{1s}^i\}$ and the predetermined policy instruments $\{B_i^t, \rho^i, \nu^i\}$.

$$\max_{k_{1s}^i, d_{2s}^i} (A + 1 - \gamma) k_{1s}^i - d_{2s}^i$$

subject to the collateral constraint in $t = 1$ and the period one budget constraint

$$d_{2s}^i \leq \theta (1 - \gamma) k_{1s}^i \quad [\lambda_{2s}^i]$$

$$k_{1s}^i F'(k_{1s}^i) + d_{1s}^i \leq (F'(k_{1s}^i) + a_{1s} - \gamma) k_0^i + B_s^i + d_{2s}^i \quad [z_{1s}^{i1}]$$

The first order condition with respect to $k_{1s}^i$ can be re-written as

$$z_{1s}^{i1} = \frac{A + 1 - \gamma + \lambda_{2s}^i \theta (1 - \gamma)}{F'(k_{1s}^i) + \frac{1}{N} F''(k_{1s}^i) k_{1s}^i F'} > 1 \quad (12)$$

The fact that $z_{1s}^{i1} > 1$ comes from the assumptions that $\gamma < q_{1s} = F'(k_{1s}^t) \leq 1$, $\frac{1}{N} F''(k_{1s}^t) k_{1s}^i F' \leq 0$ and $A - \gamma > 0$ and from the fact that $\lambda_{2s}^i > 0$. The first order condition with respect to $d_{2s}^i$ (Euler equation) is

$$-1 - \lambda_{2s}^i + z_{1s}^{i1} = 0$$

Next I prove that $\lambda_{2s}^i > 0$. Since $z_{1s}^{i1} > 1$, then $\lambda_{2s}^i = z_{1s}^{i1} - 1 > 0$ which implies that the period one collateral constraint always binds and $d_{2s}^i = \theta (1 - \gamma) k_{1s}^i$.\footnote{One can re-write equation 12 as

$$z_{1s}^{i1} = \frac{A + (1 - \theta) (1 - \gamma)}{F'(k_{1s}^i) + \frac{1}{N} F''(k_{1s}^i) k_{1s}^i F' - \theta (1 - \gamma)}$$}

At the beginning of $t = 1$, if the low state is realized, the policy maker chooses $B_i^t$ given the state variables $\{k_0^i, d_{1s}^i\}$ and the policy instruments predetermined in $t = 0$, $\{\rho^i, \nu^i\}$. He also takes into account that the first order conditions of banker $i$ at the end of $t = 1$ which are a function of $B_i^t$. Assuming that the Central Planner maximizes the welfare of all consumers who are risk neutral, the objective function of the policy maker in $t = 1$ in the low state is to maximize second period output, which is also the consumption of the consumers. The optimization problem of the policy maker in the beginning of $t = 1$ in the low state is (taking into account the fact that $\lambda_{2s}^i > 0$)
First order condition with respect to \( k_{1t}^i \)

\[
F''(k_{1t}^i) k_{1t}^i + A + (1 - \theta) (1 - \gamma) + z_{1t}^{i,1,P} (\theta (1 - \gamma) - F'(k_{1t}^i)) = F''(k_{1t}^i) \sum_{j=1}^{N} \frac{1}{N} z_{1t}^{j,1,P} k_{1t}^j
\]

(13)

Since equation 13 holds for every \( i \) and using the market clearing condition \( \sum_{j=1}^{N} \frac{1}{N} k_{1t}^{j,F} = k_{1t}^i \), \( z_{1t}^{i,1,P} \) is the same for every \( i \) and one can simplify equation 13 as

\[
z_{1t}^{i,1,P} = z_{1t}^{1,P} = \frac{F''(k_{1t}^i) k_{1t}^i + A + (1 - \theta) (1 - \gamma)}{F''(k_{1t}^i) k_{1t}^i + F'(k_{1t}^i) - \theta (1 - \gamma)} > 1
\]

(14)

where \( z_{1t}^{1,P} > 1 \) comes from the assumptions \( \gamma < F'(k_{1t}^i) \leq 1, \frac{1}{N} F''(k_{1t}^i) k_{1t}^{i,F} < 0 \) and \( A - \gamma > 0 \).

The first order condition with respect to \( B_t^i \) is

\[
1 + \frac{\partial \delta(B_t, \chi)}{\partial B_t^i} = z_{1t}^{1,P}(k_{1t}^i)
\]

(15)

The policy maker is indifferent whom to give the bail-out to. He wants to control the size of the fire-sale given the cost of the marginal bail-out but he is indifferent whether he gives all the money to a single bank or splits it equally among all banks. I will consider the symmetric equilibrium in order to solve the model, i.e. \( B_t^i = B_t \). At the end of \( t = 0 \), banker \( i \) chooses \( \{k_0^i, d_{1s}^i\} \), taking as given \( \{\rho^i, \nu^i\} \), and internalizing his effect on \( B_t^i \) and on his own future actions. I plug in for \( d_{2s}^i \) and take into account that \( k_{1s}^i \) is pinned down by the period one budget constraint while \( B_t^i \) is pinned down by equation 15. At the end of period \( t = 0 \), banker \( i \) optimizes the expected value of period two dividends given by

\[
\max_{k_0^i, d_{1s}^i, k_{1s}^i} \sum_s \pi_s (A + (1 - \theta) (1 - \gamma)) k_{1s}^i
\]

subject to the period one budget constraint, the period zero budget constraint, the period zero borrowing constraint, the minimum bank capital requirement and subject to the limit on the promised payment in the crisis state by banker \( i \)

\[
k_{1s}^i (F'(k_{1s}^i) - \theta (1 - \gamma)) + d_{1s}^i \leq (F'(k_{1s}^i) + a_{1s} - \gamma) k_0^i + B_t^i \quad [\pi_s z_{1s}^i]
\]

(16)

\[
k_0^i - n \leq \sum_s \pi_s d_{1s}^i \quad [z_0^i]
\]

\[
d_{1s}^i \leq \theta (F'(k_{1s}^i) - \gamma) k_0^i \quad [\pi_s \lambda_{1s}^i]
\]
\[ \rho^i k_0^i \leq n \]
\[ d^i_{1l} \leq \nu^i \]

First order condition with respect to \( k_0^i \)

\[
\sum_s \pi_s z^i_{1s} \left( F' (k_{1s}^T) a_{1s} + \frac{1}{N} F'' (k_{1s}^T) k_{1s}^i F + \frac{\partial B^i_s}{\partial k_0^i} \right)
- \rho^i \nu^i + \sum_s \pi_s \lambda^i_{1s} \theta \left( F' (k_{1s}^T) - \gamma + \frac{1}{N} F'' (k_{1s}^T) k_0^i \right) = 0 \]  
(17)

First order condition with respect to \( k_{1s}^i \)

\[
A + (1 - \theta) (1 - \gamma) + z^i_{1s} \left[ - \frac{1}{N} F'' (k_{1s}^T) k_{1s}^i F + \frac{\partial B^i_s}{\partial k_{1s}^i} - (F' (k_{1s}^T) - \theta (1 - \gamma)) \right]
+ \lambda^i_{1s} \theta \left[ - \frac{1}{N} F'' (k_{1s}^T) k_0^i \right] = 0 \]  
(18)

First order condition with respect to \( d^i_{1l} \)

\[
-z^i_{1l} + \rho^i \nu^i - \lambda^i_{1l} - \varphi^i = 0 \]

First order condition with respect to \( d^i_{1h} \)

\[
-z^i_{1h} + z^i_0 - \lambda^i_{1h} = 0 \]

where using equation 15 one can show that \( \frac{\partial B^i_s}{\partial k_{1s}^i} = \frac{\partial B^i_l}{\partial k_0^i} = \frac{\partial B^i_l}{\partial k_{1l}^i} = \frac{1}{\theta''(B^i_l)} \frac{\partial z^i_{1l} \varphi^i}{\partial k_{1d}^i} \).

### A.3 Constrained Central Planner’s Problem – No Commitment

Since I assume no commitment, I solve the problem backwards. In period one the Central Planner maximizes the welfare of the consumers assuming a symmetric equilibrium

\[
\max_{k_{1s}, B_s, d_{2s}} 2e - (\delta (B_s) + B_s) + F (k_{1s}^T) - F' (k_{1s}^T) k_{1s}^T + d_{1s} + (A + 1 - \gamma) k_{1s} - d_{2s} 
\]

subject to the collateral constraint in \( t = 2 \) and to the period one budget constraint

\[
d_{2s} \leq \theta (1 - \gamma) k_{1s} \]

\[
k_{1s} F' (k_{1s}^T) + d_{1s} \leq (F' (k_{1s}^T) + a_{1s} - \gamma) k_0 + B_s + d_{2s} \]

40
From the first order conditions with respect to \( k_{1s}, d_{2s} \) and \( B_s \)

\[
\begin{align*}
\frac{\partial z_{1s}^{1, CP}}{\partial k_{1s}} &= \frac{F'' (k_{1s}^T) k_{1s}^T + A + 1 - \gamma + \lambda_{2s}^{-1} \theta (1 - \gamma)}{F' (k_{1s}^T) + F'' (k_{1s}^T) k_{1s}^T} \\
\frac{\partial z_{1s}^{1, CP}}{\partial B_s} &= 1 + \lambda_{2s}^{-1} \theta (1 - \gamma) \geq 1
\end{align*}
\]

(19)

First, I prove that \( \lambda_{2s}^{1, CP} > 0 \). Since \( F'(k_{1s}^T) \leq 1 \) and \( A + 1 - \gamma + \lambda_{2s}^{1, CP} \theta (1 - \gamma) > 1 \), then \( z_{1s}^{1, CP} > 1 \). From equation 19 \( \lambda_{2s}^{1, CP} = z_{1s}^{1, CP} - 1 > 0 \) which completes the proof that \( \lambda_{2s}^{1, CP} > 0 \). Hence, \( d_{2s} = \theta (1 - \gamma) k_{1s} \). Re-writing the first order condition with respect to \( k_{1s} \) and using the fact that \( \lambda_{2s}^{1, CP} = z_{1s}^{1, CP} - 1 \)

\[
\frac{\partial z_{1s}^{1, CP}}{\partial k_{1s}} = \frac{F'' (k_{1s}^T) k_{1s}^T + A + (1 - \gamma) (1 - \theta)}{F' (k_{1s}^T) + F'' (k_{1s}^T) k_{1s}^T - \theta (1 - \gamma)}
\]

(20)

The Central Planner’s optimization problem in \( t = 0 \) becomes (taking into account that in equilibrium \( p_{2s} = p_{1s} = 1 \))

\[
\max_{k_0, \{k_{1s}, d_{1s}\}_{s=l, h}} \quad 3 e - \sum \pi_{1s} d_{1s} + \sum \pi_s \left[ \frac{d_{1s} - d_{2s} + d_{2s} + F (k_{1s}^T)}{-F' (k_{1s}^T) k_{1s}^T - B_s - \delta (B_s) + (A + 1 - \gamma) k_{1s} - d_{2s}} \right]
\]

