SUPPLEMENT TO:
TEMPORARY versus PERMANENT SHOCKS:
EXPLAINING CORPORATE FINANCIAL POLICIES

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First version: September 2006
This version: October 2009

This Supplement is not for publication
S.I Multiple transitory shocks

In a motivating example of Section I.B, only one transitory shock could hit firm’s cash flow. Assume now that the economy is subject to infinite number of shocks in expectation, even though for simplicity we continue to assume that at any point in time only one temporary shock may effect earnings and the level of the permanent cash flow is fixed. Specifically, we assume that the state of the economy follows a 2-state continuous-time Markov chain. In the first state the permanent cash flow is the only component of the cash flow, while in the second state both shock types are present. The probability per unit of time of switching from permanent-only state to both-shock state is $\lambda_s$ and the probability of switching from both-shock state to permanent-only state is $\lambda_r$. Thus, on average, a fraction $\lambda_r/(\lambda_s + \lambda_r)$ of time firm’s cash flow is affected by transitory shocks.

As before, $T_s$ denotes any date when the temporary shock hits and $T_r$ – any date when the shock mean reverts. At any $T_s$, the magnitude of the shock is drawn randomly from the same double exponential distribution function (7). In other words, the expected magnitude of the new shock is the same but its actual realization upon arrival is generally different. The firm expects infinite number of shocks to arrive and get reversed in the future.\(^1\)

To value any security, note that the economy at any time is in one of two states depending on whether current earnings are affected by the transitory shock. For the after-tax value of the unlevered firm, $V$, denote $V(\delta)$ to be the value in the no-shock state and $V(\delta, \epsilon)$ to be the value when shock of size $\epsilon$ is present. Note that $V(\delta, 0)$ and $V(\delta)$ represent values in two different states of the world and generally are not equal to each other.\(^2\)

In both states these values can be written intuitively, using recursive formulation, as

\[
V(\delta, \epsilon) = \begin{cases} 
\mathbb{E}\left[(1 - \tau_\epsilon) \int_0^{T_r} (\delta + \epsilon)e^{-rt}dt + V(\delta)e^{-rT_r}\right], & \epsilon > \epsilon_A, \\
0, & \epsilon \leq \epsilon_A,
\end{cases}
\]

\[
V(\delta) = \mathbb{E}\left[(1 - \tau_\epsilon) \int_0^{T_s} \delta e^{-rt}dt + V(\delta, \epsilon)e^{-rT_s}\right].
\]

(S.1)

As before, $\epsilon_A$ is the abandonment level of the shock, below which firmholders find it in their interest to liquidate the project to mitigate losses. In the formula for $V(\delta, \epsilon)$ above, the first term is

\(^1\)This is one of a number of potential “shock structures” that we could consider. For example, shocks can be also reversed by the arrival of the new shock. Alternatively, shocks can be reversed only by new shocks. While all these models would lead to similar implications, they do not in general allow disentangling two quantities with vastly different intuition: waiting times of shocks and their mean-reversion speed. In our model $\lambda_s$ is responsible for the first effect and $\lambda_r$ reflects the speed of mean-reversion. This distinction is helpful in understanding economic intuition.

\(^2\)Since shock of size zero happens on a probability measure of zero, we do not consider informational aspects of the shock existence, even though managers most probably would not know for certain whether the firm is in the shock or no-shock state, when the realization of the shock is zero.
the present value of cash flows until the shock reverts to the long-run mean and the second term is the firm value at the moment of shock’s mean reversion discounted to take into account future mean reversion.

To simplify the exposition, we use the same canonic security $A(\delta, g(\epsilon_T), \lambda, b)$ that we introduced in the motivating example. The only important distinction is that its value now depends on whether transitory shock is present. For example, we can re-write the unlevered value in the absence of the transitory shock as

$$V(\delta) = \mathbb{E} \left[ (1 - \tau_{\epsilon}) \int_0^{T_s} \delta e^{-rt} dt + e^{-rT_s} A(\delta, V(\delta, \epsilon), \lambda_{\epsilon}, \epsilon_A) \right]. \quad (S.2)$$

Since this value is larger than the value of perpetuity paying the permanent cash flow, once again firmholders possess a real option in the presence of transitory shocks.

