Exploiting Short-Run Predictability

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Abstract

This paper measures the utility gains from exploiting short-run predictability in the volatility of stock returns in a dynamic model in the presence of transaction costs, short-selling constraints and estimation risk. We find that utility gains are quite significant, both ex ante and out-of-sample.

JEL Classification: G11.

Keywords: Volatility Timing, Dynamic Portfolio Choice, Transaction Costs.

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1 Introduction

In this paper we investigate the potential utility gains generated by volatility timing. We perform this analysis by solving for the optimal dynamic asset allocation decision in the presence of short-term time variation in the volatility of equity returns. Given the dynamic nature of the model, we include an intermediate consumption decision every period, and we are able to model realistic constraints, namely transaction costs and short-selling restrictions, directly into the optimization problem. Therefore, the optimal asset allocation decision explicitly takes into account for the presence of those constraints. We consider different levels of transaction costs, scenarios with and without short-selling constraints, and we control for parameter uncertainty.

Our results show that the gains from exploiting autocorrelation in volatility can be quite large. Since volatility is very persistent the trading strategy does not require significant rebalancing, and therefore it is hardly affected by transaction costs. The corresponding annualized certainty equivalent gain is 8.00%, 3.22% and 1.79%, for investors with coefficients of relative risk aversion of 2, 5 and 10, respectively. If we introduce fully binding short-selling constraints, which is clearly an extreme assumption, the gains are only reduced for the investor with a risk aversion of 2, and they are still 5.26%. Moreover, since the volatility persistence parameter is estimated with high precision, these certainty equivalents are robust to parameter uncertainty. The utility gains are much lower in an out-of-sample analysis, but they are still significant: 1.87%, 1.69% and 1.45%. The volatility timing strategy generates returns that have a slightly lower mean, but compensates for this by offering a significant reduction in volatility.

The hedging demands induced by time variation in volatility have been shown to be quite small (Ait-Sahalia and Brandt (2001) and Chako and Viceira (2001)), and naturally in this paper we obtain the same result. Nevertheless, solving the intertemporal asset allocation problem is very important since it allows us to incorporate transaction cost directly into the maximization problem. In the presence of fixed transaction costs, the policy functions in a dynamic model can be significantly different than those obtained in a static setting, as the investor must anticipate the likelihood of future trades when making his/her optimal
decision today. To the best of our knowledge, this is the first paper that simultaneously considers parameter uncertainty and frictions in the context of a dynamic model.

Recently, Liu (1999), Chako and Viceira (2000), and Liu, Longstaff and Pan (2003) study the impact of volatility hedging on the optimal decision rules.\textsuperscript{1} They do not allow for frictions (transaction costs or short-selling constraints) or parameter uncertainty, and they do not measure utility gains. Tamayo and Shanken (2002) extend this analysis by taking into account for parameter uncertainty. Avramov and Chordia (2005) model time-varying betas in a similar context (although they focus on individual stocks) and therefore also capture volatility timing. Marquering and Verbeek (2000), Fleming, Kirby and Ostdiek (2001 and 2003) and concurrent work by Johannes, Polson and Stroud (2003) measure the utility gains in the context of a static asset allocation model, while taking into account for estimation risk. As a result, these papers do not include transaction costs or intermediate consumption in the optimization problem.

The rest of the paper is organized as follows. In section 2 we present the portfolio choice problem and discuss the estimation and calibration procedures. In section 3 we discuss the in-sample results, while in section 4 we study the out-of-sample performance. Finally, we offer some concluding remarks in section 5.

\section{Portfolio choice problem}

This paper considers the portfolio choice problem of an investor with power utility and a finite horizon ($T$ periods). The investor faces both a consumption decision and an asset allocation decision. Every period ($t$) she allocates her financial wealth ($W_t$) between a risky asset (stocks, which yield a random return $R_t$) and a riskless asset (T-Bills, with a constant return $R^f$). The notation $\alpha_t$ will be used to define the share of wealth invested in the risky asset at time $t$. The rebalancing horizon (the time interval between any two periods) is set at one month, and we will consider different investment horizons ($T$).

\textsuperscript{1}Recent work by Busse (1999) shows that some mutual funds often earn higher risk-adjusted returns by successfully performing “volatility timing”.