Using the fact that \( d_{2s} = \theta (1 - \gamma) k_{1s} \) and simplifying, one can re-write the optimization problem as

\[
\max_{k_0, \{k_{1s}, d_{1s}\}_{s=l, h}} \quad 3 e + \sum \pi_s \left[ F (k_{1s}^T) - F' (k_{1s}^T) k_{1s}^T - B_s - \delta (B_s) + (A + 1 - \gamma) (1 - \theta) \right]
\]

subject to the budget constraint in \( t = 1 \) and \( t = 0 \) and the period one collateral constraint

\[
k_{1s} \left( F' (k_{1s}^T) - \theta (1 - \gamma) \right) + d_{1s} \leq \left( F' (k_{1s}^T) + a_{1s} - \gamma \right) k_0 + B_s \quad \left[ \pi_s \lambda_{1s}^{CP} \right]
\]

(21)

The first order condition with respect to \( k_0 \) is

\[
\sum \pi_s \left( \frac{-F'' (k_{1s}^T) k_{1s}^T - \frac{\partial B_s}{\partial k_0} (1 + \delta' (B_s)) + \lambda_{1s}^{CP} \theta \left[ F' (k_{1s}^T) - \gamma + F'' (k_{1s}^T) k_{1s}^T \right]}{1 + \lambda_{1s}^{CP} \theta (1 - \gamma) + \lambda_{1s}^{CP} \theta (1 - \gamma) + F'' (k_{1s}^T) k_{1s}^T} \right) = z_{0}^{CP}
\]

(22)

where \( \frac{\partial B_s}{\partial k_0} = \frac{1}{\delta' (B_s)} \frac{\partial z_{1s}^{1, CP}}{\partial k_{1s}} \). The first order condition with respect to \( k_{1s} \) is
\[ F'' (k^T_{1s}) k^T_{1s} - \frac{\partial B_s}{\partial k_{1s}} (1 + \delta' (B_s)) + A + (1 - \gamma) (1 - \theta) \]

\[ + z_{1s}^{CP} \left[ -F'' (k^T_{1s}) k^T_{1s} + \frac{\partial B_s}{\partial k_{1s}} - (F' (k^T_{1s}) - \theta (1 - \gamma)) \right] - \lambda_{1s}^{CP} \theta F'' (k^T_{1s}) k_0 = 0 \]

where \( \frac{\partial B_s}{\partial k_{1s}} = -\frac{1}{\delta'' (B_s)} \frac{\partial z_{1s}^{CP}}{\partial k_{1s}} \) and the first order condition with respect to \( d_{1s} \) is \( z_0^{CP} - z_{1s}^{CP} - \lambda_{1s}^{CP} = 0 \).

A.4 Proofs

A.4.1 Proposition 1

**Proposition 1:** See text. Assumption 8 is given by

\[
\left( \frac{1}{N} + 1 \right) F'' (k^T_{1l}) + \frac{1}{N} F''' (k^T_{1l}) k^T_{1l} + \left[ \frac{1}{\delta'' (B_l)} \frac{\partial z_{1l}^{1,P}}{\partial k_{1l}} - \frac{\delta'' (B_l)}{\left( \delta'' (B_l) \right)^2} \right] < 0
\]

Before I prove Proposition 1 I prove Lemmas 1 and 2.

**Lemma 1** Conditional on Assumptions 1-6 and conditional on a fire-sale in the low state, \( k^T_{1l} > 0 \), then \( \frac{\partial B_s (k^T_{1l})}{\partial k^T_{1l}} > 0 \) and \( \frac{\partial z_{1l}^{1,P}}{\partial k^T_{1l}} > 0 \).

**Proof of Lemma 1.** Differentiating the first order condition of the policy maker in the beginning of \( t = 1 \) given by equations 14 and 15, and using the fact that \( z_{1l}^{1,P} > 1 \) (see Section A.2), Assumption 4 and the assumption \( F'' (k^T_{1l}) < 0 \) then

\[
\frac{\partial z_{1l}^{1,P}}{\partial k^T_{1l}} = \frac{F'' (k^T_{1l}) + F''' (k^T_{1l}) k^T_{1l} \left( 1 - z_{1l}^{1,P} (k^T_{1l}) \right) - z_{1l}^{1,P} (k^T_{1l}) F'' (k^T_{1l})}{F'' (k^T_{1l}) k^T_{1l} + F' (k^T_{1l}) - \theta (1 - \gamma)} > 0
\]

Since \( \frac{\partial z_{1l}^{1,P}}{\partial k^T_{1l}} > 0 \) and since \( \delta'' (B_l) > 0 \), one can show that larger fire-sale leads to a larger optimal bail-out

\[
\frac{\partial B_l (k^T_{1l})}{\partial k^T_{1l}} = \frac{1}{\delta'' (B_l)} \frac{\partial z_{1l}^{1,P}}{\partial k^T_{1l}} > 0
\]

**Lemma 2** Given Assumptions 1-7 and considering a symmetric equilibrium, there is never a fire-sale in the high state, \( q_{1h} = 1 \) and there is a fire-sale in the low state, \( q_{1l} < 1 \).

**Proof of Lemma 2.** Assuming a symmetric equilibrium, first, I show that \( q_{1h} = 1 \). In Section A.2, I proved that \( \lambda_{2s} > 0 \) and \( d_{2s} = \theta (1 - \gamma) k_{1s} \). I can re-write the budget constraint in \( t = 1 \) and state \( s \) as
Similarly, the case is impossible. Case 4) (Type 2 equilibrium). From the first order condition with respect to $q_{1s}$ and also from Assumption 5, $a_{1h} > \gamma$, then one can show that

$$(k_{1h} - k_0)(q_{1h} - \theta (1 - \gamma)) = (a_{1h} - \gamma + \theta (1 - \gamma)) k_0 - d_{1h} \geq (a_{1h} - \gamma + \theta (1 - q_{1h})) k_0 > 0$$

Since $k_{1h} - k_0 > 0$, there is no fire-sale in the high state, $q_{1h} = 1$.

The proof that $q_{1l} < 1$ proceeds in two steps.

**Step 1)** First, I show that, given Assumption 2, if there is no fire-sale in $t = 1$ in the low state (i.e. $q_{1l} = 1$), the only possible equilibrium is the corner one where the banker borrows to the maximum in $t = 0$ (Type 2 equilibrium). This implies that if the equilibrium is not the corner equilibrium of Type 2, then there must be a fire-sale in the low state.\(^{53}\)

**Step 2)** The second step is to show that given Assumption 6, even if the Type 2 equilibrium is the optimal one, there will be always a fire-sale in $t = 1$ in the low state.

Steps 1 and 2 are sufficient to prove that there is always a fire-sale in equilibrium in the crisis state given the assumptions made. Finally, via simulations I prove that the set of parameters for which the Type 1 equilibrium is the optimal one is non-empty. Assumption 6 is the most general assumption which guarantees the presence of a fire-sale in the crisis state.

**Proof of Step 1** I will prove that conditional on Assumption 2 being satisfied, $q_{1h} = 1$ and if I assume that $q_{1l} = 1$, then $z_0 > z_{1s}$. This will imply that the only possible equilibrium if there are no fire-sales is of Type 2 (the corner equilibrium). Since $q_{1s} = 1$, then $\frac{\partial B_i}{\partial k_{1s}} = 0$ and $F''(k_{1s}^T) = 0$, and one can re-write the first order condition with respect to $k_0$ as $\sum_s \pi_s [z_{1s}(1 + a_{1s} - \gamma) + \lambda_{1s} \theta (1 - \gamma)] = z_0$. Let’s consider all the possible cases based on all the possible combinations of $\lambda_{1h}$ and $\lambda_{1l}$. Case 1) $\lambda_{1s} = 0$. If $\lambda_{1s} = 0$, then $z_{1s} = z_0$ which is impossible since if Assumption 2 was satisfied then $z_{1s} \sum_s \pi_s (1 + a_{1s} - \gamma) = z_0 > z_{1s}$, which is a contradiction. Case 2) $\lambda_{1l} = 0$ and $\lambda_{1h} > 0$. This case is impossible since it implies that $z_0 = z_{1l} > z_{1h} = z_0 - \lambda_{1h}$. However, from the first order condition with respect to $k_{1s}$, $z_{1l} = z_{1h}$ which is a contradiction. Case 3) $\lambda_{1h} = 0$ and $\lambda_{1l} > 0$. Similarly, the case $\lambda_{1h} = 0$ and $\lambda_{1l} > 0$ is impossible due to the same argument as to why Case 3 is impossible. Case 4) $\lambda_{1s} > 0$. This is the case where banker $i$ borrows to the maximum in $t = 0$ (Type 2 equilibrium). From the first order condition with respect to $d_{1s}$, $z_{1s} + \lambda_{1s} = z_0 > z_{1s}$. One can re-write $z_0$ as

$$z_0 = z_{1s} \sum_s \pi_s [\frac{(1 - \gamma)(1 - \theta) + a_{1s}}{1 - \theta (1 - \gamma)}]$$

**Proof of Step 1** I will prove that conditional on Assumption 2 being satisfied, $q_{1h} = 1$ and if I assume that $q_{1l} = 1$, then $z_0 > z_{1s}$. This will imply that the only possible equilibrium if there are no fire-sales is of Type 2 (the corner equilibrium). Since $q_{1s} = 1$, then $\frac{\partial B_i}{\partial k_{1s}} = 0$ and $F''(k_{1s}^T) = 0$, and one can re-write the first order condition with respect to $k_0$ as $\sum_s \pi_s [z_{1s}(1 + a_{1s} - \gamma) + \lambda_{1s} \theta (1 - \gamma)] = z_0$. Let’s consider all the possible cases based on all the possible combinations of $\lambda_{1h}$ and $\lambda_{1l}$. Case 1) $\lambda_{1s} = 0$. If $\lambda_{1s} = 0$, then $z_{1s} = z_0$ which is impossible since if Assumption 2 was satisfied then $z_{1s} \sum_s \pi_s (1 + a_{1s} - \gamma) = z_0 > z_{1s}$, which is a contradiction. Case 2) $\lambda_{1l} = 0$ and $\lambda_{1h} > 0$. This case is impossible since it implies that $z_0 = z_{1l} > z_{1h} = z_0 - \lambda_{1h}$. However, from the first order condition with respect to $k_{1s}$, $z_{1l} = z_{1h}$ which is a contradiction. Case 3) $\lambda_{1h} = 0$ and $\lambda_{1l} > 0$. Similarly, the case $\lambda_{1h} = 0$ and $\lambda_{1l} > 0$ is impossible due to the same argument as to why Case 3 is impossible. Case 4) $\lambda_{1s} > 0$. This is the case where banker $i$ borrows to the maximum in $t = 0$ (Type 2 equilibrium). From the first order condition with respect to $d_{1s}$, $z_{1s} + \lambda_{1s} = z_0 > z_{1s}$. One can re-write $z_0$ as

$$z_0 = z_{1s} \sum_s \pi_s [\frac{(1 - \gamma)(1 - \theta) + a_{1s}}{1 - \theta (1 - \gamma)}]$$

(25)

$z_0 > z_{1s}$ implies

\(^{53}\) Also since $k_0 = 0$ implies no fire sale in equilibrium (no borrowing in period zero and only lending to the maximum), proving the first step automatically implies that the corner solution $k_0 = 0$ is impossible, which justifies why I ignored the $k_0 \geq 0$ constraint when solving for the banker’s problem.
\[
\sum_{s} \pi_s [(1 - \gamma) (1 - \theta) + a_{1s}] \frac{1}{1 - \theta (1 - \gamma)} > 1 \tag{26}
\]

One can show that the condition \( z_0 > z_{1s} \) is satisfied as long as Assumption 2 is satisfied which completes the proof that as long as Assumption 2 is satisfied and there is no fire-sale, the banker always optimally borrows to the maximum in \( t = 0 \) (the equilibrium is of Type 2).