Appendix B derives the closed-form expressions for the values of this and all other securities introduced below. To find $\epsilon_A$, we solve the following abandonment condition:

$$V(\delta, \epsilon_A) = 0. \quad (S.3)$$

Turning now to the value of equity, and assuming for the moment that the coupon level is given, the value of equity in two states can be written as follows:

$$E(\delta, \epsilon) = \begin{cases} \mathbb{E} \left[ (1 - \tau_{\epsilon}) \int_0^{T_s} (\delta + \epsilon - c)e^{-rt} dt + E(\delta)e^{-rT_s} \right] & \epsilon > \epsilon_B, \\ 0, & \epsilon \leq \epsilon_B, \end{cases}$$

$$E(\delta) = \mathbb{E} \left[ (1 - \tau_{\epsilon}) \int_0^{T_s} (\delta - c)e^{-rt} dt + E(\delta)e^{-rT_s} \right]. \quad (S.4)$$

In both expressions above, the first term is the equity payout in the current state. The equity-holders will choose the default threshold level of $\epsilon_B$ by solving for the negative root of the following value-matching condition:

$$E(\delta, \epsilon_B) = 0. \quad (S.5)$$

The value of debt consists of the sum of two components, the present value of coupon payments until default and the liquidating payment in default. The first component, which we denote in two
states by $C(\delta, \epsilon)$ and $C(\delta)$, respectively, can be written as:

$$
C(\delta, \epsilon) = \begin{cases} 
    \mathbb{E} \left[ (1 - \tau_i) \int_0^{T_s} ce^{-rt} dt + C(\delta)e^{-rT_r} \right], & \epsilon > \epsilon_B, \\
    0, & \epsilon \leq \epsilon_B,
\end{cases}
$$

$$
C(\delta) = \mathbb{E} \left[ (1 - \tau_i) \int_0^{T_s} ce^{-rt} dt + C(\delta, \epsilon)e^{-rT_s} \right].
$$

(S.6)

It is easy to see that the value of debt before bankruptcy $C(\delta, \epsilon)$ does not depend on the realization of the temporary shock, as long as the shock does not lead to default.

The expected continuation value of liquidating payments, which we denote $L(\delta, \epsilon)$ and $L(\delta)$, respectively, can analogously be written as:

$$
L(\delta, \epsilon) = \begin{cases} 
    \mathbb{E} \left[ L(\delta)e^{-rT_r} \right], & \epsilon > \epsilon_B, \\
    V((1 - \alpha)\delta, \epsilon), & \epsilon \leq \epsilon_B,
\end{cases}
$$

$$
L(\delta) = \mathbb{E} \left[ L(\delta, \epsilon)e^{-rT_s} \right].
$$

(S.7)

To gauge the intuition behind the expression above note that when the realized shock does not lead to default, there are no liquidating payments until the next shock.

The optimal coupon choice follows from the optimization problem

$$
\max_c D(\delta) + E(\delta),
$$

(S.8)
given expressions for $\epsilon_B(c)$ and $\epsilon_A(c)$ obtained above. In the above expression, we explicitly assume that at the time of optimal coupon choice there are no transitory shocks, although the results are without loss of generality.

Figure 1 shows the comparative statics of optimal coupon and leverage ratio implied by the problem above and, for comparison, by the single-shock example of Section I.B. As shocks arrive more frequently (the arrival intensity, $\lambda_s$, is larger), the firm finds financial flexibility more valuable and reduces optimal coupon. As the magnitude of shocks is smaller (i.e., the parameter $\gamma$ is larger) or shocks expect to reverse sooner ($\lambda_r$ is larger), equityholders increasingly behave as if there were no jumps at all and choose the optimal coupon equal to the permanent cash flow level to realize full benefits of tax benefits. On the other hand, smaller values of $\gamma$ (or $\lambda_r$) lead to shocks of larger present value via a larger magnitude (or longer influence). In particular, as $\gamma$ goes to zero, the variance of jumps is so high that shocks of substantial size, both positive and negative, arrive almost certainly, magnifying the limited liability option of equityholders and leading to higher optimal coupon levels. This results in behavior that we call “finance substitution” in Section I.B. In between these two extremes, both optimal coupon and leverage are substantially lower than in the absence of temporary shocks. Figure 1 also shows that, the finance substitution scenario aside, the expectation
of future shocks makes firms behave more conservatively relative to the single-shock scenario. For example, when \( \lambda_s = 1.2 \) (see Section III.A), the optimal leverage ratio is lower by almost 20%.

