Aggregation of Information and Beliefs in Prediction (& Financial) Markets

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Prediction Markets

Incentive-based mechanisms that pool information about future events:

• **Betting markets:**
  – Iowa electronic markets
  – Parimutuel derivative auctions on non-farm payroll at Goldman Sachs

• **Corporate prediction markets:**
  – Sales of HP printers
  – LCD TV Futures, developed by Newsfutures for Corning [forecasting contest]
  – Launch of *secret product* at Google [design akin to financial market]

• Public policy applications:
  – Avian flu outbreak by Gerson Lehrman Group
  – Medical observations at Sermo
Our Interest

- Arguably, source of accurate and cheap *information*
  - complementing expert judgements and opinion pools
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- Observable outcomes, hence ideal setting for testing theories—below we discuss main empirical finding: favorite-longshot bias
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- Intriguing, information as byproduct of markets — also useful inside firm — we investigate implications for more general financial markets

- Market design questions; manipulation concerns — not in this talk!
Our Aim

Traders have **no liquidity needs** in these markets—so what motivates trade?

- Simple model with:
  1. Heterogeneous prior beliefs—hence, there is trade
  2. Private information—hence, valuable prediction tool

- Stack cards **in favor of information aggregation**—focus on fully revealing REE

- Main question: **How does market price relate to traders’ posterior beliefs?**

Here we do **not** study effect of trading rules on market outcomes
Contributions

1. Bridge two literatures:

   — Literature on prediction markets, **heterogeneous priors**, no info

   — REE, common prior, **heterogeneous information**
     Grossman (1976), Radner (1979)
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   — REE, common prior, heterogeneous information
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2. Heterogeneous priors & concordant beliefs, like Milgrom and Stokey (1981)
   — Positive characterization of REE in first period of Milgrom and Stokey
   — Compared to Varian (1989), relax CARA—we focus on wealth effect!
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   - REE, common prior, **heterogeneous information**
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2. Heterogeneous priors & concordant beliefs, like Milgrom and Stokey (1981)
   - Positive characterization of REE in **first period** of Milgrom and Stokey
   - Compared to Varian (1989), relax CARA—we focus on **wealth effect**!

3. With limited exposure (or wealth) and/or DARA preferences, price underreacts to information, possibly explaining:
   - Favorite-longshot bias
   - Post-event drift/momentum
Outline

1. What are prediction markets?

2. **Fact: favorite-longshot bias**

3. Model 1 (prediction markets): bounded positions

4. Model 2 (financial markets): positions bounded by risk aversion

5. Extensions
Favorite-Longshot Bias

**Empirical literature:**

- Group horses by *market probability*
- Compute *empirical probability* of winning in races for group
- Compare

**Main findings:**

- Market probability close to empirical probability
- But:
  - Market prob $> \text{empirical prob}$ for longshots
  - Market prob $< \text{empirical prob}$ for favorites
Data: Asch, Malkiel and Quandt (1982)

Subjective and objective probabilities of winning in 729 Atlantic City (NJ) races in 1978 (total number of horses = 5805).

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favorites  

longshots

empirical market  
prob. prob.
Outline

1. What are prediction markets?

2. Fact: favorite-longshot bias

3. **Model 1 [prediction markets]: bounded positions**

4. Model 2 [financial markets]: positions bounded by risk aversion

5. Extensions
Limited-Wealth Model

- Events $E$ and $E^c$, two Arrow-Debreu assets (pay 1 or 0 depending on event)
- Focus on $p := p_E$ of asset paying in event $E$. [By arbitrage, $p_E + p_{E^c} = 1$]
- Trader $i \in [0, 1]$, with subjective prior $q_i$, updated to posterior $\pi_i$
- Risk-neutral traders max expected wealth $\pi_i w_i(E) + (1 - \pi_i) w_i(E^c)$
- Designer gives each trader $w_{i0}$ units of each asset, $w_{i0}(E) = w_{i0}(E^c)$
- Traders cannot hold less than 0 of any asset
- Initial distribution of assets: $G(q)$, continuous and strictly increasing
- Private information: signals $s_i$
- Likelihood ratio $L(s) = f(s_1, \ldots, s_I | E) / f(s_1, \ldots, s_I | E^c)$
- Concordant beliefs: heterogeneous priors, common $L(s)$
Individual Optimization