3
2.1 Investment Opportunity Set

The stock return process is specified as

$$\text{Ln}R_t = \mu + \varepsilon_t \quad (1)$$

$$\varepsilon_t \sim N(0, \sigma_t^2) \quad (2)$$

$$\sigma_t^2 = \theta_0 + \theta_1 \varepsilon_{t-1}^2 + \theta_2 \sigma_{t-1}^2 \quad (3)$$

We will refer to model A as the i.i.d. model (i.e. with $\theta_1 = \theta_2 = 0$) while model B denotes the general case. Using a GARCH(1,1) process to model time variation in the volatility is a relatively standard choice, as this model fits the data quite well (see Campbell, Lo and MacKinlay (1997) or Bollerslev, Chou and Kroner (1992) for detailed surveys). However we do not claim that this is the best statistical model. We consider the GARCH(1,1) for tractability reasons and because it is not the objective of this paper to identify the best trading strategy, but rather to compute the utility gains obtained by following a “relatively plausible” volatility timing strategy.

2.2 Transaction Costs

Whenever the investor rebalances her portfolio she must pay transaction costs. Define $\hat{\alpha}_t$ as the share of wealth invested in the risky asset at time $t$, assuming that the investor does not rebalance her portfolio in that period. So this is the inherited $\alpha$, given the previous period’s allocation ($\alpha_{t-1}$), and this period’s stock return. If $\alpha_{t-1} = 1$ or if $R_t = R_f$, then $\hat{\alpha}_t = \alpha_{t-1}$ so that the beginning of period portfolio allocation is exactly equal to last-period’s portfolio allocation. However, in general, $\hat{\alpha}_t$ is given by:

$$\hat{\alpha}_t = \frac{\alpha_{t-1} R_t}{\alpha_{t-1} R_t + (1 - \alpha_{t-1}) R_f} \quad (4)$$

If the investor chooses $\alpha_t = \hat{\alpha}_t$, then she is keeping her portfolio allocation unchanged and does not have to pay any transaction cost. For $\alpha_t \neq \hat{\alpha}_t$ the transaction costs are given by:

$$k_t = v |\alpha_t - \hat{\alpha}_t| + f I(\alpha_t \neq \hat{\alpha}_t) \quad (5)$$
where the parameters $v$ and $f$ represent respectively a variable (proportional) transaction cost and a fixed transaction cost, while $I(\cdot)$ is the indicator function. Just like Balduzzi and Lynch (1999) we assume that the fixed cost is proportional to the level of wealth to reduce the number of state variables (we will get back to this point later).

### 2.3 Maximization problem and decision rules

The investor’s decision problem is given by

$$\max_{\{\alpha_t\}_{t=1}^T,\{C_t\}_{t=1}^T} E_0 \sum_{t=1}^T \delta^{t-1} \frac{C_t^{1-\gamma}}{1-\gamma}$$

subject to

$$W_{t+1} = [\alpha_t(1-k_t)R_{t+1} + (1-\alpha_t)R^f](W_t - C_t)$$

with $k_t$ given by equations (4) and (5), and $R_{t+1}$ given by equations (1), (2), and (3).

We let $\{\alpha^i, C^i\}$ denote respectively the vectors containing the optimal portfolio rules and the optimal consumption allocations, under the specification given by model $i$ ($i = A, B$). The relevant state variables are $E_t(\sigma_{t+1})$ (which will denote by $\tilde{\sigma}_{t+1}$ from now on), $\tilde{\alpha}$ and $t$. The model is solved numerically and the details are given in the appendix.

### 2.4 Calibration

#### 2.4.1 Return process

This section presents the estimation results for the return processes given by equations (1) to (3), under the different specifications: $A$ and $B$. The return on the risky asset is taken to be the value weighted return on the NYSE, including dividends, while the return on the safe asset is given by the average return on 3-month Treasury Bills over the sample. Both series were obtained from CRSP. The sample starts in January 1926 and ends in December 1999. Each specification is estimated using the excess return data. The estimation results

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\(^2\)We do not impose short-sales constraints to keep the model more realistic. However, since we have CRRA preferences and a discrete-time model this could cause problems in the numerical solution. Details are given in the appendix.
are reported in table 1. The parameters $\theta_1$ and $\theta_2$ are quite large and strongly significant, therefore suggesting a strong case for volatility timing.

2.4.2 Preferences

We will consider different values for the risk aversion parameter ($\gamma$), from 2 to 10, and we set the discount rate equal to 0.96 in annual terms ($\delta = 0.96^{1/12}$).