### Appendix S.A Proofs: Model with multiple transitory shocks

**Proposition S.1** The unlevered before-tax value of the firm in two states (with and without temporary shock), \( V(\delta, \epsilon) \) and \( V(\delta) \), is given by:

\[
V(\delta) = \frac{\delta + \frac{r + \lambda_s}{r + \lambda_s} (\delta A(1, \lambda_s, \epsilon_A) + A(\epsilon, \lambda_s, \epsilon_A))}{(r + \lambda_s) \left[ 1 - \frac{\lambda_s}{r + \lambda_s} A(1, \lambda_s, \epsilon_A) \right]} ; \quad \text{(S.A1)}
\]

\[
V(\delta, \epsilon) = \begin{cases} 
\frac{\delta + \lambda_s}{r + \lambda_s} V(\delta), & \epsilon \geq \epsilon_A, \\
0, & \epsilon_A > \epsilon. 
\end{cases} \quad \text{(S.A2)}
\]

**Proof.** From

\[
V(\delta, \epsilon | \epsilon \geq \epsilon_A) = \mathbb{E} \left[ \int_0^{T_s} (\delta + \epsilon)e^{-rt} dt + V(\delta) e^{-rT_s} \right]
\]

and

\[
V(\delta) = \mathbb{E} \left[ \int_0^{T_s} \delta e^{-rt} dt + V(\delta, \epsilon | \epsilon \geq \epsilon_A) e^{-rT_s} \right],
\]

where \( \epsilon_A \) is the negative temporary shock realization at which the present value of unlevered firm’s cash flows...
becomes zero, we obtain the recursive formulation

\[
V(\delta, \epsilon | \epsilon \geq \epsilon_A) = \frac{\delta + \epsilon}{r + \lambda_r} + \frac{\lambda_r}{r + \lambda_r} V(\delta);
\]

\[
V(\delta) = \frac{\delta}{r + \lambda_s} + \frac{\lambda_s}{r + \lambda_s} \int_{\epsilon_A}^{\infty} V(\delta, \epsilon) dF(\epsilon).
\] (S.A3)

To exclude the integral term from the second equation above, we treat \(\epsilon\) as a random variable in the first equation and take expectations of both parts of the equation with respect to \(\epsilon\).

\[
\int_{\epsilon_A}^{\infty} V(\delta, \epsilon) dF(\epsilon) = \frac{r + \lambda_s}{\lambda_s} \left[ A(1, \lambda_s, \epsilon_A) \left( \frac{\delta}{r + \lambda_r} + V(\delta) \frac{\lambda_r}{r + \lambda_r} \right) + \frac{A(\epsilon, \lambda_s, \epsilon_A)}{r + \lambda_r} \right].
\] (S.A4)

Now, inserting this into the second equation of (S.A3), we obtain

\[
V(\delta) = \frac{\delta + \frac{r + \lambda_s}{r + \lambda_r} (A(1, \lambda_s, \epsilon) + \lambda_s) \int_{\epsilon_A}^{\infty} V(\delta, \epsilon) dF(\epsilon)}{(r + \lambda_s) \left( 1 - \frac{\lambda_s}{r + \lambda_r} A(1, \lambda_s, \epsilon_A) \right)}.
\] (S.A5)

The second formula of the statement follows directly from (S.A3) and (S.A5).

**Proposition S.2** The value of equity in two states (with and without temporary shock), \(E(\delta, \epsilon)\) and \(E(\delta, \epsilon)\), is given by

\[
E(\delta) = \left( 1 - \tau e \right) \frac{\delta - c}{r + \lambda_r} + \frac{\lambda_r}{r + \lambda_r} (A(1, \lambda_s, \epsilon_B) + A(\epsilon, \lambda_s, \epsilon_B)) \frac{\epsilon - c}{r + \lambda_r} + \lambda_s \int_{\epsilon_B}^{\infty} E(\delta, \epsilon) dF(\epsilon);
\]

\[
E(\delta, \epsilon) = \begin{cases} 
\left( 1 - \tau e \right) \frac{\delta - c}{r + \lambda_r} + \frac{\lambda_r}{r + \lambda_r} E(\delta), & \epsilon \geq \epsilon_B, \\
0, & \epsilon_B > \epsilon.
\end{cases}
\] (S.A7)