- A trader with posterior $\pi_i > p$
  - sells entire endowment of asset $E^c$ to get $(1 - p)w_i0$
  - invests all proceeds to buy $(1 - p)\frac{w_i0}{p}$ units of asset $E$
  - achieving final portfolio of $\langle \frac{w_i0}{p}, 0 \rangle$ of assets $\langle E, E^c \rangle$

- Conversely, final portfolio of trader with $\pi_i < p$ is $\langle 0, \frac{w_i0}{1 - p} \rangle$

- **Note:** price-dependent bound on number of assets an agent can hold!
Equilibrium

• In a fully revealing REE, \( p(L) \) price is a sufficient statistic for likelihood \( L(s) \)
• Each trader’s posterior \( \pi_i(L) \) incorporates information \( L \)
• Market clearing requires that net trades are equal to zero
• Asset \( E^c \) is demanded in amount \( w_i0/(1-p) \) by individual with prior \( q_i < p/[(1-p)L + p] \)

• Aggregate demand, \( \frac{G\left(\frac{p}{(1-p)L+p}\right)}{1-p} \), is equal to aggregate supply, 1, if

\[
p = 1 - G\left(\frac{p}{(1-p)L+p}\right)
\]

(1)

• Note that this price reveals \( L \), because it’s strictly increasing in \( L \)
• This equation has a unique solution at which \( G \notin \{0,1\} \), because \( G \) is non-decreasing
Bias

• Using Bayes’ rule \( \frac{p}{1-p} = \frac{\pi}{1-\pi} = \frac{q}{1-q} L \), we can interpret the market price as a posterior belief of a hypothetical “market” agent with prior belief equal to \( \frac{p}{(1-p)L + p} \), call this the “market prior”

Result: The “market prior” \( \frac{p}{(1-p)L + p} \) is strictly decreasing in \( L \)

Proof: When \( L \) increases, \( p \) also increases. By the equilibrium condition

\[
p = 1 - G \left( \frac{p}{(1-p)L + p} \right),
\]

when \( p \) increases, \( \frac{p}{(1-p)L + p} \) must fall!
Intuition

• Higher $L$ increases price $p$
  – Optimistic traders can buy less units of asset $E$
  – Pessimistic traders buy more units of asset $E^c$

• To equilibrate the market, price must change to move agents from pessimistic to optimistic side

• The indifferent trader (that determines equilibrium price) is someone with more pessimistic prior belief

• Hence, although $p$ rises with $L$, it rises less fast than a posterior belief, because of this negative effect on the market prior
Illustration

Suppose prior beliefs uniformly distributed across traders, \( G(q) = q \)
Equilibrium \( 1 - \frac{p}{(1-p)L+p} = p \); average prior \( \frac{1}{2} \) equal to price with \( L = 1 \)

Associated to price \( p \), posterior belief \( \frac{L}{1+L} = \frac{p^2}{p^2 + (1-p)^2} \)
Literature on FLB [1]: Heterogeneous Beliefs

- Theories: mis-calculation of probabilities, risk-loving bettors, market power...

- Ali (1977): heterogeneous prior beliefs $q_i \sim F(q_i)$, but no information

- Each bettor places a fixed amount on the preferred horse


- Market probability $p > 1/2$ is below median bettor’s belief $m$:

  $$1 - F(p) = p > 1/2 = 1 - F(m) \Rightarrow p < m$$

- FLB explained by identifying median bettor’s belief with empirical chance

- Differences with Ali: **informational** bias; risk aversion, endogenous positions...
Literature on FLB [2]: Asymmetric Information

- Ottaviani and Sørensen (2004, 2005) study parimutuel betting
- Bettors have **private information**, but **common prior** belief
- FLB results in equilibrium if there is more information than noise
- Key difference: *Bayes-Nash equilibrium* of simultaneous betting $\neq$ REE
Outline