2.4.3 Transaction Costs

Trading costs are usually decomposed into two major components: bid-ask spreads and brokerage fees. Total transaction costs for large stocks typically correspond to 1% of the transaction amount or slightly less (see, for example, Keim and Madhavan (1998)). Since our investor is trading a market-wide index we expect her to face even lower trading costs. Balduzzi and Lynch (1999) consider values of the proportional costs between 0.1% and 0.5%, and values of the fixed cost between 0% and 0.1%. Marquering and Verbeek (2000) ignore fixed costs but consider a similar range for the proportional costs, namely between 0.1% and 1%. Based on these numbers we will consider values of the proportional cost ($v$) up to 0.5%, and values of the fixed cost ($f$) up to 0.1%. This is probably a conservative measure since transaction costs have fallen significantly in recent years and, in addition, investors can also implement these strategies by trading in futures markets where they would face lower fees (Fleming, Kerby and Ostdiek (2003) estimate that trading costs in the futures market correspond to 0.01% of the transaction amount).
3 In-sample results

3.1 Model without frictions

We start by considering the case without transaction costs \((v = f = 0)\) and without parameter uncertainty, but both will be added later on.

3.1.1 Policy functions and hedging demands

Figure 1 plots the portfolio rules implied both by the GARCH(1,1) model \((\alpha_1^B)\), and the passive rule \((\alpha_1^A)\). Results are shown for the one-year investment horizon and for an investor with relative risk aversion \((\gamma)\) equal to 10. The qualitative properties of the policy functions remain unchanged by considering different values of \(\gamma\), or different investment horizons. The horizontal axis covers approximately 95% of the in-sample realizations of \(\tilde{\sigma}_t\), under the GARCH(1,1) model. Consistent with the results of Chako and Viceira (2000), volatility timing has a significant impact on the optimal portfolio allocation: when expected volatility falls significantly below its unconditional mean, the investor will double or even triple her exposure to stocks.

Next we measure the hedging demands implied by the GARCH(1,1) model. The period\(-t\) hedging demand, denoted by \(h_t^B\), is given by

\[
h_t^B(R_t) = \alpha_t^B(R_t) - \alpha_{T-1}^B(R_t) \tag{8}
\]

Table 2.1 reports the average hedging demand as a fraction of the average portfolio allocation:

\[
\bar{h}_t^B = \frac{\int h_t^B(R_t)dF(R_t)}{\int \alpha_t^B(R_t)dF(R_t)} \tag{9}
\]

where \(F(R_t)\) is the unconditional density function for \(R_t\), implied by equations (1)-(3) and the estimation results from section 2. Results are shown for different time horizons (1, 2, and 5 years) and different degrees of risk aversion (2, 5 and 10). The hedging demands are quite negligible, regardless of the degree of risk aversion.
3.1.2 Conditional Utility Gains

To evaluate the utility gains associated with each rule we start by computing certainty equivalent wealth levels. Given a decision rule and a return process, this measures the certain level of wealth that would give the investor the same expected utility. Since the policy rules are scale-independent (we have CRRA preferences) we can normalize initial (current) wealth to 1, so that the certainty equivalent actually measures a percentage increase.

In the context of a static model (for $t = T - 1$), if we let $W^{CE,i}$ denote the certainty equivalent for portfolio rule $i$, then:

$$\left(\frac{W^{CE,i}}{1 - \gamma}\right)^{1-\gamma} = E_{T-1} \left[ \left( (\alpha^i_{T-1} R_T + (1 - \alpha^i_{T-1}) R^f) \right)^{1-\gamma} \right] \Leftrightarrow$$

$$1 = E_{T-1} \left[ \left( \frac{(\alpha^i_{T-1} R_T + (1 - \alpha^i_{T-1}) R^f)}{W^{CE,i}} \right)^{1-\gamma} \right]$$

and naturally the certainty equivalent will be a function of the state variables. More precisely, we would write $W^{CE,B}(\tilde{\sigma}^*_t)$ to denote the certainty equivalent of portfolio rule $\alpha^B$ given $\tilde{\sigma} = \tilde{\sigma}^*_t$.