**Proof of Step 2)** Next, I prove that given Assumption 6 and if the banker borrows to the maximum in \( t = 0 \) (the equilibrium is of Type 2), there is always a fire-sale in \( t = 1 \) in the low state. To do that, I show that if the banker borrows to the maximum, there exists an unique \( k_{1l}^{T, \text{max}} > 0 \) where the superscript \( \text{max} \) stands for the fire-sale if the equilibrium is of Type 2.

In order to solve for \( k_0^{\text{max}} \) as a function of \( k_{1l}^{T, \text{max}} \), I use the period zero budget constraint (imposing the condition that all the borrowing constraints are binding),

\[
k_0^{\text{max}} = \frac{n}{1 - \pi_b \theta (1 - \gamma) - \pi_t \theta \left( F' (k_{1l}^{T, \text{max}}) - \gamma \right)} \tag{27}
\]

First, I prove by contradiction that if the equilibrium is of Type 2 and if \( k_{1l}^{T, \text{max}} \) exists, then \( k_{1l}^{T, \text{max}} > 0 \). Assume that banker \( i \) borrows to the maximum in \( t = 0 \) and that \( k_{1l}^{T, \text{max}} = 0 \) (i.e. \( q_{1s} = 1 \)). Re-writing the budget constraint in the low state in \( t = 1 \)

\[
(k_{1l}^{\text{max}} - k_0^{\text{max}}) (1 - \theta (1 - \gamma) = \frac{(a_{1l} - \gamma) n}{1 - \theta (1 - \gamma)} + (\delta')^{-1} \left( \frac{A - \gamma}{1 - \theta (1 - \gamma)} \right) < 0
\]

where the last inequality follows from Assumption 6 and it implies that \( k_{1l}^{\text{max}} - k_0^{\text{max}} = -k_{1l}^{T, \text{max}} < 0 \). This contradicts the assumption that \( k_{1l}^{T, \text{max}} = 0 \). Therefore, if \( k_{1l}^{T, \text{max}} \) exists and the banker borrows to the maximum in \( t = 0 \), then \( k_{1l}^{T, \text{max}} > 0 \).

Next, I prove existence and uniqueness of \( k_{1l}^{T, \text{max}} \). Using the fact that \( d_{2s} = \theta (1 - \gamma) k_{1s} \) and \( d_{1s} = \theta (q_{1s} - \gamma) k_0 \) and using the period one budget constraint in the low state, define

\[
H \left( k_{1l}^T; k_0 = k_0^{\text{max}} \right) = (a_{1l} - \gamma + \theta (1 - \gamma) - \theta \left( F'(k_{1l}^T) - \gamma \right) ) k_0^{\text{max}} + B_{l} (k_{1l}^T) + (F'(k_{1l}^T) - \theta (1 - \gamma)) k_{1l}^T \tag{28}
\]

Next, I prove that \( H \left( k_{1l}^{T, \text{max}} = k_0^{\text{max}}, k_0 = k_0^{\text{max}} \right) > 0, H \left( k_{1l}^T = 0; k_0 = k_0^{\text{max}} \right) < 0 \) and \( \frac{\partial H(k_{1l}^T)}{\partial k_{1l}^T} > 0 \). This is sufficient to guarantee that there exists an unique \( k_{1l}^{T, \text{max}} > 0 \).

\[
H \left( k_{1l}^T = k_0^{\text{max}}, k_0 = k_0^{\text{max}} \right) = (a_{1l} + (F'(k_0^{\text{max}}) - \gamma) (1 - \theta)) k_0^{\text{max}} + B_{l} (k_0^{\text{max}}) > 0
\]

Given Assumption 6

\[
H \left( k_{1l}^T = 0; k_0 = k_0^{\text{max}} \right) = \frac{(a_{1l} - \gamma) n}{1 - \theta (1 - \gamma)} + (\delta')^{-1} \left( \frac{A - \gamma}{1 - \theta (1 - \gamma)} \right) < 0 \tag{29}
\]

Since \( F'' \left( k_{1l}^T \right) < 0, B_{l}' \left( k_{1l}^T \right) > 0 \) (derived in Lemma 1), also using Assumption 4, and since Assumption 3 implies \( \theta (1 - F'(k_{1l}^T))) < \theta (1 - \gamma) < \gamma - a_{1l} \), then one can show that
\[
\frac{\partial H}{\partial k_{1l}^T} = -\theta F''(k_{1l}^T) k_{0}^{\text{max}} + (a_{1l} - \gamma + \theta (1 - F'(k_{1l}^T))) \frac{\partial k_{0}^{\text{max}}}{\partial k_{1l}^T} + B'_l(k_{1l}^T) + F'(k_{1l}^T) - \theta (1 - \gamma) + F''(k_{1l}^T) k_{1l}^T > 0
\]

where after differentiating equation 27
\[
\frac{\partial k_{0}^{\text{max}}}{\partial k_{1l}^T} = \frac{\pi_i \theta F''(k_{1l}^T) k_{0}^{\text{max}}}{1 - \pi_i \theta (1 - \gamma) - \pi_i \theta (F'(k_{1l}^T) - \gamma)} < 0 \text{ if } k_{1l}^T > 0
\]

The fire-sale that will emerge in equilibrium if the bank borrows to the maximum in \(t = 0\) is pinned down by \(H(k_{1l}^{T,\text{max}}; k_0 = k_{0}^{\text{max}}) = 0\). This completes the proof that there exists an unique equilibrium \(k_{1l}^{T,\text{max}} > 0\) (i.e. \(q_{1l}\left(k_{1l}^{T,\text{max}}\right) < 1\)).

**Proof of Proposition 1.**

**Part 1) First, I prove that the only two types of equilibria that can emerge are of Type 1 and Type 2.**

In order to characterize the equilibrium, I consider all four combinations of whether \(\lambda_{1h}\) and \(\lambda_{1l}\), are greater than or equal to zero.\(^{54}\) First let’s consider the case \((z_{1h} = z_0 = z_{1l})\), and let’s prove that given Assumption 2 this case will never be an equilibrium. If the policy maker does not have an access to ex-ante regulatory instruments, from the first order condition with respect to \(k_0\)

\[
z_{1l} = \frac{A + (1 - \theta)(1 - \gamma)}{1} + \frac{1}{\delta''(B_l) N} \frac{\partial z_{1l}^{1.P}}{\partial k_{1l}^T} + 1 - \theta (1 - \gamma) = z_{1h}
\]

which implies

\[
1 - F'(k_{1l}^T) = \frac{1}{N} \left[ F''(k_{1l}^T) k_{1l}^T + \frac{1}{\delta''(B_l)} \frac{\partial z_{1l}^{1,P}}{\partial k_{1l}^T} \right]
\]

From the first order condition with respect to \(k_{1s}\)

\[
A + (1 - \theta)(1 - \gamma) + z_0 \sum_s \pi_s (\theta (1 - \gamma) + a_{1s} - \gamma) = z_0
\]

Plugging \(z_{1h} = z_{1l} = z_0 = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta (1 - \gamma)}\) in equation 31 and simplifying implies \(1 = \sum_s \pi_s (1 + a_{1s} - \gamma)\). However, given Assumption 2, the equation above will not be satisfied and hence \(\lambda_{1s} = 0\) will not be an equilibrium given the assumptions made. Next, I show that the case \(\lambda_{1h} = 0, \lambda_{1l} > 0\),

\(^{54}\)I already proved that the period 2 borrowing constraints are always binding \(\lambda_{2s} > 0\).
which implies \((z_{1h} = z_0 > z_{1l})\), is impossible. Re-writing the first order conditions of the banker, \(\lambda_{1l} = z_0 - z_{1l}, z_{1l} = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta(1 - \gamma)}\) and

\[
z_{1l} = \frac{A + (1 - \theta)(1 - \gamma) - z_{1l} \theta \frac{1}{N} F''(k_{1l}^T) k_0}{\varpi(k_{1l}^T)} \tag{32}
\]

where \(\varpi(k_{1l}^T) = \left[ \frac{1}{N} F''(k_{1l}^T) k_{1l}^T + \frac{1}{\delta'(B_1)N} \frac{\partial z_{1l}^{1,p}}{\partial k_{1l}^T} + F'(k_{1l}^T) - \theta (1 - \gamma) - \theta \frac{1}{N} F''(k_{1l}^T) k_0 \right] > 0\). In order for \(z_{1h} > z_{1l}\), it will have to be the case \(z_{1h} > \frac{A + (1 - \theta)(1 - \gamma)}{\varpi(k_{1l}^T)}\) which implies

\[
\frac{1}{N} F''(k_{1l}^T) k_{1l}^T + \frac{1}{\delta'(B_1)N} \frac{\partial z_{1l}^{1,p}}{\partial k_{1l}^T} + F'(k_{1l}^T) - 1 > 0 \tag{33}
\]

Re-writing the first order condition with respect to \(k_0\) and using the fact that \(z_{1h} = z_0\)

\[
z_0 = \frac{\pi_1 z_{1l} \left( \varpi(k_{1l}^T) + \theta (1 - F'(k_{1l}^T)) + a_{1l} - \gamma \right)}{[1 - \pi_1 \theta (F'(k_{1l}^T) - \gamma + \frac{1}{N} F''(k_{1l}^T) k_0) - \pi_h (1 + a_{1h} - \gamma)]} \tag{34}
\]

Combining equations 34 and 32 one can solve for \(z_0\)

\[
z_0 = \frac{\pi_1 \left[ A + (1 - \theta)(1 - \gamma) \right] \left[ 1 + \frac{\left( \theta(1 - F'(k_{1l}^T)) + a_{1l} - \gamma \right)}{\varpi(k_{1l}^T)} \right]}{[1 - \pi_1 \theta (F'(k_{1l}^T) - \gamma - \pi_h (1 + a_{1h} - \gamma) + \pi_1 \theta \frac{1}{N} F''(k_{1s}^T) k_0 \left( \theta(1 - F'(k_{1l}^T)) + a_{1l} - \gamma \right)}{\varpi(k_{1l}^T)}]} \tag{35}
\]

Next I prove that given Assumption 2, Assumption 3 and the inequality 33, it will be impossible that \(z_0 = z_{1h} = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta(1 - \gamma)}\) where \(z_0\) is given by equation 35. Equating \(z_0 = z_{1h}\) and simplifying one gets

\[
0 < LHS = \left[ \sum_s \pi_s (1 + a_{1s} - \gamma) \right] - 1 = \frac{\left( \frac{1}{N} F''(k_{1l}^T) k_{1l}^T + \frac{1}{\delta'(B_1)N} \frac{\partial z_{1l}^{1,p}}{\partial k_{1l}^T} + F'(k_{1l}^T) - 1 \right) \pi_1 \left( \theta (1 - F'(k_{1l}^T)) + a_{1l} - \gamma \right)}{\varpi(k_{1l}^T)} = RHS < 0
\]

where \(\left( \theta (1 - F'(k_{1l}^T)) + a_{1l} - \gamma \right] < \theta (1 - \gamma) + a_{1l} - \gamma < 0\). Since it is impossible for both \(LHS > 0\) and \(RHS < 0\), the case \(\lambda_{1h} = 0, \lambda_{1l} > 0\) will never be an equilibrium outcome. Next, I consider the remaining two cases which can occur in equilibrium.