1. What are prediction markets?
2. Fact: favorite-longshot bias
3. Model 1 (prediction markets): bounded positions
4. Model 2 (financial markets): positions bounded by risk aversion
5. Extensions
Risk-Aversion Model

• Events $E$ and $E^c$, two Arrow-Debreu assets (pay 1 or 0 depending on event)

• Focus on $p := p_E$ of asset paying in event $E$. [By arbitrage, $p_E + p_{E^c} = 1$]

• Trader types $i = 1, \ldots, I$ with heterogeneous priors $q_i$

• Subjective EU $\pi_i u_i(w_i(E)) + (1 - \pi_i) u_i(w_i(E^c))$ with heterogeneous $\pi_i$

• General risk aversion, $u'_i > 0$ and $u''_i < 0$

• Initial wealth $w_{i0}$, constant with respect to events $E$ and $E^c$

• Private information: signals $s_i$

• Likelihood ratio $L(s) = f(s_1, \ldots, s_I | E) / f(s_1, \ldots, s_I | E^c)$

• Concordant beliefs: heterogeneous priors, common $L(s)$
Trader’s Problem

• Given info with likelihood ratio $L$, subjective posterior $\pi_i(L)$ satisfies

$$\frac{\pi_i(L)}{1 - \pi_i(L)} = \frac{q_i}{1 - q_i}L$$

• Portfolio $(x_{i1}, x_{i2}) \in \mathbb{R}^2$ gives, through net position $\Delta x_i = x_{i1} - x_{i2}$,

$$w_i(E) = w_{i0} + x_{i1} - px_{i1} - (1 - p)x_{i2} = w_{i0} + (1 - p)\Delta x_i$$

$$w_i(E^c) = w_{i0} + x_{i2} - px_{i1} - (1 - p)x_{i2} = w_{i0} - p\Delta x_i$$

• Given $p$ and $L$, maximize $\pi_i(L)u_i(w_i(E)) + (1 - \pi_i(L))u_i(w_i(E^c))$

• Necessary FOC for optimal $\Delta x_i$ is

$$\frac{\pi_i(L)}{1 - \pi_i(L)} \frac{u'_i(w_{i0} + (1 - p)\Delta x_i)}{u'_i(w_{i0} - p\Delta x_i)} = \frac{p}{1 - p} \quad \text{(FOC)}$$
Fully Revealing Equilibrium

- Consider fully revealing REE, ideal for info aggregation—noisy REE later
- One-to-one map between price $p$ and likelihood ratio $L$; correct inference
- Two assets supplied in equal amount (as in Iowa Electronic Markets or parimutuel)
- Given $L$, $p(L)$ clears the market: $0 = \sum_{i=1}^{I} \Delta x_i$

**Lemma:** With DARA or CARA preferences, demand $\Delta x_i$ is strictly decreasing in $p$

**Proposition:** If demand $\Delta x_i$ is strictly decreasing in $p$, there exists a unique fully revealing REE $p(L)$.

- The price is a *generalized average of the trader’s posterior beliefs*: strictly increasing in $q_i$ and $L$, strictly inside the convex hull of the traders’ $\pi_i(L)$

Now, let’s look at some special cases
Special Case 1: Common Prior

- Suppose that all traders share the same prior belief \( q_i = q \)

- The price \( p \) must then be equal to the common posterior belief that event \( E \) is true

\[
\frac{\pi_i(L)}{1 - \pi_i(L)} \cdot \frac{u'_i(w_i(E))}{u'_i(w_i(E^c))} = \frac{p}{1 - p}
\]