This concept is extended to a dynamic model by using the value function instead of the utility function. Define $V^A$ as the expected utility from following the decision rules $\{C^A_t\}_{t=1}^T$ and $\{\alpha^A_t\}_{t=1}^{T-1}$:

$$V^A \equiv E_0 \sum_{t=1}^T \delta^{t-1} C^A_t^{1-\gamma}$$

We can then define a constant certainty-equivalent consumption level $C^{CE,A}$ by solving the following equation

$$V^A = \sum_{t=1}^T \delta^{t-1} \frac{C^{CE,A} A^{1-\gamma}}{1 - \gamma}$$

Since $C^{CE,A}$ is riskless we can now compute the initial level of wealth required to finance it, which will be our certainty equivalent level of wealth:

$$W^{CE,A} = \sum_{t=1}^T \frac{C^{CE,A}}{(R^f)^t}$$

Now $W^{CE,A}$ is also a function of time, and therefore we will write $W^{CE,A}_t$ to denote the certainty equivalent level of wealth at time $t$. Likewise we can define $W^{CE,B}$ in the same manner.
Given the certainty equivalents corresponding to the two different decision rules, we use their ratio to compute the gain from one relative to the other. This measures the percentage increase in (riskless initial) wealth that would make the investor indifferent between the two strategies. Figure 2 plots the annualized certainty equivalent gain from volatility timing (using the previous notation, $W_{t}^{CE,B} (\tilde{\sigma}_t)/W_{t}^{CE,A} (\tilde{\sigma}_t)$), under the assumption that model B is indeed the correct data generating process. These results are again for the one year investment horizon ($t = T - 12$) case, and for different values of risk aversion (2, 5 and 10). By definition, the portfolio rule $\alpha^B$ must do at least as well as the passive rule ($\alpha^A$), for any realization of the state variable. Naturally, as volatility increases the investor is worse off in both cases. However, the volatility-timing rule allows her to reduce her portfolio allocation to stocks when this happens, and therefore utility does not fall as much as it would if she followed rule $\alpha^A$.

### 3.1.3 Unconditional Utility Gains

The results in figure 2 show that, if the GARCH(1,1) model is correct, then for extreme values of the state variable investors stand to gain a lot by rebalancing their portfolios. However, if such realizations are highly unlikely, this will only generate modest expected utility gains. Therefore, we now compute a more comprehensive metric, by integrating over $W_{t}^{CE,B}$ and $W_{t}^{CE,A}$ using the relevant distribution of stock returns. Denoting this unconditional expected utility gain by $\overline{W_{t}^{CE,B-A}}$, we have define

$$\overline{W_{t}^{CE,B-A}} = \frac{\int W_{t}^{CE,B}(R_t) dF(R_t)}{\int W_{t}^{CE,A}(R_t) dF(R_t)}$$  \hspace{1cm} (14)$$

where, as before, $F(R_t)$ is the unconditional density function for $R_t$, implied by equations (1)-(3) and the estimation results from section 2.\(^3\)

Table 2.2 reports the corresponding mean certainty equivalent gains ($\overline{W_{t}^{CE,B-A}}$), for different values of risk aversion (2, 5 and 10) and different investment horizons (1, 2, and 5 years). For the range of $\gamma$ considered in table 2.2, the certainty equivalent gain from timing volatility decreases with risk aversion. For example, for the one-year investment horizon

\(^3\)We perform these numerical integrations using Monte-Carlo simulation with 100,000 draws of each random variable (the same results were obtained with Gaussian quadrature).
case, they increase from 1.79% for the more risk averse investor ($\gamma = 10$) to 8.00% for the more risk tolerant one ($\gamma = 2$). At first sight this might seem counter-intuitive since more risk averse investors should benefit the most from a reduction in the volatility of returns. However table 2.2 does not report the utility gains generated by an exogenous reduction in volatility. Investors can only reduce volatility by decreasing their exposure to stocks, and therefore reducing the expected return on their portfolio. The optimal trade-off between these two differs across investors, depending on their risk aversion. We know that, as the investor’s risk aversion converges to infinity, she will completely avoid stocks and therefore the utility gains generated by all portfolio rules will converge to zero. So, for very large coefficients of risk aversion we expect a decreasing pattern for the certainty equivalents, consistent with table 2.2. On the other hand, if we let risk aversion converge to zero, the investor will not care about changes in volatility. As a result, the certainty equivalent gains from volatility timing should eventually become an increasing function of $\gamma$ as it approaches zero. In table 2.2 we find that, for the range of $\gamma$ that we consider, the decreasing pattern prevails.