**Proof.** From

\[
E(\delta, \epsilon | \epsilon \geq \epsilon_B) = \mathbb{E} \left[ \int_0^{T_r} (1 - \tau e)(\delta + \epsilon - c)e^{-rt} dt + E(\delta)e^{-rT_r} \right] \quad \text{and},
\]

\[
E(\delta) = \mathbb{E} \left[ \int_0^{T_r} (1 - \tau e)(\delta - c)e^{-rt} dt + E(\delta, \epsilon | \epsilon \geq \epsilon_B)e^{-rT_r} \right],
\] (S.A8)

where \(\epsilon_B\) is the negative temporary shock realization at which the value of equity becomes zero, we obtain the recursive formulation

\[
E(\delta, \epsilon | \epsilon \geq \epsilon_B) = \left( 1 - \tau e \right) \frac{\delta + \epsilon - c}{r + \lambda_r} + \frac{\lambda_r}{r + \lambda_r} E(\delta);
\]

\[
E(\delta) = \left( 1 - \tau e \right) \frac{\delta - c}{r + \lambda_s} + \frac{\lambda_s}{r + \lambda_s} \int_{\epsilon_B}^{\infty} E(\delta, \epsilon) dF(\epsilon).
\] (S.A9)

Using the same technique as in Proposition S.1, we treat \(\epsilon\) as a random variable in the first equation and take
expectations of both parts of the equation with respect to \( \epsilon \).

\[
\int_{\epsilon_B}^{\infty} E(\delta, \epsilon) dF(\epsilon) = \frac{r + \lambda_s}{\lambda_s} \left[ (1 - \tau_c) \delta - c \right] A(1, \lambda_s, \epsilon_B) + \frac{\lambda_r}{r + \lambda_r} E(\delta) A(1, \lambda_s, \epsilon_B) + (1 - \tau_c) \frac{1}{r + \lambda_r} A(\epsilon_T, \lambda_s, \epsilon_B).
\]

(S.A10)

Inserting this into the second equation of (S.A9), we obtain

\[
E(\delta) = (1 - \tau_c) \left( \delta - c + \frac{r + \lambda_s}{r + \lambda_r} (A(1, \lambda_s, \epsilon_B)(\delta - c) + A(\epsilon_T, \lambda_s, \epsilon_B)) \right) \left( r + \lambda_s \right) \left( 1 - \frac{\lambda_r}{r + \lambda_r} A(1, \lambda_s, \epsilon_B) \right).
\]

(S.A11)

The value of the equity in presence of the temporary shock is then the combination of (S.A9) and (S.A11).

**Proposition S.3** The value of debt in two states (with and without temporary shock), \( D(\delta, \epsilon) \) and \( \bar{D}(\delta) \), is given by

\[
D(\delta) = (1 - \tau_c) \left( \frac{c + A(1, \lambda_s, \epsilon_B) e^{\frac{r + \lambda_s}{r + \lambda_r}}}{r + \lambda_s} \left( 1 - \frac{\lambda_r}{r + \lambda_r} A(1, \lambda_s, \epsilon_B) \right) \right) + (1 - \tau_c) \left( A(1, \lambda_s, \epsilon_B) - A(1, \lambda_s, \epsilon_B) \right) \left( \frac{1 - (1 - \alpha)\delta}{1 - \frac{\lambda_r}{r + \lambda_r} A(1, \lambda_s, \epsilon_B)} \right) + (1 - \tau_c) \left( \delta - c \right) A(1, \lambda_s, \epsilon_B) + (1 - \tau_c) \frac{1}{r + \lambda_r} A(\epsilon_T, \lambda_s, \epsilon_B).
\]

(S.A12)

\[
D(\delta, \epsilon) = \begin{cases} 
(1 - \tau_c) \frac{c + A(1, \lambda_s, \epsilon_B) e^{\frac{r + \lambda_s}{r + \lambda_r}}}{r + \lambda_s} D(\delta), & \epsilon \geq \epsilon_B, \\
V (1 - (1 - \alpha)\delta, \epsilon), & \epsilon_B > \epsilon \geq \epsilon_L, \\
0, & \epsilon > \epsilon_L.
\end{cases}
\]

(S.A13)

**Proof.** To calculate the value of debt, we will first separately find expressions for continuation values of debt before and after bankruptcy.