**Type 1 equilibrium:** \(\lambda_{1h} > 0, \lambda_{1l} = 0\) \((z_{1l} = z_0 > z_{1h})\). Using the fact that \(\lambda_{1h} = z_0 - z_{1l}\) and rewriting equations 17 and 18, one can show that \(z_{1h} = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta(1 - \gamma)}\),

\[
z_0 = \frac{\pi_h z_{1h} [(1 - \gamma)(1 - \theta) + a_{1h}] + \pi_1 (A + (1 - \theta)(1 - \gamma)) + \pi_1 (a_{1l} + \theta (1 - \gamma) - \gamma) z_{1l}}{[1 - \pi_h \theta (1 - \gamma)]} \tag{36}
\]

\[
z_{1l} = \frac{A + (1 - \theta)(1 - \gamma)}{\frac{1}{N} F''(k_{1h}^T) k_{1l}^T + \frac{1}{\delta'(B_1)N} \frac{\partial z_{1l}^{1,p}}{\partial k_{1l}^T} + F'(k_{1l}^T) - \theta (1 - \gamma)} \tag{37}
\]
Using the equations above, one can solve for $k^T_{1l}$. In order for $z_{1l} > z_{1h}$, it has to be the case that $z_{1l} > \frac{A + (1-\theta)(1-\gamma)}{1-\theta(1-\gamma)}$ which implies

$$\frac{1}{N} F''(k^T_{1l}) k^T_{1l} + \frac{1}{\delta''(B_l) N} \frac{\partial z^1_P}{\partial k^T_{1l}} + F'(k^T_{1l}) - 1 < 0 \quad (38)$$

The rest of the endogenous variables are given by the following system of equations.

$$k_0 = \frac{\pi_l B_l + n + [F'(k^T_{1l}) - \theta (1 - \gamma)] k^T_{1l} \pi_l}{(1 - \pi_h \theta (1 - \gamma) + (\gamma - a_{1l} - \theta (1 - \gamma)) \pi_l)} \quad (39)$$

$$k_{1l} = k_0 - k^T_{1l} \quad (40)$$

$$k_{1h} = \frac{((1 - \theta) (1 - \gamma) + a_{1h}) k_0}{[1 - \theta (1 - \gamma)]} \quad (41)$$

$$d_{2s} = \theta (1 - \gamma) k_{1s}; \quad d_{1l} = \frac{1}{\pi_l} [(k_0 - n) - \pi_h \theta (1 - \gamma) k_0] \quad (42)$$

$$d_{1h} = \theta (1 - \gamma) k_0 \quad (43)$$

and $B_l = (\delta')^{-1} \left( z^1_P (k^T_{1l}) - 1 \right)$. Finally $p_{1s} = p_{2s} = 1$ and $q_0 = q_{2s} = q_{1h} = 1, q_{1l} = F' (k^T_{1l})$.

**Type 2 equilibrium:** $\lambda_{1s} > 0$ ($z_0 > z_{1s}$). It will be the case that $\lambda_{1s} = z_0 - z_{1s}$ and using the first order conditions with respect to $k_0$ and $k_{1s}$, equations 17 and 18, one can solve for $z_{1s}$ and $z_0$ as a function of $k^T_{1l}$. The rest of the endogenous variables $d_{2s}, d_{1s}, k_{1l}, k_{1h}$ and $B_l$ are pinned down by the following system of equations: $k^T_{1l}^{\text{max}}$ is determined by the solution to the equation $H \left( k^T_{1l}^{\text{max}}; k_0 = k_0^{\text{max}} \right) = 0$ where $H (\cdot)$ is given by equation 28. $k_0^{\text{max}}$ is given by equation 27. Also $d_{1s} = \theta (q_{1s} - \gamma) k_0, d_{2s} = \theta (1 - \gamma) k_{1s}$ and

$$k_{1l} = \frac{((F'(k^T_{1l}) - \gamma) (1 - \theta) + a_{1l}) k_0 + B_l}{F'(k^T_{1l}) - \theta (1 - \gamma)} \quad (44)$$

$$k_{1h} = \frac{((1 - \gamma) (1 - \theta) + a_{1h}) k_0}{1 - \theta (1 - \gamma)} \quad (45)$$

and $B_l = (\delta')^{-1} \left( z^1_P (k^T_{1l}) - 1 \right)$. Finally $p_{1s} = p_{2s} = 1$ and $q_0 = q_{2s} = q_{1h} = 1, q_{1l} = F' (k^T_{1l})$.

**Part 2) Existence and Uniqueness**

The proof of existence and uniqueness proceeds in two steps:

**Step 1)** Solve for the equilibrium by solving for $k_0$. First, I show that for every $k_0 \in [0, k_0^{\text{max}}]$ there exists an unique $k^T_{1l}$. I will consider two regions for $k_0$ separately. If the equilibrium $k_0$ is such that $k_0 \in [0, \hat{k}_0]$ then there will be no-fire-sale, $k^T_{1l} = 0$, where I will derive $\hat{k}_0$ as a function of exogenous variables. If the equilibrium $k_0$ is such that $k_0 \in (\hat{k}_0, k_0^{\text{max}}]$ then there will be a fire-sale $k^T_{1l} > 0$ and $k^T_{1l}$ is unique. Also I prove that if $k_0 \in (\hat{k}_0, k_0^{\text{max}}]$, then $\frac{\partial k^T_{1l}(k_0)}{\partial k_0} > 0$. 47
Step 2) Prove existence and uniqueness using Step 1.

Proof of Step 1) Since I proved that the only possible case is \( \lambda_{1l} > 0, \lambda_{2s} > 0 \) (which encompasses the Type 1 and 2 equilibria), from the period zero budget constraint and the period zero borrowing constraint
\[
d_{1l} (k_{1l}^T; k_0) = \min \left\{ \frac{1}{\pi_l} (k_0 (1 - \pi_l \theta (1 - \gamma)) - n), \theta (F' (k_{1l}^T) - \gamma) k_0 \right\}
\]
Using the budget constraint in the low state in \( t = 1 \), define the following function
\[
H (k_{1l}^T; k_0) = (a_{1l} - \gamma + \theta (1 - \gamma)) k_0 + (F' (k_{1l}^T) - \theta (1 - \gamma)) k_{1l}^T + B_l (k_{1l}^T) - d_{1l} (k_{1l}^T; k_0)
\]
Next, consider how the function \( H (k_{1l}^T; k_0) \) behaves in the range \( k_{1l}^T \in [0, k_0] \). First, I show that \( H (k_{1l}^T = k_0; k_0) > 0 \) for every \( k_0 \)
\[
H (k_{1l}^T = k_0; k_0) = (a_{1l} - \gamma + F' (k_0)) k_0 + B_l (k_0) - d_{1l} (k_{1l}^T; k_0) \geq (a_{1l} + (F' (k_0) - \gamma) (1 - \theta)) k_0 + B_l (k_0) > 0
\]
where for the first inequality I used the fact that \( d_{1l} (k_{1l}^T; k_0) \leq \theta (q_{1l} - \gamma) k_0 \). Next, I show that \( H (k_{1l}^T = 0; k_0) > 0 \) if \( k_0 \in [0, \hat{k}_0] \) and \( H (k_{1l}^T = 0; k_0) < 0 \) if \( k_0 \in (\hat{k}_0, k_0^\max) \). Since I already showed that if \( k_0 = k_0^\max \), \( H (k_{1l}^T = 0; k_0 = k_0^\max) < 0 \) (inequality 29), here consider only the case \( k_0 \in [0, k_0^\max) \) which implies that \( d_{1l} (k_{1l}^T; k_0) = \frac{1}{\pi_l} (k_0 (1 - \pi_l \theta (1 - \gamma)) - n) \)
\[
H (k_{1l}^T = 0; k_0) = (a_{1l} - \gamma + \theta (1 - \gamma)) k_0 + B_l (0) - \frac{1}{\pi_l} (k_0 (1 - \pi_l \theta (1 - \gamma)) - n)
\]
\[
= \frac{1}{\pi_l} ((a_{1l} - \gamma) \pi_l - 1 + \theta (1 - \gamma)) k_0 + n) + (\delta')^{-1} \left( \frac{A - \gamma}{1 - \theta (1 - \gamma)} \right)
\]
Since \((a_{1l} - \gamma) \pi_l - (1 - \theta (1 - \gamma)) < 0, \frac{\partial H (k_{1l}^T = 0; k_0)}{\partial k_0} < 0 \). One can show that
\[
H (k_{1l}^T = 0; k_0) \begin{cases} 
\geq 0 \text{ if } k_0 \leq \hat{k}_0 \\
< 0 \text{ if } k_0 > \hat{k}_0
\end{cases}
\]
where
\[
\hat{k}_0 = \frac{n + \pi_l (\delta')^{-1} \left( \frac{A - \gamma}{1 - \theta (1 - \gamma)} \right)}{1 - \theta (1 - \gamma) + (\gamma - a_{1l}) \pi_l} \tag{44}
\]
I prove that \( H (k_{1l}^T; k_0) \) is continuous in \( k_{1l}^T \) and \( \frac{\partial H (k_{1l}^T; k_0)}{\partial k_{1l}^T} > 0 \). The continuity of \( H (k_{1l}^T; k_0) \) follows from \( B_l (k_{1l}^T) \) and \( F' (k_{1l}^T) \) being continuous with respect to \( k_{1l}^T \). From Assumption 4 and Lemma 1,
\[
\frac{\partial H (k_{1l}^T; k_0)}{\partial k_{1l}^T} = \begin{cases} 
F' (k_{1l}^T) - \theta (1 - \gamma) + F'' (k_{1l}^T) k_{1l}^T + \frac{\partial B_l (k_{1l}^T)}{\partial k_{1l}^T} > 0 & \text{if } k_{1l}^T > 0 \\
0 & \text{if } k_{1l}^T = 0
\end{cases}
\] (45)

Since in the region \( k_0 \in [0, \hat{k}_0] \), \( H (k_{1l}^T = 0; k_0) \geq 0 \) and \( H (k_{1l}^T = k_0; k_0) > 0 \) and since \( \frac{\partial H (k_{1l}^T; k_0)}{\partial k_{1l}^T} \geq 0 \), it follows that \( k_{1l}^T (k_0) = 0 \) if \( k_0 \in [0, \hat{k}_0] \). In the region \( k_0 \in (\hat{k}_0, k_{0}^{\max}] \), \( H (k_{1l}^T = 0; k_0) < 0 \), \( H (k_{1l}^T = k_0; k_0) > 0 \) and \( \frac{\partial H (k_{1l}^T; k_0)}{\partial k_{1l}^T} > 0 \). As a result, there exists an unique \( k_{1l}^T (k_0) > 0 \) if \( k_0 \in (\hat{k}_0, k_{0}^{\max}] \). This completes the proof that for every \( k_0 \in [0, k_{0}^{\max}] \) there exists an unique \( k_{1l}^T \geq 0 \).