### 3.2 Transaction costs

These strategies require portfolio rebalancing on a frequent (monthly) basis. Therefore, it is important to determine whether the utility gains are large enough to compensate the investor for the required transactions costs.\textsuperscript{4} Based on our previous discussion, we will consider values of the proportional cost ($v$) up to 0.5%, and values of the fixed cost ($f$) up to 0.1%. In all cases, the investor’s initial portfolio allocation is set equal to the optimal allocation for the case of no transaction costs and i.i.d. returns (decision rule $\alpha^A$ with $v = f = 0$).

Table 2.3 reports the certainty equivalent gains from following rule $\alpha^B$ (GARCH(1,1) model), for different values of the transaction cost parameters, and different degrees of risk aversion. With moderate transaction costs ($v = 0.5\%$ and $f = 0.0\%$, or $v = 0.25\%$ and $f = 0.1\%$) the certainty equivalent gains from the volatility timing are only marginally

\textsuperscript{4}Note that the benchmark rule also requires rebalancing since it sets a fixed share of wealth invested in stocks, rather than a fixed level of stock holdings, but in this case the transaction costs will clearly be quite small.
affected. If we consider the one-year investment horizon, for the less risk averse investors they are still above 6%, and even for $\gamma = 5$ they still exceed 2.5%.

### 3.3 Short-selling constraints

Most investors face limits on the amount of short selling that they are allowed to perform. Investors will never short the risky asset, as the expected equity returns is always positive, but the more risk tolerant ones will short the riskless asset when they expect volatility to be quite low. We measure the costs of potential short-selling restrictions by adding one additional constraint to the model in section 2:

$$\alpha_t \in [0, 1]$$

(15)

Naturally this is a very strong and counterfactual constraint, as it completely rules-out short selling of either asset, and therefore it should be considered as a limiting case.

Table 2.4 reports the mean certain equivalent gains for this case. As expected, investors with a moderate degree of risk aversion are not affected by the presence of the short-selling constraint. Even for $\gamma = 5$ the values in table 2.4 are almost identical to the ones in table 2.3. Only the less risk averse investors are significantly affected. Both because they would like to hold more extreme positions, and because their unconditional optimal portfolio allocation to stocks ($\alpha^A$) is already very close to 100%. Again it is important to remember that constraint (15) is a limiting case, as investors are typically allowed to implement some degree of short-selling. In any case it becomes clear that the more extreme utility gains are vulnerable to this assumption.

### 3.4 Parameter Uncertainty

The parameter estimates are subject to estimation error, and moreover the return processes might be mis-specified, which motivates the explicit incorporation of parameter uncertainty, for example in a Bayesian context.\footnote{See, among others, Cremers (2003), Lewellen and Shanken (2002), Avramov (2002), Barberis (2000), Tamayo (2000), Xia (2000), or Kandell and Stambaugh (1996) or, in settings without return predictability, Pástor and Stambaugh (2000) and Pástor (2000)} Given the presence of transaction costs, and the non-
trivial dimension of the state space, modelling parameter uncertainty and learning explicitly in a Bayesian approach is problematic from a computational perspective. As a result we will not take into account for learning, and we will only allow for parameter uncertainty in the most crucial parameter, which is $\theta_2$.

Once we take into account for parameter uncertainty, the expected utility gains is given by

$$\mathbb{EW}_{t}^{CE,B-A} = \int W_{t}^{CE,B-A}(\theta_2) dG(\theta_2)$$

where $G(\theta_2)$ is the posterior distribution of $\theta_2$, assuming a flat prior. We compute this integral using quadrature methods where each term is evaluated from:

$$W_{t}^{CE,B-A}(\theta_2) = \frac{\int W_{t}^{CE,B}(R_t; \theta_2) dF(R_t; \theta_2)}{\int W_{t}^{CE,A}(R_t; \theta_2) dF(R_t; \theta_2)}$$

and where $F(R_t; \theta_2)$ is the unconditional density function for $R_t$, for a given $\theta_2$. Naturally, when considering different values of $\theta_2$, we adjust the parameter $\theta_0$ so that the unconditional mean volatility remains unaffected.

The results are shown in table 2.5. The certainty equivalents are reduced, but not too much. Net of transaction costs, the investor with $\gamma = 5$ still has a certainty equivalent gain of 1.74%, and the less risk aversion investor, which cares less about parameter uncertainty, is still facing a gain of 4.7%.