Continuation values of debt before bankruptcy

\[
C(\delta, \epsilon | \epsilon \geq \epsilon_B) = \mathbb{E} \left[ \int_0^{T_r} (1 - \tau_i) c e^{-rt} dt + C(\delta) e^{-rT_r} \right]
\]

and

\[
C(\delta) = \mathbb{E} \left[ \int_0^{T_r} (1 - \tau_i) c e^{-rt} dt + C(\delta, \epsilon | \epsilon \geq \epsilon_B) e^{-rT_r} \right].
\]

(S.A14)

give us the system of recursive equations

\[
C(\delta, \epsilon | \epsilon \geq \epsilon_B) = (1 - \tau_c) \frac{c}{r + \lambda_r} + \frac{\lambda_r}{r + \lambda_r} C(\delta);
\]

\[
C(\delta) = (1 - \tau_c) \frac{c}{r + \lambda_s} + \frac{\lambda_s}{r + \lambda_s} \int_{\epsilon_B}^{\infty} C(\delta, \epsilon) dF(\epsilon).
\]

(S.A15)

From the first equation,

\[
\int_{\epsilon_B}^{\infty} C(\delta, \epsilon) dF(\epsilon) = \frac{r + \lambda_s}{\lambda_s} A(1, \lambda_s, \epsilon_B) \left( (1 - \tau_c) \frac{c}{r + \lambda_r} + \frac{\lambda_r}{r + \lambda_r} C(\delta) \right),
\]

(S.A16)
and, consequently,
\[ C(\delta) = (1 - \tau_i) \frac{c + \frac{r + \lambda s}{r + \lambda r} cA(1, \lambda s, \epsilon_B)}{(r + \lambda s) \left(1 - \frac{\lambda}{r + \lambda r} A(1, \lambda s, \epsilon_B)\right)}. \]  
(S.A17)

Continuation values of debt after bankruptcy
\[
L(\delta, \epsilon | \epsilon_B \geq \epsilon \geq \epsilon_L) = V((1 - \alpha)\delta, \epsilon),
\]
\[
L(\delta, \epsilon | \epsilon \geq \epsilon_B) = \mathbb{E}\left[L(\delta)e^{-rT}\right]
\]
and
\[
L(\delta) = (1 - \tau e)\lambda s r + \lambda s \int_{\epsilon_L}^{\epsilon_B} V((1 - \alpha)\delta, \epsilon)dF(\epsilon)
\]
\[
\int_{\epsilon_B}^{\infty} L(\delta, \epsilon)dF(\epsilon).
\]  
(S.A18)

where liquidation threshold \( \epsilon_L \) is the solution of the equation \( V((1 - \alpha)\delta, \epsilon_L) = 0 \), give the system of equations
\[
L(\delta, \epsilon | \epsilon \geq \epsilon_B) = \lambda r r + \lambda r L(\delta);
\]
\[
L(\delta) = (1 - \tau e)\frac{\lambda s}{r + \lambda s} \int_{\epsilon_L}^{\epsilon_B} V((1 - \alpha)\delta, \epsilon)dF(\epsilon) + \frac{\lambda s}{r + \lambda s} \int_{\epsilon_B}^{\infty} L(\delta, \epsilon)dF(\epsilon).
\]  
(S.A19)

The transformation of equations gives
\[
L(\delta) = (1 - \tau e)\frac{\lambda s}{r + \lambda s} \int_{\epsilon_L}^{\epsilon_B} V((1 - \alpha)\delta, \epsilon)dF(\epsilon)
\]
\[
\frac{1}{(r + \lambda s) \left(1 - \frac{\lambda}{r + \lambda r} A(1, \lambda s, \epsilon_B)\right)},
\]  
(S.A20)