- No trade (Milgrom and Stokey 1982)
Special Case 2: CARA

- Constant absolute risk aversion, $u_i(w) = -\exp(-w/t_i)$
- Risk tolerance $t_i$, relative risk tolerance $\tau_i = t_i/\left(\sum_{j=1}^{I} t_j\right)$
- Demand is
  $$\Delta x_i = t_i \log \left( \frac{1 - p_1(L)}{p_1(L)} \frac{\pi_i(L)}{1 - \pi_i(L)} \right)$$
- Fully revealing REE price $p(L)$ satisfies
  $$\log \left( \frac{p(L)}{1 - p(L)} \right) = \log L + \sum_{i=1}^{I} \tau_i \log \left( \frac{q_i}{1 - q_i} \right)$$
  $$= \log \left( \frac{q}{1 - q} \right) \text{ prior aggregation, Wilson (1968)}$$

With CARA preferences, market price is a posterior belief!
Main Result: Underreaction

By Bayes’ rule, \(price = posterior\) (for some prior \(q\)) if

\[
\log \left( \frac{p(L)}{1 - p(L)} \right) = \log L + \log \left( \frac{q}{1 - q} \right),
\]

i.e., if \(\log \left( \frac{p(L)}{(1 - p(L))} \right) - \log L\) is constant wrt \(L\)

Under DARA (Decreasing Absolute Risk Aversion) and heterogeneous beliefs, the price \(p(L)\) underreacts to information \(L\): 
\[
\log \left( \frac{p(L)}{(1 - p(L))} \right) - \log L \text{ is strictly decreasing in } L
\]

- There is no market prior \(q\) that is not linked to information \(L\)
- Optimists are given less weight when the information becomes more favorable—and pessimists are given more weight
- It’s as if the prior were automatically adjusted against the information—\textbf{why?}
Underreaction: Intuition

- $L \uparrow \Rightarrow p \uparrow \Rightarrow$ wealth effects:
  1. negative wealth effect on optimistic individuals (with $\pi_i > p$), who are net demanders ($\Delta x_i > 0$)
  2. pessimistic traders benefit from the price increase

- By DARA:
  1. optimists become more risk averse, and so take on smaller risky positions
  2. pessimists become less risk averse, and so take larger positions

- Downward price adjustment to clear market

Summary: $L \uparrow$ shifts weight from optimists to pessimists when the market price is calculated as a belief average
Illustration: Log Preferences

- Suppose $u_i(w) = \log(w)$. Demand is

$$\Delta x_i = w_{i0} \frac{\pi_i(L) - p(L)}{p(L)(1 - p(L))}$$

- Fully revealing REE price is

$$p(L) = \sum_{i=1}^{I} \omega_i \pi_i(L) = \sum_{i=1}^{I} \omega_i \frac{q_i L}{q_i L + (1 - q_i)}$$

where $\omega_i = \frac{w_{i0}}{\sum_{j=1}^{I} w_{j0}}$ is relative wealth

- The price is an increasing function of $L$ and hence revealing

- Candidate for market prior is $q := p(1) = \sum_{i=1}^{I} \omega_i q_i$

- Market posterior: $\pi(L) = qL / (qL + (1 - q))$
Illustration

Log utility traders, priors uniformly distributed over \([0, 1]\)
Underreaction: Implications

Possible explanation for:

- Favorite-longshot bias observed in betting markets
  - Favorable information leads to a high market price that nevertheless understates true posterior probability
  - Opposite for the longshot—the market overbets the longshot

- Post-event drift and momentum observed in financial markets
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Price Drift

- After one round of trade, price is $p$, while market posterior is $\pi$
- Extend model to allow new information $L_t$ to arrive to the market at time $t$
- Under the true posterior $\pi$, we can show that

$$E[p_t - p] = (\pi - p) \int \frac{(1 - p)p(L_t - 1)}{pL_t + 1 - p} (L_t - 1) f(L_t|E^c) dL_t$$

which has the same sign of $(\pi - p)$ since the integrand is positive

- We also have

$$E[(p_t(L_t, L) - p(L))(p(L) - p(1))] > 0$$

i.e., later price changes are positively correlated to first-round price change
Literature

- Milgrom and Stokey (1982): once market is in equilibrium, price responds correctly to new info (with no trade)
- Here focus on first formed equilibrium

- Grossman (1976): information aggregation with CARA & common prior
- Varian (1989): heterogeneous priors but still CARA—we allow for general risk attitudes