I totally differentiate \( H (k_0) = 0 \) with respect to \( k_0 \) to solve for \( \frac{\partial k_{1l}^T (k_0)}{\partial k_0} \) in the relevant range \( k_0 \in (\hat{k}_0, k_{0}^{\max}] \) where there is a fire-sale. (This is the only relevant range since I already proved that given the assumptions made, there is a fire-sale in the low state in \( t = 1 \)). In that range, for a given \( k_0 \), \( k_{1l}^T \) is pinned down by setting \( H (k_{1l}^T; k_0) = 0 \).

Totally differentiate \( H (k_{1l}^T; k_0) = 0 \) with respect to \( k_0 \) to solve for \( \frac{\partial k_{1l}^T (k_0)}{\partial k_0} \). From Lemma 1 and also from Assumption 3 and Assumption 4

\[
\frac{\partial k_{1l}^T (k_0)}{\partial k_0} = \begin{cases} 
\frac{1}{F'(k_{1l}^T) - \theta (1 - \gamma) + F'' (k_{1l}^T) k_{1l}^T + \frac{\partial B_l (k_{1l}^T)}{\partial k_{1l}^T}} > 0 & \text{if } k_0 \in (\hat{k}_0, k_{0}^{\max}] \\
0 & \text{if } k_0 \in [0, \hat{k}_0]
\end{cases}
\]

**Proof of Step 2)** I already proved that given the assumptions made, the only two types of equilibria are an equilibrium of Type 1 (interior equilibrium) and Type 2 (corner equilibrium). In order to prove existence and uniqueness I define the following function

\[
\psi (k_0) = z_{1l} (k_0) - z_0 (k_0)
\]

where \( z_{1l} (k_0) \) and \( z_0 (k_0) \) are the marginal value of wealth in the crisis state and in period zero as perceived by the banker and as defined in the equilibrium of Type 1 (equations 37 and 36). I will prove that \( \psi (k_0) \) is strictly increasing and crosses the zero line at most once, which will be sufficient to prove existence and uniqueness.

In Step 1, I proved that there is a one-to-one mapping from \( k_{1l}^T \) to \( k_0 \) and one can solve for \( k_0 \) as a function of \( k_{1l}^T \) using equation 39 if the equilibrium is of Type 1. Also if there is a fire-sale in the crisis state, which is the relevant region, then \( \frac{\partial k_{1l}^T (k_0)}{\partial k_0} > 0 \). Since both \( z_{1l} \) and \( z_0 \) are functions only of \( k_{1l}^T \), which, in turn, is a function of \( k_0 \), I can re-write \( \psi \) as

\[
\psi (k_{1l}^T (k_0)) = z_{1l} (k_{1l}^T (k_0)) - z_0 (k_{1l}^T (k_0))
\]

Define

\[
M (k_{1l}^T) = \frac{1}{N} F'' (k_{1l}^T) k_{1l}^T + \frac{1}{\delta'' (B_l)} \frac{\partial z_{1l}}{\partial k_{1l}^T} + F'' (k_{1l}^T) - 1
\]
Inequality 38 implies that if the equilibrium is of Type 1, then $M(k_{11}^T) < 0$. In order to derive the support of the $\psi(k_{11}^T(k_0))$ function, let’s investigate the properties of $M(k_{11}^T)$ in the range $k_{11}^T \in [0,k_{11}^{T,\text{max}}]$ so that we can derive for what values of $k_{11}^T$, $M(k_{11}^T) < 0$.

One can show that given Assumption 8, then $M'(k_{11}^T) < 0$. Since $M'(k_{11}^T) < 0$, define the range for $k_{11}^T$ over which the equilibrium is of Type 1 as $[\tilde{k}_{11}^T, k_{11}^{T,\text{max}}]$. If $M(\cdot) < 0$ for every $k_{11}^T \in [0,k_{11}^{T,\text{max}}]$ then $\tilde{k}_{11}^T = 0$. If $M(\cdot) > 0$ for every $k_{11}^T \in (0,k_{11}^{T,\text{max}}]$ then $\tilde{k}_{11}^T = k_{11}^{T,\text{max}}$. Otherwise $\tilde{k}_{11}^T$ is pinned down by $M(\tilde{k}_{11}^T) = 0$. $\tilde{k}_{11}^T$ is unique since $M'(k_{11}^T)$ is strictly decreasing due to Assumption 8.

Therefore the relevant range one needs to consider for the $\psi(k_0)$ function is $k_0 \in [\tilde{k}_0,k_{0\text{max}}]$ where $\tilde{k}_0 = k_0(\tilde{k}_{11}^T)$ if $0 < \tilde{k}_{11}^T < k_{11}^{T,\text{max}}$. Also $\tilde{k}_0 = \hat{k}_0$ if $\tilde{k}_{11}^T = 0$ (where $\tilde{k}_0$ is pinned down by equation 44) and $\hat{k}_0 = k_{0\text{max}}$ if $\tilde{k}_{11}^T = k_{11}^{T,\text{max}}$. Also $k_{0\text{max}}$ is given by equation 27. Notice that in the case of a continuum of banks, $N \to \infty$, $\tilde{k}_{11}^T = 0$ which implies $\hat{k}_0 = k_0$.

Given Assumption 8 which implies

$$\frac{\partial}{\partial k_{11}^T} \left( \frac{1}{N} F''(k_{11}^T) k_{11}^T + \frac{1}{\sigma^2(\beta)N} \frac{\partial z_{11}}{\partial k_{11}^T} + F'(k_{11}^T) \right) < 0$$

and also the result

$$\frac{\partial z_{11}(k_0)}{\partial k_0} > 0$$

I can differentiate equation 37 to prove that

$$\frac{\partial z_{11}(k_0)}{\partial k_0} = \frac{\partial}{\partial k_{11}^T} \left( \frac{1}{N} F''(k_{11}^T) k_{11}^T + \frac{1}{\sigma^2(\beta)N} \frac{\partial z_{11}}{\partial k_{11}^T} + F'(k_{11}^T) \right) \frac{\partial k_{11}^T(k_0)}{\partial k_0}$$

Combining equations 37 and 36

$$\psi(k_{11}^T(k_0)) = z_{11} (1 - \theta (1 - \gamma) - \pi_1 (a_{11} (1 - \gamma))) - z_{11} (1 - \theta (1 - \gamma) + \pi_1 (1 - \gamma)) + (A + (1 - \theta) (1 - \gamma) \pi_1)$$

$$\frac{\partial \psi(k_{11}^T(k_0))}{\partial k_0} = \frac{\partial z_{11}(k_0)}{\partial k_0} - \frac{\partial z_{11}(k_0)}{\partial k_0} \left[ 1 - \theta (1 - \gamma) + \pi_1 (1 - \gamma) \right] > 0$$

Next, evaluate $\psi(k_{11}^T)$ at $k_{11}^T = \tilde{k}_{11}^T$ and $k_{11}^T = k_{11}^{T,\text{max}}$ which is the relevant range for the Type 1 equilibrium. If $k_{11}^T = \tilde{k}_{11}^T$, then one can show that $\psi(\tilde{k}_{11}^T) = \frac{A + (1 - \theta) (1 - \gamma) \pi_1}{1 - \theta (1 - \gamma)}$. Given Assumption 2

$$\psi(\tilde{k}_{11}^T) = \frac{A + (1 - \theta) (1 - \gamma) \pi_1}{1 - \theta (1 - \gamma)} \frac{1 - \sum_s \pi_1 (a_{11} (1 - \gamma) + 1)}{1 - \pi_1 (1 - \gamma)} < 0$$

$k_{11}^{T,\text{max}}$ is pinned down by setting equation 28 equal to zero. (I already proved that $k_{11}^{T,\text{max}}$ exists and is unique.)
If $\psi(k_{1l}^T) < 0$ for every $k_{1l}^T \in [\tilde{k}_{1l}^T, k_{1l}^{T,\text{max}}]$, then the equilibrium is a corner equilibrium of Type 2. If $\psi(k_{1l}^T) > 0$, the equilibrium is an interior equilibrium of Type 1. This completes the proof of existence and uniqueness.

A.4.2 Proposition 2

**Proposition 2:** See text. Assumption 9 is given by

$$ [\pi_h (a_{1h} - \gamma) + 1 - \theta (1 - \gamma)] A + (1 - \gamma) (1 - \theta) \quad \text{(Assumption 9)} $$

$$ < \frac{F''(k_{1l}^T) k_{1l}^T + A + (1 - \gamma) (1 - \theta)}{F'(k_{1l}^T) + F''(k_{1l}^T) k_{1l}^T - \theta (1 - \gamma)} [\pi_t (\gamma - a_{1l}) + 1 - \theta [1 - \gamma]] \quad \text{(47)} $$

Before I prove Proposition 2 I prove Lemmas 3 and 4.

**Lemma 3** Conditional on Assumptions 1-6 and conditional on a fire-sale in the low state, $\frac{\partial B_t(k_{1l}^T)}{\partial k_{1l}^T} > 0$ and $\frac{\partial z_{11}^{1CP}(k_{1l}^T)}{\partial k_{1l}^T} > 0$.

**Proof of Lemma 3.** The proof is identical to the proof of Lemma 1 since $z_{11}^{1CP}(k_{1l}^T) = z_{1l}^{1CP}(k_{1l}^T)$.

**Lemma 4** Given Assumption 5, considering a symmetric equilibrium, there is never a fire-sale in the high state, $q_{1h} = 1$. Given Assumption 2 and also given the additional Assumption 6, it is always the case that there is a fire-sale in the low state, $q_{1l} < 1$.

**Proof of Lemma 4.** The proof that $q_{1h} = 1$ is identical to the one in Lemma 1. The proof that $q_{1l} < 1$ is very similar since one can re-write the first order conditions of the Central Planner’s problem, assuming that $q_{1l} = 1$, as

$$ \sum \pi_s \left( - \frac{\partial B_s}{\partial k_0} (1 + \delta' (B_s)) + z_{1s}^{CP} \left( 1 + a_{1s} - \gamma + \frac{\partial B_s}{\partial k_0} \right) + \lambda_{1s}^{CP} \theta (1 - \gamma) \right) = z_{1s}^{CP} $$

Since $z_{1l}^{1CP}(k_{1l}^T) = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta (1 - \gamma)}$ implies that $\frac{\partial z_{1l}^{1CP}}{\partial k_{1l}^T} = 0$, then $\frac{\partial B_s}{\partial k_0} = \frac{1}{\delta'(B_s)} \frac{\partial z_{1s}^{1CP}}{\partial k_{1s}} = 0$ and $\frac{\partial B_s}{\partial k_{1s}} = -\frac{1}{\delta'(B_s)} \frac{\partial z_{1s}^{1CP}}{\partial k_{1s}} = 0$. As a result,

$$ \sum \pi_s \left( z_{1s}^{CP} (1 + a_{1s} - \gamma) + \lambda_{1s}^{CP} \theta (1 - \gamma) \right) = z_{1s}^{CP} $$

where $z_{1s}^{CP} = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta (1 - \gamma)}$ and $z_{1s}^{CP} - z_{1s}^{CP} = \lambda_{1s}^{CP}$. The equations above coincide with the equations in the proof of Lemma 2, Step 1. The rest of the proof is identical to the proof in Lemma 2.