and the integral above can be calculated explicitly in terms of securities \( A(\cdot) \), from expressions (S.A3) and (S.A5). Finally,
\[
D(\delta) = (1 - \tau_i) \frac{c + \frac{r + \lambda s}{r + \lambda r} cA(1, \lambda s, \epsilon_B)}{(r + \lambda s) \left(1 - \frac{\lambda}{r + \lambda r} A(1, \lambda s, \epsilon_B)\right)}
\]
\[
+ (1 - \tau e)\lambda s \int_{\epsilon_L}^{\epsilon_B} V((1 - \alpha)\delta, \epsilon)dF(\epsilon)
\]
\[
\frac{1}{(r + \lambda s) \left(1 - \frac{\lambda}{r + \lambda r} A(1, \lambda s, \epsilon_B)\right)},
\]  
(S.A21)

and summing first equations of (S.A15) and (S.A19),
\[
D(\delta, \epsilon) = (1 - \tau_i) \frac{c}{r + \lambda r} + \frac{\lambda r}{r + \lambda r} D(\delta).
\]  
(S.A22)

\[\blacksquare\]

**Appendix S.B  Extended details of numerical procedures**

**S.B.1 Approximate liquidation and default boundaries**

The model with both permanent and multiple transitory shocks in Section II.G allows to characterize a solution to optimal liquidation and default boundaries as non-linear functions of transitory shock size and coupon. For example, the functional form of the default boundary \( \delta_B(c, \epsilon_i) \) is obtained by applying the standard smooth-
pasting condition (49) to the Hamilton-Jacobi-Bellman (HJB) equation for equity value (48):

\[
\sum_{i=0}^{\infty} \left( a_{s,i}^+ \delta_{B,t}^+ \log^i \delta_{B,t}^+ + a_{s,i}^- \delta_{B,t}^- \log^i \delta_{B,t}^- + b_{s,i}^+ \delta_{B,t}^+ \log^i \delta_{B,t}^+ + b_{s,i}^- \delta_{B,t}^- \log^i \delta_{B,t}^- \right) + c_{i} \delta_{B,t} + d_{s} + c_{e} \epsilon_{t} = 0,
\]

(S.B1)

if the transitory shock is present and

\[
\sum_{i=0}^{\infty} \left( a_{r,i}^+ \delta_{B,t}^+ \log^i \delta_{B,t}^+ + a_{r,i}^- \delta_{B,t}^- \log^i \delta_{B,t}^- + b_{r,i}^+ \delta_{B,t}^+ \log^i \delta_{B,t}^+ + b_{r,i}^- \delta_{B,t}^- \log^i \delta_{B,t}^- \right) + c_{i} \delta_{B,t} + d_{r} = 0,
\]

(S.B2)

if the transitory shock is absent. A similar functional form is obtained for the liquidation boundary by solving the HJB equation (46)–(47) for \(V((1-\alpha)\delta_{t},\epsilon_{t})\).

However, the closed form representation of these functions in terms of model primitives does not exist: \(\{a_{s,i}^+, a_{s,i}^-, a_{r,i}^+, a_{r,i}^-, b_{s,i}^+, b_{s,i}^-, b_{r,i}^+, b_{r,i}^-, c_{s}, d_{s}, d_{r}, c_{e}\}\) are unknown functions of the coupon \(c\) and the distribution of shock size. To solve for optimal liquidation and default boundaries approximately and then control for approximation quality, we develop the following numerical procedures.

Consider the following functional form for both liquidation and default boundaries:

\[
H_{s}^{-}(\epsilon_{t}, \delta_{X,t}) = b_{s,0}^+ \tilde{\delta}_{X,t}^+ + c_{s}^+ \delta_{X,t} + d_{s}^+ + (\epsilon_{t} - \bar{\epsilon}) = 0, \quad \epsilon < \bar{\epsilon},
\]

\[
H_{s}^{+}(\epsilon_{t}, \delta_{X,t}) = a_{s,0}^- \tilde{\delta}_{X,t}^- + c_{s}^- \delta_{X,t} + d_{s}^- + (\epsilon_{t} - \bar{\epsilon}) = 0, \quad \epsilon \geq \bar{\epsilon},
\]

\[
H_{r}(\delta_{X,t}) = \delta_{X,t} - d_{r} = 0,
\]

(S.B3)

where \(\delta_{X,t}\) is either liquidation (\(\delta_{L,t}\)) or default (\(\delta_{D,t}\)) boundary, and the choice of the switching point \(\bar{\epsilon}\) is explained below. First two equations define the approximate boundary in presence of the transitory shock and therefore depend on the realization of temporary cash flows. The last equation defines the exact boundary in absence of the transitory shock which for the given coupon is a constant. We maximize the values of post-bankruptcy debt and levered equity over the set of parameters \(\Omega_{X} = \{b_{s,0}^+, c_{s}^+ , d_{s}^+, a_{s,0}^- , c_{s}^- , d_{s}^- , d_{r}\}\) and find approximate solutions \(\tilde{\delta}_{L,t}(\epsilon_{t})\) and \(\tilde{\delta}_{D,t}(c, \epsilon_{t})\).