4 Out-of-sample analysis

In the previous sections the utility gains are measured under the assumption that the structure of data generating process is known (even though the specific parameter values are uncertain). This naturally overstates the performance of these investment strategies. We can always increase the investor’s within sample expected utility by specifying a more complex data generating process for returns, but we might just be over-fitting the observed data. In this section we address this concern by studying the out-of-sample performance of the alternative strategies.

The two alternative specifications for the return process were estimated using the sample from January 1926 until December 1969, while the sample from January 1970 until December
1990 was used to perform the out-of-sample evaluation of the investment rules.\textsuperscript{6} The investor is not allowed to re-estimate the model at any given time during the out-of-sample period, as otherwise the optimal policy functions would have to take this into account, and there would be an additional hedging demand driven by learning (as in Xia (2001)).\textsuperscript{7} By not allowing the investor to re-estimate the parameters over time, and thus ignoring this additional source of learning, the policy functions are suboptimal relative to the more complete specification. As a result, the analysis in the paper actually understates the potential utility gains from volatility timing, and it is biased against finding positive results.\textsuperscript{8}

Table 3 shows the out-of-sample mean returns, standard deviations and the corresponding certainty equivalent gains for the different policy rules, and for the different values of the coefficient of relative risk aversion (respectively $\gamma = 2, 5$ and 10). The volatility timing strategy generates returns that have a slightly lower mean, but compensates for this by offering a significant reduction in volatility. The utility gains are much lower in an out-of-sample analysis but they are still positive and significant: 1.87, 1.69 and 1.45.

5 Conclusion

This paper measures the utility gains from short-run portfolio rebalancing, in the presence of time variation in return volatility. We use a GARCH process for returns and compute

\textsuperscript{6}We have abstained from using the 1990s bull market in the out-of-sample period as this could bias the results.

\textsuperscript{7}For tractability reasons it is not possible to add this feature to the model. This is already the first model to add estimation risk to a dynamic set-up with transaction costs (or transaction costs to a dynamic set-up with estimation risk), but this is only computationally feasible if we omit this extra hedging demand.

Johannes et al. (2003) perform such a study but they “compensate” by solving a mean-variance optimization problem instead, and thus ignoring the “standard” hedging demands. In addition they also ignore transaction costs. In Xia (2001) the model is dynamic but again there are no transaction costs. So those authors have to simplify the model along other dimensions to deal with this additional hedging demand.

\textsuperscript{8}We have also considered an intermediate case in which we ignore the hedging demand from learning and yet allow the investor to re-estimate (every year) the return process in the out-of-sample evaluation. The utility gains were not significantly affected. Since from the previous discussion this is not theoretically consistent, those results have not been included in the paper.
the implied optimal trading strategy. Our strategy is optimal given the GARCH process and the paper does not attempt to contribute to the debate concerning the best model for stock price dynamics. We acknowledge that there are alternative plausible representations and they would yield different policy functions. We have settled on a more modest goal: compute the utility gains obtained by following a “relatively plausible” model of volatility timing. This analysis takes into account for realistic frictions, namely transaction costs and short-selling constraints, and parameter uncertainty. Overall there is a strong case in favor of volatility timing. The corresponding utility gains are robust to frictions and parameter uncertainty, and they are still present in an out-of-sample analysis.
Appendix: Numerical Solution

The relevant state variables for the different portfolio rules are:

\[ \alpha^A = \alpha^A(\hat{\alpha}, t) \]
\[ \alpha^B = \alpha^B(E_t(\sigma_{t+1}), \hat{\alpha}, t) \]

Since we assume that the fixed cost is proportional to the level of wealth we do not need to consider \( W \) as an additional state variable. Naturally, the corresponding consumption functions are dependent on the same state variables.

The model is solved numerically, using the Bellman equation and backward induction to obtain the decision rules and the value functions at each point in time. We discretize the state-space using (100) equally spaced grid points for each of the two continuous state variables \( (E_t(\sigma_{t+1}), \hat{\alpha}) \). In the case of the exogenous stochastic state variable \( (E_t(\sigma_{t+1})) \) the upper and lower bounds on the state-space were chosen so that the corresponding interval includes all of the in-sample realizations. The upper and lower bounds on the grid for \( \hat{\alpha} \) were confronted with the optimal policy functions, to determine whether they should be increased even further. As a further robustness check, we solved the model with an expanded version of the state space and with finer grids, and the results remained unchanged. We use Gaussian quadrature to compute all the relevant expectations, and combine a grid search algorithm with a bi-section algorithm to solve for the optimal decision rules. Including a grid search component in the algorithm is computational painful but always advisable in the presence of fixed transaction costs, as these generate discontinuities in the decision rules.