- Harrison and Kreps (1978): heterogeneous priors; risk neutrality; no short sales
- But prediction market allows short positions: no equilibrium
- Hence, our limited budget or risk aversion assumption
- There is no information in their model—our result hinges on information
Model with General \# States

- Retain complete markets, and suppose there are 3 states and traders have log utility

- Two new phenomena:
  1. **Information interdependence/contagion**: price of asset $C$ is affected by information $L_{AB}$ (about $A$ relative to $B$)
  2. **Overreaction to information** $L_{AB}$ (about $A$ relative to $B$) results if:
     (a) Relative to trader 2, trader 1 is more optimistic about state $A$ relative to state $B$:
         $$\frac{q_{A1}}{q_{B1}} > \frac{q_{A2}}{q_{B2}}$$

     (b) Relative to trader 2, trader 1 is less optimistic about state $A^C$ relative to state $B^C$:
         $$\frac{q_{A1}}{q_{A2}} < \frac{1-q_{B1}}{1-q_{B2}}$$

     (c) Strong good news for state $C$, $L_{CB} > 1 + \frac{(q_{A1} - q_{A2})}{(q_{A1}q_{B2} - q_{B1}q_{A2}) - (q_{A1} - q_{A2})}$
Noisy REE: Uncertain Priors

- Suppose CARA utility with common risk tolerance $1$
- Two states; conditionally i.i.d. signals
- Before we found

$$\log \left( \frac{p}{1-p} \right) = \sum_{i=1}^{I} \log \left( \frac{f(s_i|E)}{f(s_i|E^c)} \right) + \sum_{i=1}^{I} \frac{1}{I} \log \left( \frac{q_i}{1-q_i} \right)$$

- Now, suppose prior beliefs are private information, with

$$\log \left( \frac{q_i}{1-q_i} \right) \sim N \left( 0, \sigma^2 \right) \text{ and } \log \left( \frac{f(s_i|E)}{f(s_i|E^c)} \right) \sim N \left( 1_E - 1_{E^c}, \nu^2 \right)$$

- Linear noisy REE:

$$\log \left( \frac{p}{1-p} \right) = \frac{\sigma^2/I + \nu^2}{\sigma^2 + \nu^2} \sum_{i=1}^{I} \log \left( \frac{q_i}{1-q_i} \frac{f(s_i|E)}{f(s_i|E^c)} \right)$$

- Cannot separate traders’ priors and signals
Liquidity-Motivated Trade (w/ Common Prior)

- Traders with **common prior**, but **heterogeneous endowments** in the 2 states

- **Result**: If all traders have **HARA** (Hyperbolic Absolute Risk Aversion, i.e., linear risk tolerance $T(w) = \alpha_i + \beta_i w$) with **common cautiousness parameters** (i.e., $\beta_i = \beta$), there is **no bias** [i.e., $\ln \left( \frac{p(L)}{1-p(L)} \right) - \ln L$ is constant in $L$]

- Intuition for different condition compared to heterogeneous priors: next slide
Liquidity versus Heterogeneous Priors

Example: Log utility, agent $i$'s endowment $(a_i, b_i)$, aggregate endowment $(A, B)$

- Trader $i$’s equilibrium net trade in first asset

$$\Delta x_i = \begin{cases} \frac{\pi}{p} \left( \frac{b_i - \frac{B}{A}}{a_i} \right) & \text{if } B > A \\ \uparrow L \text{ if } B > A \end{cases}$$

- All traders take more [or less if $B < A$] extreme positions, as $L$ increases.
- With liquidity motivated trade, those who buy, buy more, those who sell, sell more

- With heterogeneous beliefs, optimists (who buy) buy less, while pessimists (who sell) sell less—so the price had to equilibrate against direction of the information
Conclusion

1. Bridge literatures on REE and prediction markets by analyzing first-period trading
2. Belief aggregation interacts with information aggregation due to income effects
3. Underreaction to information with two states (and complete markets)
4. Subtler effects and contagion with more than two states