The approximation works well, because true functional forms of both liquidation and default boundaries are very similar to those in the model with single temporary shock. The approximate functional form (S.B3) is chosen to exploit this similarity and obtain the same properties (continuous, decreasing, concave function with asymptotic behavior as \(\epsilon_{t} \to -\infty\)) of \(\tilde{\delta}_{L,t}, \tilde{\delta}_{D,t}\) as in the single-shock model, while keeping the number of parameters small to achieve convergence to optimum in reasonable time. In particular, \(H_{s}^{-}(\epsilon_{t}, \delta_{X,t})\) approximates small curvature and asymptotic behavior as \(\delta_{X,t} \to \infty\) of the boundary for sufficiently negative \(\epsilon_{t}\) because the term with \(\theta_{L}^+\) is included in its functional form. \(H_{s}^{+}(\epsilon_{t}, \delta_{X,t})\) approximates large curvature for positive and slightly negative \(\epsilon_{t}\) due to the inclusion of the term with \(\theta_{L}^+\) in its functional form.

For the liquidation boundary, the approximation simplifies because \(d_{r} = 0\): there is no liquidation in absence of the temporary shock. In addition, despite having non-linear functional form in the model with multiple shocks, in practice the liquidation boundary quickly becomes approximately linear for sufficiently negative \(\epsilon_{t}\), as numerical results show. With proper choice of \(\bar{\epsilon}\), the negative part \(H_{s}^{-}(\epsilon_{t}, \delta_{L,t})\) of the liquidation boundary can therefore be approximated by a functional form that is linear in \(\delta_{L,t}\): \(b_{s,0}^+ = 0\). This limits the set \(\Omega_{L} = \{c_{s}^+, d_{s}^+, a_{s,0}^- , c_{s}^- , d_{s}^-\}\) to five parameters.
Using our knowledge of model solution, we introduce additional constraints on sets of parameters \(\Omega_L\) and \(\Omega_B\) to further decrease the dimensionality of the optimization problem. For the abandonment boundary,

\[
H^*_L(\epsilon_L, 0) = 0, \quad (S.B4)
\]

\[
\frac{\partial H^*_L(\epsilon_t, \delta_L, t)}{\partial \delta_L, t} \bigg|_{\epsilon_t = \tilde{\epsilon}, \delta_L = \delta_L, t(\tilde{\epsilon})} = \frac{\partial H^*_L(\epsilon_t, \delta_L, t)}{\partial \delta_L, t} \bigg|_{\epsilon_t = \tilde{\epsilon}, \delta_L = \delta_L, t(\tilde{\epsilon})}. \quad (S.B6)
\]

For the default boundary,

\[
H^*_B(\epsilon_B, 0) = 0, \quad (S.B7)
\]

\[
\frac{\partial H^*_B(\epsilon_t, \delta_B, t)}{\partial \delta_B, t} \bigg|_{\epsilon_t = \tilde{\epsilon}, \delta_B = \delta_B, t(\tilde{\epsilon})} = \frac{\partial H^*_B(\epsilon_t, \delta_B, t)}{\partial \delta_B, t} \bigg|_{\epsilon_t = \tilde{\epsilon}, \delta_B = \delta_B, t(\tilde{\epsilon})}. \quad (S.B9)
\]

Finally, an additional constraint, as in Proposition 5, establishes the relationship between the slopes of liquidation and default boundaries as \(\epsilon_t\) becomes sufficiently negative:

\[
\frac{\partial \delta_B, t(\epsilon_t)}{\partial \epsilon_t} \bigg|_{\epsilon_t \to -\infty} = - \frac{\partial H^*_B(\epsilon_t, \delta_B, t)}{\partial \epsilon_t, \delta_B, t(\epsilon_t, \delta_B, t)} \bigg|_{\epsilon_t \to -\infty} = (1 - \alpha) \frac{\partial \delta_L, t(\epsilon_t)}{\partial \epsilon_t} \bigg|_{\epsilon_t \to -\infty} = s_{-\infty}, \quad (S.B10)
\]

where \(s_{-\infty}\) stands for “slope” as \(\epsilon_t \to -\infty\).