Since we do not impose short-sales constraints and we have both CRRA preferences and a discrete-time model this could cause problems: the limit on any short positions is determined by the lower bound on the grid for the realizations of the equity return. Therefore, we have experimented with different lower bounds to check that the results are not particularly sensitive to this choice.
References


The return on the risky asset is taken to be the value weighted return on the NYSE, including dividends, while the return on the safe asset is given by the return on 3-month Treasury Bills. Both series were obtained from CRSP. The sample starts in January 1926 and ends in December 1999. Each specification is estimated using the excess return data, and the return on the safe asset is set equal to the mean return on the 3-month T-Bill. Standard errors are reported below in parenthesis.

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Table 2.1 – Hedging Demands for the GARCH(1,1) Model

Mean hedging demand (in terms of share invested in stocks), as a fraction of the mean overall share invested in stocks, from using the GARCH(1,1) model to forecast future stock returns, assuming that this is indeed the correct data generating process, and without transaction costs, parameter uncertainty or any other frictions. Results are shown for different investment horizons and different coefficients of relative risk aversion.

<table>
<thead>
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<th>2 years</th>
<th>5 years</th>
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<td>2</td>
<td>0.23%</td>
<td>0.31%</td>
<td>0.45%</td>
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<tr>
<td>5</td>
<td>0.40%</td>
<td>0.55%</td>
<td>0.73%</td>
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<tr>
<td>10</td>
<td>0.49%</td>
<td>0.61%</td>
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Table 2.2 – Mean Certainty Equivalent Gains for the GARCH(1,1) Model

Mean certainty equivalent gains (measured as a percentage increase in initial wealth) from using the GARCH(1,1) model to forecast future stock returns, assuming that this is indeed the correct data generating process, and without transaction costs, parameter uncertainty or any other frictions. Results are shown for different investment horizons and different coefficients of relative risk aversion.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>1 year</th>
<th>2 years</th>
<th>5 years</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8.00%</td>
<td>7.72%</td>
<td>5.29%</td>
</tr>
<tr>
<td>5</td>
<td>3.22%</td>
<td>3.17%</td>
<td>2.57%</td>
</tr>
<tr>
<td>10</td>
<td>1.79%</td>
<td>1.78%</td>
<td>1.35%</td>
</tr>
</tbody>
</table>
Table 2.3 – Mean Certainty Equivalent Gain for the GARCH(1,1) Model
(In the presence of transaction costs)

Mean certainty equivalent gains from using the GARCH(1,1) model to forecast future stock returns, assuming that this is indeed the correct data generating process. Results are shown for different investment horizons, different coefficients of relative risk aversion, and different values for the proportional transaction costs \( v \) and for the fixed transaction costs \( f \) (both are reported in basis points).

\[
\begin{array}{cccc|cccc}
\text{1-year horizon} & & & & \text{5-year horizon} & & & \\
\hline
v & f & \gamma = 2 & \gamma = 5 & \gamma = 10 & \gamma = 2 & \gamma = 5 & \gamma = 10 \\
\hline
0 & 0 & 8.00\% & 3.22\% & 1.79\% & 5.92\% & 2.57\% & 1.35\% \\
25 & 0 & 7.64\% & 3.19\% & 1.75\% & 5.71\% & 2.51\% & 1.19\% \\
50 & 0 & 7.29\% & 2.99\% & 1.67\% & 5.62\% & 2.39\% & 1.15\% \\
0 & 10 & 6.33\% & 2.63\% & 1.55\% & 5.46\% & 2.34\% & 1.06\% \\
25 & 10 & 6.21\% & 2.51\% & 1.43\% & 5.33\% & 2.27\% & 1.04\% \\
\end{array}
\]

Table 2.4 - Mean Certainty Equivalent Gain for the GARCH(1,1) Model
(In the presence of transaction costs and short-selling constraints)