**Proof of Proposition 2:**

*Part 1*) First, I prove that the only two types of equilibria are of Type 1 and 2. The proof is similar to the proof in Proposition 1. In order to characterize the equilibrium, I consider
all four possible combinations of whether $\lambda_{1h}$ and $\lambda_{1l}$ are greater than or equal to zero. If $\lambda_{1h} = 0$ and $\lambda_{1l} = 0$, then $z_{1h}^{CP} = z_{1l}^{CP} = z_{0}^{CP}$. Plugging equation 20 in equation 23a one gets $z_{1s}^{CP} (k_{1s}^{T}) = \frac{F''(k_{1s}) k_{1s}^{T} + A + (1 - \theta)(1 - \gamma)}{F''(k_{1s}) k_{1s}^{T} + \theta(1 - \gamma)}$. However, since there is a fire-sale only in the low state and no fire-sale in the high state, one can prove that $z_{1l}^{CP} > z_{1h}^{CP}$. This is true since $(1 - F'(k_{1l}^{T}) (A + (1 - \theta)(1 - \gamma)) > (A - \gamma) F''(k_{1l}^{T}) k_{1l}^{T}$. Hence it will never be the case that $\lambda_{1h} = 0$ and $\lambda_{1l} = 0$. If $\lambda_{1h} = 0$ and $\lambda_{1l} > 0$, then $z_{1h}^{CP} = z_{0}^{CP} = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta(1 - \gamma)}$ and $z_{0}^{CP} - \lambda_{1l} = z_{1l}^{CP} < z_{1h}^{CP}$.

where

$$z_{1l}^{CP} = \frac{F''(k_{1l}^{T}) k_{1l}^{T} - \frac{\partial h}{\partial k_{1l}} z_{1l}^{1 CP} + A + (1 - \gamma) (1 - \theta) - z_{0}^{CP} \theta F''(k_{1l}^{T}) k_{0}}{F''(k_{1l}^{T}) k_{1l}^{T} - \frac{\partial h}{\partial k_{1l}} + F'(k_{1s}^{T}) - \theta (1 - \gamma) - \theta F''(k_{1l}^{T}) k_{0}}$$

(48)

I prove by contradiction that it is impossible that $z_{1l}^{CP} < z_{1h}^{CP}$. After plugging in $z_{1l}^{1 CP}$ given by equation 20 in equation 48 and taking into account that $z_{1l}^{CP} = z_{0}^{CP}$, one can re-write the inequality $z_{1l}^{CP} < z_{1h}^{CP}$ as $F''(k_{1l}^{T}) k_{1l}^{T} + (1 - \theta)(1 - \gamma) (1 - \theta) - z_{0}^{CP} \theta F''(k_{1l}^{T}) k_{0} < z_{1h}^{CP} = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta(1 - \gamma)}$ which is a contradiction. As a result, it is impossible that $\lambda_{1h} = 0$ and $\lambda_{1l} > 0$.

**Type 1 equilibrium:** $\lambda_{1h} > 0$ and $\lambda_{1l} = 0$ ($z_{1h}^{CP} = z_{1l}^{CP} > z_{1h}^{CP}$)

Notice that $z_{0}^{CP} = z_{1l}^{CP} = z_{h}^{1 CP} + \lambda_{1h}^{CP} > z_{1h}^{CP}$ and from the first order condition with respect to $k_{1h}$, $z_{h}^{1 CP} = \frac{A + (1 - \theta)(1 - \gamma)}{1 - \theta(1 - \gamma)}$

$$z_{0}^{CP} = \frac{\pi_{h} h z_{1h}^{CP} ((1 - \gamma) (1 - \theta) + a_{1h}) + \pi_{l} (A + (1 - \theta)(1 - \gamma) + z_{1l}^{CP} [\theta (1 - \gamma) + a_{1l} - \gamma])}{(1 - \pi_{h} \theta [1 - \gamma])}$$

(49)

Plugging in for $z_{1l}^{1 CP}$, from the first order condition with respect to $k_{1l}$

$$z_{1l}^{1 CP} = z_{1l}^{CP} = \frac{F''(k_{1l}^{T}) k_{1l}^{T} + A + (1 - \theta)(1 - \gamma)}{F'(k_{1l}^{T}) - \theta (1 - \gamma) + F''(k_{1l}^{T}) k_{1l}^{T}}$$

(50)

The rest of the endogeneous variables are determined by the same set of equations as the ones in the Type 1 equilibrium in Proposition 1. The equilibrium of Type 1 is possible since I already proved that $z_{1l}^{CP} > z_{1h}^{CP}$.

**Type 2 equilibrium:** If $\lambda_{1h} > 0$ and $\lambda_{1l} > 0$ ($z_{1h}^{CP} > z_{1l}^{CP} > z_{1h}^{CP}$)

Since I already proved that $\lambda_{2h} > 0$, in this type of equilibrium the banker borrows to the maximum in $t = 0$.

Next I prove existence and uniqueness.

As in the proof of Proposition 1, one can show that for every $k_{0} \in [0, k_{0}^{max}]$ there exists an unique $k_{1l}$ and if the equilibrium $k_{0}$ is such that $k_{0} \in (k_{0}, k_{0}^{max}]$ then there will be a fire-sale, $k_{1l}^{T} > 0$, where $k_{0}$ is pinned down by equation 44. Also as in Proposition 1, one can prove that $\frac{\partial k_{1l}(k_{0})}{\partial k_{0}} > 0$ if $k_{0} \in (k_{0}, k_{0}^{max}]$.

I take into account that it will be always the case that $\lambda_{1h}^{CP} > 0, \lambda_{2l}^{CP} > 0$. Consider the interior equilibrium of Type 1 which implies $\lambda_{1l}^{CP} = 0$. Following the same steps as in the Proof of Proposition 1, define the following function which will be used to pin down the equilibrium $k_{0}$
\[\psi^{CP}(k_0) = z_1^{CP}(k_0) - z_0^{CP}(k_0) \text{ if } k_0 \in [\hat{k}_0, \hat{k}_0^{max}]\]  

(51)

where \(z_0^{CP}(k_0)\) and \(z_1^{CP}(k_0)\) are given by equations 49 and 50. If \(\psi^{CP}(k_0) = 0\), then the equilibrium is interior and of Type 1. If for every \(k_0\) in the range \([\hat{k}_0, \hat{k}_0^{max}]\), \(\psi^{CP}(k_0) < 0\) then the equilibrium is a corner equilibrium where it is optimal to borrow to the maximum in period zero against the high and the low states. Since \(\frac{\partial k^T_{1l}(k_0)}{\partial k_0} > 0\) and since \(z_1^{CP} = z_1^{1CP}\) in the interior equilibrium.

\[
\frac{\partial z_1^{1l}(k_0)}{\partial k_0} = \frac{\partial k^T_{1l}(k_0)}{\partial k_0} \frac{\partial z_1^{CP}}{\partial k^T_{1l}} > 0
\]

Finally, I prove that Assumption 9 ensures that an equilibrium of Type 2 never occurs. Consider the case where \(\lambda_{1s} > 0\) and assume that Assumption 6 is satisfied which implies that even if the equilibrium is of Type 2 there will be a fire-sale in the crisis state. It is sufficient to show that given Assumption 9, it is always the case that \(z_1^{CP} > z_0^{CP}\) and, as a result, equilibrium of Type 2 will never occur. One can show that \(z_1^{CP} - z_0^{CP} > 0\) by re-writing the first order conditions of the Central Planner.

\[
(z_0^{CP} - z_1^{CP}) = \frac{[\pi_h(a_{1l} - \gamma) + 1 - \theta(1 - \gamma)] A^{(1-\gamma)(1-\theta)} + z_1^{CP} [\pi_l(a_{1l} - \gamma) - 1 + \theta(1 - \gamma)]}{(1 - \pi_h \theta (1 - \gamma) - \pi_l \theta (1 - \gamma)} < 0
\]

(53)

In order for the inequality above to be true it will have to be the case that

\[
[\pi_h(a_{1l} - \gamma) + 1 - \theta(1 - \gamma)] A^{(1-\gamma)(1-\theta)} < z_1^{CP} [\pi_l(\gamma - a_{1l}) + 1 - \theta (1 - \gamma)]
\]

But since if \(\lambda_{1s} > 0\), then \(z_1^{CP} > z_1^{1CP}\) = \(\frac{F''(k^T_{1l})k^T_{1l} + A^{(1-\gamma)(1-\theta)}}{F'(k^T_{1l}) + F'(k^T_{1l})k^T_{1l}(1-\gamma)}\), a sufficient condition is given by Assumption 9.
A.4.3 Corollary 1

Corollary 1: See text.

Proof of Corollary 1: Combining equations 37 and 50, one can re-write $z_{1l}^{CP} > z_{1l}$ as

\[
\left(\frac{1}{N} - 1\right) F''(k_{1l}^{T}) k_{1l}^{T} + \frac{1}{\delta''(B_l)N} \frac{\partial z_{1l}^{1P}}{\partial k_{1l}^{T}} z_{1l}^{CP} > -F''(k_{1l}^{T}) k_{1l}^{T}
\]

If $N \to \infty$, $z_{1l}^{CP} > z_{1l}$ is always true since $z_{1l}^{CP} > 1$. If $N < \infty$, then in order for $z_{1l}^{CP} > z_{1l}$, Assumption 10 has to be satisfied. If $N = 2$ and the country has zero fiscal capacity a sufficient condition is $2 < A + (1 - \gamma) (1 + \theta)$ since

\[
F''(k_{1l}^{T}) k_{1l}^{T} + 2F'(k_{1l}^{T}) < 2 < A + (1 - \gamma) (1 + \theta)
\]

A.4.4 Proposition 4

Proposition 4: See text.

Proof of Proposition 4: The proof that $k_{1l}^{T}$ is unique and exists is provided in the proof of Proposition 1. Since the minimum capital requirement constraint is binding, $k_{0}(\rho) = \frac{n}{\rho}$. Let’s consider the different types of equilibria.