Mean certainty equivalent gains from using the GARCH(1,1) model to forecast future stock returns, assuming that this is indeed the correct data generating process, when the investor faces short-selling constraints on both assets. Results are shown for different investment horizons, different coefficients of relative risk aversion, and different values for the proportional transaction costs \( v \) and for the fixed transaction costs \( f \) (both are reported in basis points).

\[
\begin{array}{cccc|cccc}
\text{1-year horizon} & & & & \text{5-year horizon} & & & \\
\hline
v & f & \gamma = 2 & \gamma = 5 & \gamma = 10 & \gamma = 2 & \gamma = 5 & \gamma = 10 \\
\hline
0 & 0 & 5.26\% & 3.22\% & 1.79\% & 4.00\% & 2.55\% & 1.35\% \\
25 & 0 & 4.66\% & 3.18\% & 1.75\% & 3.73\% & 2.48\% & 1.19\% \\
25 & 10 & 3.94\% & 2.50\% & 1.43\% & 3.39\% & 2.21\% & 1.04\% \\
\end{array}
\]
Table 2.5 - Mean Certainty Equivalent Gain for the GARCH(1,1) Model
(With parameter uncertainty)

Mean certainty equivalent gains from using the GARCH(1,1) model to forecast future stock returns and taking into account for parameter uncertainty in the volatility persistence parameter, $\theta_1$ (while assuming a flat prior). Results are shown for a one-year investment horizon, different values of transaction costs (proportional transaction costs ($v$) and fixed transaction costs ($f$), both reported in basis points), and different coefficients of relative risk aversion.

<table>
<thead>
<tr>
<th>$v$</th>
<th>$f$</th>
<th>$\gamma = 2$</th>
<th>$\gamma = 5$</th>
<th>$\gamma = 10$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>6.17%</td>
<td>2.24%</td>
<td>1.04%</td>
</tr>
<tr>
<td>25</td>
<td>10</td>
<td>4.79%</td>
<td>1.74%</td>
<td>0.83%</td>
</tr>
</tbody>
</table>

Table 3 - Out-of-sample performance of the different rules, with proportional transaction costs equal to 25 basis points and fixed transaction costs equal to 10 basis points, and different coefficients of relative risk aversion (out-of-sample horizon: Jan 1970-Dec 1990)

<table>
<thead>
<tr>
<th>$\alpha^A$</th>
<th>$\alpha^B$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma = 2$</td>
<td>$\gamma = 2$</td>
</tr>
<tr>
<td>$\gamma = 5$</td>
<td>$\gamma = 5$</td>
</tr>
<tr>
<td>$\gamma = 10$</td>
<td>$\gamma = 10$</td>
</tr>
<tr>
<td>Mean</td>
<td>11.65%</td>
</tr>
<tr>
<td></td>
<td>9.62%</td>
</tr>
<tr>
<td></td>
<td>8.95%</td>
</tr>
<tr>
<td>Std. Dev.</td>
<td>14.81%</td>
</tr>
<tr>
<td></td>
<td>5.95%</td>
</tr>
<tr>
<td></td>
<td>3.04%</td>
</tr>
<tr>
<td>C.E. Gain</td>
<td>N.A.</td>
</tr>
<tr>
<td></td>
<td>N.A.</td>
</tr>
<tr>
<td></td>
<td>N.A.</td>
</tr>
</tbody>
</table>
Figure 1 plots two alternative portfolio rules (share invested in stocks) for a CRRA investor with a coefficient of relative risk aversion of 10, which is able to rebalance his/her portfolio on a monthly basis. One portfolio rule assumes that he/she uses a GARCH(1,1) model to forecast stock return volatility, while the other is derived under the assumption of i.i.d. stock returns. The horizontal axis covers approximately 95% of the in-sample realizations of the standard deviation of stock returns. The vertical lines mark a two standard deviation band around the unconditional mean (of the variance).
Figure 2 plots the certainty equivalent gain from using a GARCH(1,1) model to forecast future stock returns, relative to assuming i.i.d. returns. Results are shown for CRRA investors with different coefficients of relative risk aversion (2, 5 and 10), which are able to rebalance his/her portfolio on a monthly basis. The certainty equivalent gain is measured as a percentage increase in initial wealth, and the investment horizon is one year. The horizontal axis covers approximately 95% of the in-sample realizations of the standard deviation of stock returns. The vertical lines mark a two standard deviation band around the unconditional variance.