Equilibrium of Type 1: If $\lambda_{1h} > 0$ and $\lambda_{1l} = 0$ ($z_{1l} = z_{0} > z_{1h}$), $k_{1l}^{T}(\rho)$ is pinned down from the budget constraint in the low state in $t = 1$

\[
[1 - \theta (1 - \gamma) + \pi_{l} (\gamma - a_{1l})] \frac{n}{\rho} - n = \pi_{l} [k_{1l}^{T} (F'(k_{1l}^{T}) - \theta (1 - \gamma)) + B_{l}(k_{1l}^{T})]
\]

(54)

The rest of the equations are:

\[
k_{1l}(\rho) = k_{0}(\rho) - k_{1l}^{T}(\rho); \quad k_{1h}(\rho) = \frac{(1 - \theta) (1 - \gamma) + a_{1h})}{[1 - \theta (1 - \gamma)]}
\]

\[
d_{1h}(\rho) = \theta (1 - \gamma) k_{0}(\rho); \quad d_{1l}(\rho) = \frac{1}{\pi_{l}} [k_{0}(\rho) - n - \pi_{l}\theta (1 - \gamma) k_{0}(\rho)]
\]

\[
d_{2s}(\rho) = \theta (1 - \gamma) k_{1s}(\rho)
\]

$k_{1l}^{T}$ in the Type 1 equilibrium is determined by equation 54 where the condition $z_{1l} > z_{1h}$ has to be satisfied which implies $M (k_{1l}^{T}) = \frac{1}{N} F''(k_{1l}^{T}) k_{1l}^{T} + \frac{1}{\delta''(B_l)N} \frac{\partial z_{1l}^{1P}}{\partial k_{1l}^{T}} + F'(k_{1l}^{T}) - 1 < 0$ (see the proof of Proposition 1) and also the borrowing constraint in the low state has to be non-binding i.e.

\[
d_{1l}(\rho) = \frac{1}{\pi_{l}} \left[(1 - \pi_{l}\theta (1 - \gamma)) \frac{n}{\rho} - n\right] < \theta (F'(k_{1l}^{T}) - \gamma) \frac{n}{\rho}
\]

\[
1 - \pi_{l}\theta (1 - \gamma) - \theta \pi_{l} (F'(k_{1l}^{T}) - \gamma) < \rho
\]
Equilibrium of Type 2: If $\lambda_{1s} > 0$ ($z_0 > z_{1s}$). This will be the optimal equilibrium only if $\rho = \frac{n}{k_0}$. The rest of the equations are the same as the equations in the equilibrium of Type 2 in Proposition 1.

Equilibrium of Type 3: If $\lambda_{1s} = 0$ ($z_{1l} = z_0 = z_{1h}$). $k_{1l}^T$ is pinned down by $M(k_{1l}^T) = 0$ and $k_{1l}(\rho) = \frac{n}{\rho} - k_{1l}^T$

$$d_{1l}(\rho) = (F'(k_{1l}^T) + a_{1l} - \gamma) \frac{n}{\rho} + B_l + [\theta (1 - \gamma) - F'(k_{1l}^T)] k_{1l}(\rho)$$

$$d_{1h}(\rho) = \frac{n - n - \pi_1 d_{1l}}{\pi_h}; k_{1h}(\rho) = \frac{d_{1h} - (1 + a_{1h} - \gamma) \frac{n}{\rho}}{[\theta (1 - \gamma) - 1]}$$

In order for the equilibrium to be of Type 3 the borrowing constraints in $t = 0$ against the high and low state should not be binding. One has to check that

$$d_{1l} < \theta (F'(k_{1l}^T) - \gamma) k_0; d_{1h} < \theta (1 - \gamma) k_0$$

Equilibrium of Type 4: If $\lambda_{1h} = 0, \lambda_{1l} > 0$ ($z_{1h} = z_0 > z_{1l}$). $k_{1l}^T$ is pinned down by the budget constraint in $t = 1$ in the low state

$$0 = (a_{1l} - \gamma + \theta (1 - F'(k_{1l}^T))) \frac{n}{\rho} + B_l(k_{1l}^T) - k_{1l}^T [\theta (1 - \gamma) - F'(k_{1l}^T)]$$

The rest of the equations are given by

$$d_{1h} = \frac{1}{\pi_h} \left[ \frac{n}{\rho} (1 - \pi_1 \theta (F'(k_{1l}^T) - \gamma)) - n \right]$$

$$k_{1h} = \frac{(1 + a_{1h} - \gamma) \frac{n}{\rho} - d_{1h}}{1 - \theta (1 - \gamma)}; d_{1l} = \theta (F'(k_{1l}^T) - \gamma) \frac{n}{\rho}$$

In order for the equilibrium to be of Type 4, the following conditions also have to be satisfied, $z_0 > z_{1l}$ which implies $M(k_{1l}^T) > 0$ and also $d_{1h} < \theta (1 - \gamma) k_0$ which implies

$$\frac{n}{\rho} (1 - \pi_1 \theta (F'(k_{1l}^T) - \gamma)) - \pi_h \theta (1 - \gamma) - n < 0$$

A.4.5 Corollary 2

Corollary 2: See text.

Proof of Corollary 2. Totally differentiate the budget constraint in the low state with respect to $k_0^l$ holding $d_{1l}^\tau$ fixed to solve for $\frac{\partial k_{1l}^T}{\partial k_0^l}$

$$\frac{\partial k_{1l}^T}{\partial k_0^l} = \frac{F'(k_{1l}^T) + a_{1l} - \gamma + F''(k_{1l}^T) k_{1l}^T \frac{1}{N} + \frac{\partial B_l^T}{\partial k_0^l}}{F'(k_{1l}^T) - \theta (1 - \gamma) + \frac{1}{N} F''(k_{1l}^T) k_{1l}^T \frac{1}{N} F'(k_{1l}^T) - \theta (1 - \gamma) k_0^l}$$

Totally differentiate the budget constraint in the low state with respect to $d_{1l}^\tau$ holding $k_0^l$ fixed to solve for $\frac{\partial k_{1l}^T}{\partial d_{1l}^\tau}$

55
Assumption 4. From the first order condition with respect to every statics with respect to fiscal capacity follows from the fact that the larger \( \lim \) is, the larger the moral hazard is. Also notice that for a given \( k \) either 
\[ (1 - \gamma) \theta + \frac{1}{N} F''(k_{1l}^T) k_{1l}^{i,F} \]  

Proof of Proposition 5: Before I start with the proof, it is useful to note that if \( k_{1l}^T \), the fewer the banks are and the larger the fiscal capacity is, the larger the moral hazard is, the smaller \( \delta''(B_l) \) is.

A.4.6 Proposition 5

Proposition 5: See text.

Proof of Proposition 5: I already proved that given Assumption 8, \( M'(k_{1l}^T) < 0 \) and also \( BC_{1l}(k_{1l}^T) > 0 \) follows from Assumption 4 and Lemma 1. The system of equations pins down either \( \{k_{1l}^{1,CP}, \chi^*(N)\} \) or \( \{k_{1l}^{T*, N^*(\chi)}\} \) if \( \chi^*(N) \) and \( N^*(\chi) \) are interior. \( k_{1l}^{T*} \) is determined by \( BC_{1l}(k_{1l}^{T*}) = 0 \) and the solution exists and is unique given that we proved existence and uniqueness of the constrained Central Planner's allocation, which is the allocation being replicated, (i.e. \( k_{1l}^{T*} = k_{1l}^{T,CP} \)).

Before I start with the proof, it is useful to note that if \( \chi > 0 \), then \( \lim_{k_{1l}^T \to 0} M(k_{1l}^T) > 0 \) since 
\[ \lim_{k_{1l}^T \to 0} \frac{1}{\sigma^2(B_l, \chi)} N \frac{\partial^2 v_{1l}^{i,P}}{\partial k_{1l}^{i,F} \partial x} > 0. \]

Also note that for a given \( N < \infty \), \( \lim_{\chi \to \infty} M(k_{1l}^T, \chi) \to \infty \) for every \( k_{1l}^T \) and \( M(k_{1l}^T, \chi) = 0 \) < 0 for every \( k_{1l}^T \).

---

\[ ^{56} \text{This will be the case since I proved that } z_{1l}^{i,P}(k_{1l}^T) > 1. \text{ I also assumed that } F''(k_{1l}^T) < 0 \text{ for every } k_{1l}^T > 0 \text{ in addition to Assumption 4 and the assumption } \frac{\partial^2 v_{1l}^{i,(B_l, \chi)}}{\partial M_{1l}^2} > 0. \]
The decentralized equilibrium will be of Type 1 as long as \( M(k_{1l}^T) < 0 \) and in that case only a minimum bank capital requirement is needed to decentralize the constrained Central Planner’s allocation since both the Central Planner and the banker value wealth more in the crisis state than in the high state in \( t = 1 \). If \( N \to \infty \) or \( \chi = 0 \) then \( M(k_{1l}^T) < 0 \) for every \( k_{1l}^T \in \left[ 0, k_{1l}^{T, \text{max}} \right] \) which is why the equilibrium is always of Type 1.

Next, I prove that if \( \rho_* = \frac{\rho}{k_{0l}} \) and \( \chi \leq \chi^* (N) \) then the decentralized equilibrium is of Type 1. The figure below depicts \( M(k_{1l}^T) \) and the budget constraint in the crisis state, \( BC_{1l}(k_{1l}^T) \) for two different values of fiscal capacity where the decentralized equilibrium is of Type 1.\(^57\) As the fiscal capacity increases, \( BC_{1l}(k_{1l}^T) \) shifts up (since \( \frac{\partial B_i(\chi; k_{1l}^T)}{\partial \chi} > 0 \) for a given \( k_{1l}^T \))\(^58\) and so does \( M(k_{1l}^T) \) (since I assumed that \( \frac{\partial^2 B_i(\chi)}{\partial \chi^2} < 0 \) for a given \( k_{1l}^T \)).

Therefore, to solve for the maximum possible fiscal capacity such that Type 1 equilibrium is achieved, one has to solve the following system of two equations and two unknowns, \( k_{1l}^T \) and \( \chi^* \) (for a given \( N \)).

\[
\begin{align*}
M(k_{1l}^T, \chi^*(N)) &= 0 \\
BC_{1l}(k_{1l}^T, \chi^*(N)) &= 0 
\end{align*}
\]

It is possible that \( \rho_* = \frac{\rho}{k_{0l}} \) for every \( k_{1l}^T \) in which case the equilibrium will never be of Type 1 unless \( \chi = 0 \) implying that \( \chi^* = 0 \). If \( \chi > \chi^*(N) \) then \( BC_{1l} \) will cross the zero line at a point where \( M(k_{1l}^T) > 0 \) and hence the requirement \( M(k_{1l}^T) < 0 \) will be violated and the equilibrium will not be of Type 1. In the figure below \( \chi^*(N = 3) = .77 \).

Next, I prove that if \( \rho_* = \frac{\rho}{k_{0l}} \) and \( N \geq N^* (\chi) \), then the decentralized equilibrium is of Type 1 and the minimum bank capital requirement is a sufficient instrument to replicate the constrained

\[
\lim_{k_{1l}^T \to +0} \frac{\partial z_{1l}^1, \rho}{\partial k_{1l}^T} = \frac{F''(k_{1l}^T)}{F'(k_{1l}^T) - \theta (1 - \gamma)} > 0
\]  \(^{(59)}\)

\(^{57}\) The parameters are the same as in Figure 3 (where \( N = 3 \)).

\(^{58}\) See the Proof of 6 for derivation.
Central Planner’s allocation. Since BC\textsubscript{I} is not a function of N directly one can solve for \( k_{I1}^{T*} \) from \( BC_{i1} \left( k_{I1}^{T*} \right) = 0 \) and \( k_{I1}^{T*} \) will not depend on N. If for the equilibrium value \( k_{I1}^{T*} \),

\[
\Lambda \left( k_{I1}^{T*} \right) = \left[ F'' \left( k_{I1}^{T*} \right) k_{I1}^{T*} + \frac{1}{\sigma_s \left( \sigma_s^2 - \lambda \right)} \frac{\partial_s \left( k_{I1}^{T*} \right)}{\sigma_s^2} \right] > 0 \text{ then } \frac{\partial M \left( k_{I1}^{T*} \right)}{\partial N} \big|_{k_{I1}^{T*}=k_{I1}^{T*}} = -\frac{1}{N^2} \Lambda \left( k_{I1}^{T*} \right) < 0,
\]

which implies that as N increases M shifts down and leads to the result that for \( N > N^* (\chi) \), the equilibrium is of Type 1 where we can solve explicitly for \( N^* (\chi) \) from \( M \left( k_{I1}^{T*}, N^* \right) = 0 \), \( N^* (\chi) = \frac{1}{1-F' \left( k_{I1}^{T*} \right)} \Lambda \left( k_{I1}^{T*} \right) > 0 \). If \( \Lambda \left( k_{I1}^{T*} \right) < 0 \), then \( M \left( k_{I1}^{T*} \right) < 0 \) and the Type 1 equilibrium will be achieved for any N, i.e. \( N^* (\chi) = 1 \).

A.4.7 Proposition 6

**Proposition 6:** See text.

**Proof of Proposition 6:** Consider an interior equilibrium for the Central Planner. By setting \( \rho^* = \frac{n}{k_0^*} \) and \( \nu^* = \phi_{CP}^\prime \) (if necessary) the policy maker can replicate the constrained Central Planner’s allocation. Since I proved that \( \frac{\partial \phi^{CP} \left( k_{I1}, \chi \right)}{\partial \phi_0} > 0 \), it is sufficient to prove that holding \( k_0 \) constant, \( \frac{\partial \phi^{CP} \left( \chi, k_0 \right)}{\partial \chi} < 0 \) (partial derivative) in order to prove that \( \frac{\partial \phi_0}{\partial \chi} > 0 \). The budget constraint of the banker is

\[
H \left( \chi; k_0 \right) = \pi_l B_l \left( k_{I1}^{T}, \chi \right) + n + \left[ F' \left( k_{I1}^{T} \right) - \theta (1-\gamma) \right] k_{I1}^{T} \pi_l
- k_0 \left(1-\theta (1-\gamma) + \gamma - a_{II} \right) = 0
\]

For a given \( k_0 \), differentiate \( H \left( \chi; k_0 \right) = 0 \) with respect to \( \chi \) (partial derivative) in order to derive the relationship between fiscal capacity and fire-sale size, for a given level of ex-ante investmen, i.e. \( \frac{\partial k_{I1}^{T} \left( \chi; k_0 \right)}{\partial \chi} \)

\[
\frac{\partial k_{I1}^{T} \left( \chi; k_0 \right)}{\partial \chi} = \frac{- \frac{\partial B_l \left( \chi, k_{I1}^{T} \right)}{\partial \chi} }{\left[ \frac{\partial B_l \left( k_{I1}^{T}, \chi \right)}{\partial k_{I1}^{T}} + F' \left( k_{I1}^{T} \right) - \theta (1-\gamma) + F'' \left( k_{I1}^{T} \right) k_{I1}^{T} \right] } < 0
\]

The denominator is positive because of Assumption 4 and 1. One can solve for \( \frac{\partial B_l \left( \chi, k_{I1}^{T} \right)}{\partial \chi} \) by totally differentiating the first order condition that pins down \( B_l \), holding \( k_{I1}^{T} \) constant. The derivative is given by

\[
\frac{\partial B_l \left( \chi, k_{I1}^{T} \right)}{\partial \chi} = - \frac{\partial^2 \delta \left( B_l, \chi \right)}{\partial B_l \partial \chi} / \frac{\partial^2 \delta \left( B_l, \chi \right)}{\partial B_l^2} = \frac{B_l}{(\eta-1) \chi} > 0
\]

where I assumed that \( \frac{\partial^2 \delta \left( B_l, \chi \right)}{\partial B_l \partial \chi} < 0 \) and \( \frac{\partial^2 \delta \left( B_l, \chi \right)}{\partial B_l^2} > 0 \). The result \( \frac{\partial k_{I1}^{T} \left( \chi; k_0 \right)}{\partial \chi} < 0 \) is intuitive and means that for a given level of period zero investment, the fire-sale will be smaller if the country had a larger fiscal capacity.
\[
\frac{\partial \psi^{CP}(\chi; k_0)}{\partial \chi} = \frac{\partial z^{CP}_{LT}}{\partial \chi} - \frac{\partial z^{CP}_0}{\partial \chi} = \left[ \frac{1 - \theta (1 - \gamma) + \pi_l (\gamma - a_{1l})}{1 - \pi_k \theta (1 - \gamma)} \right] \frac{\partial k^{T}_{LT}(\chi; k_0)}{\partial \chi} \frac{\partial z^{CP}_{LT}}{\partial k^{T}_{LT}} < 0
\]

where in Lemma 3, I proved that \( \frac{\partial z^{CP}_{LT}}{\partial k^{T}_{LT}} > 0 \) and given that the equilibrium is of Type 1, \( z^{CP}_{LT} = z^{CP}_L \).

### A.4.8 Proposition 7

**Proposition 7:** See text.

**Proof of Proposition 7:** Conditional on the policy maker having an access to two ex-ante instruments — an ex-ante tax on period zero investment ("price" instrument), \( \tau^i_{k_0} \), and a limit on the payment promised in the crisis state, \( d^i_{LT} \leq v^i \), the constrained Central Planner’s allocation can be replicated. Consider parametrization such that the equilibrium is of Type 1 for the constrained Central Planner.

The only difference between the problem of the banker where the ex-ante instrument is a "price" instrument and the problem of the banker where the ex-ante instrument is a "quantity" instrument is that the period zero budget constraints becomes

\[
k^i_0 (1 + \tau^i_{k_0}) - n + T_{k_0} \leq \sum_s \pi_s d^i_{1s} \quad [z^i_0]
\]

where \( \tau^i_{k_0} \) is the tax on period zero capital. The revenues from the proportional tax are distributed equally back to the bankers using the lump sum tax, \( T_{k_0} = -\sum_{i=1}^N \frac{1}{N} k^i_0 \pi_k \) in the form of a non-targetted tax. Banker \( i \) chooses \( k^i_0 \) at the end of \( t = 0 \), while \( \tau^i_{k_0} \) is determined in the beginning of \( t = 0 \) and banker \( i \) takes it as given. However, banker \( i \) internalizes the fact that he affects the lump sum tax \( T_{k_0} \) since he is large (not essential). Banker’s \( i \) optimization problem at the end of \( t = 0 \) is

\[
\max_{k^i_0, d^i_{1s}, k^i_{1s}} \sum_s \pi_s (A + (1 - \theta) (1 - \gamma)) k^i_{1s}
\]

subject to the period one budget constraint, the borrowing constraint

\[
k^i_{1s} (F' (k^T_{1s}) - \theta (1 - \gamma)) + d^i_{1s} \leq (F' (k^T_{1s}) + a_{1s} - \gamma) k^i_0 + B^i_s \quad [\pi_s z^i_{1s}]
\]

and to the period zero budget constraint given by equation 63. First order condition with respect to \( k^i_0 \)

\[
\sum_s \pi_s z^i_{1s} \left( F' (k^T_{1s}) + a_{1s} - \gamma + \frac{1}{N} F'' (k^T_{1s}) k^i_{1s} + \frac{\partial B^i_s}{\partial k^i_0} \right) - z^i_0 \left[ 1 + \frac{1}{N} \right] + \sum_s \pi_s \lambda^i_{1s} \theta \left( F' (k^T_{1s}) - \gamma + \frac{1}{N} F'' (k^T_{1s}) k^i_0 \right) = 0
\]
and the rest of the first order conditions are the same as in the decentralized equilibrium with a "quantity" ex-ante instrument. After imposing a symmetric equilibrium and using the fact that 
\[ z_0 = z_{1l} > z_{1h}, \lambda_{2s} > 0, \lambda_{1l} = 0 \text{ and } \lambda_{1h} > 0, \]
one can re-write the first order condition with respect to \( k_0 \) from the decentralized problem as

\[
z_{1l} \left[ 1 + \tau_{k_0} \left( 1 - \frac{1}{N} \right) - \pi_h \theta (1 - \gamma) - \pi_l (\theta (1 - \gamma) + a_{1l} - \gamma) \right] = \frac{(A + (1 - \theta) (1 - \gamma))(1 - \theta (1 - \gamma) + \pi_h (a_{1h} - \gamma))}{1 - \theta (1 - \gamma)}
\]

where \( z_{1l} \) is given by equation 37. From equation 49

\[
z_{1l}^{CP} (1 - \pi_h \theta [1 - \gamma] - \pi_l (\theta (1 - \gamma) + a_{1l} - \gamma)) = \frac{(A + (1 - \theta) (1 - \gamma))}{1 - \theta (1 - \gamma)} [\pi_h (a_{1h} - \gamma) + 1 - \theta (1 - \gamma)]
\]

Subtracting equation 66 from equation 68, since 
\[ z_{1l}^{CP} \left( k_{1l}^{T, CP} \right) - z_{1l} \left( k_{1l}^{T, CP}, \chi \right) > 0 \]
for a given \( k_{1l}^{T, CP} \), which I proved in Corollary 1,

\[
\tau_{k_0}^* = \frac{z_{1l}^{CP} \left( k_{1l}^{T, CP} \right) - 1}{z_{1l} \left( k_{1l}^{T, CP}, \chi \right)} \Phi > 0
\]

where

\[
\Phi = \frac{[1 - \theta (1 - \gamma) + \pi_l (\gamma - a_{1l})]}{(1 - \frac{1}{N})}
\]

Since the equilibrium \( k_{1l}^{T, CP} \) does not vary with \( \chi \) in the Central Planner problem and \( z_{1l}^{CP} \) is a function only of \( k_{1l}^{T, CP} \), \( \frac{\partial z_{1l}^{CP} (k_{1l}^{T, CP})}{\partial \chi} = 0 \)

\[
\frac{\partial \tau_{k_0}^*}{\partial \chi} = -\frac{\partial z_{1l} \left( k_{1l}^{T, CP}, \chi \right)}{\partial \chi} \frac{z_{1l}^{CP} \left( k_{1l}^{T, CP} \right)}{z_{1l} \left( k_{1l}^{T, CP}, \chi \right)} 2 \Phi \geq 0
\]

where

\[
\frac{\partial z_{1l} \left( k_{1l}^{T, CP}, \chi \right)}{\partial \chi} = -\frac{\partial B_l \left( k_{1l}^{T, CP} \right)}{\partial k_{0}^{2} \partial \chi} \left[ F' \left( k_{1l}^{T, CP} \right) - \theta (1 - \gamma) + \frac{1}{N} F'' \left( k_{1l}^{T, CP} \right) k_{1l}^{T, CP} + \frac{\partial B_l}{\partial k_{0}} \right] \leq 0
\]

where in Corollary 2 I proved \( \frac{\partial B_l}{\partial k_{0} \partial \chi} \geq 0 \). If \( N \rightarrow \infty \), \( z_{1l} \) is not a function of \( \chi \) because \( \frac{\partial B_l}{\partial k_{0}} = 0 \) which implies \( \frac{\partial \tau_{k_0}^*}{\partial \chi} = 0 \). If \( 1 < N < \infty \), \( \frac{\partial B_l}{\partial k_{0} \partial \chi} > 0 \) and \( \frac{\partial \tau_{k_0}^*}{\partial \chi} > 0 \).