In the following discussion of (S.B5)–(S.B6) and (S.B7)–(S.B9), the intuition is similar for constraints on both liquidation and default boundaries. For brevity we discuss constraints on the default boundary. In (S.B7), \(\epsilon_B = \min_{\epsilon \in \mathbb{R}} (\delta_B, t(\epsilon_t) = 0)\) (correspondingly, in (S.B4), \(\epsilon_L = \min_{\epsilon \in \mathbb{R}} (\delta_L, t(\epsilon_t) = 0)\)). If \(\epsilon_t > \epsilon_B\), the firm does not default for any realization of permanent cash flow. Since it must be that the switching point \(\tilde{\epsilon}\) is below \(\epsilon_B\), the boundary condition (S.B7) applies to \(H^*_B(\epsilon_t, \tilde{\delta}_B, t)\). It directly corresponds to (S.5) when \(\tilde{\delta}_t = 0\): \(\epsilon_B\) is the highest value of the temporary shock that causes shareholders to default in the model with no permanent shocks (\(\tilde{\delta}_t = 0\)). In the model with both permanent and single transitory shock, \(\epsilon_B = c\). Multiple transitory shocks create additional value under optimal stopping policy, so \(\epsilon_B < c\) in the model with multiple shocks. The corresponding equation for the liquidation boundary is (S.3) when \(\tilde{\delta}_t = 0\). Also, \(\epsilon_L < 0\) in the model with multiple shocks.

(S.B5)–(S.B6) and (S.B8)–(S.B9) follow from continuity and smoothness of liquidation and default boundaries. Finally, (S.B10) equates the appropriately scaled slopes of approximate boundaries as \(\epsilon_t \to \infty\). As in the single-shock model, the equality of these slopes is the theoretical result. This is because when \(\epsilon_t \to -\infty\), the surviving firm has \(\tilde{\delta}_t\) so high above any future exit boundary, that it renders future liquidation/bankruptcy a zero probability event. So the trade-off between the current temporary and the sum of all future permanent and temporary shocks, which shapes optimal boundaries, becomes linear in the size of the current permanent shock, and the size of the coupon \(c\) does not influence the marginal rate of substitution between the current temporary shock and all future shocks.

Finally, the choice of \(\tilde{\epsilon}\) should be such that \(\epsilon_B - \tilde{\epsilon} \geq c\), because \(H^*_B\), the part of the boundary \(\delta_L, t\) or \(\delta_B, t\) which is mostly responsible for its curvature, at positive and slightly negative \(\epsilon_t\) should be at least as large as that in the model with the single transitory shock: numerical results for different choices of \(\tilde{\epsilon}\) show that boundaries in the model with multiple shocks preserve non-zero curvature for larger interval of \(\epsilon_t\) for any given
We choose \( \tilde{\epsilon} \) so that \( \epsilon_B - \tilde{\epsilon} = 2c \) to ensure that \( H_+^s \) approximates the true curvature well for those values of \( \epsilon_t \) where it is substantial. In unreported results, a further increase of the distance between \( \epsilon_B \) and \( \tilde{\epsilon} \) does not give any improvement in the approximation quality.

(S.B4)–(S.B10) reduce the number of free parameters in \( \Omega_L \) and \( \Omega_B \). We use (S.B4)–(S.B6) to reduce \( \Omega_L(\cdot, \cdot) \) to a two-dimensional parameter set, and use (S.B7)–(S.B10) to reduce \( \Omega_B(\cdot, \cdot, \cdot) \) to a three-dimensional parameter set. In this reduced parameter space, instead of using the original set of parameters it is convenient to re-parameterize \( \Omega_L \rightarrow \hat{\Omega}_L(a - s, 0, \delta_{L,t}(\tilde{\epsilon})) \) and \( \Omega_B \rightarrow \hat{\Omega}_B(a - s, d_r, \delta_{B,t}(\tilde{\epsilon})) \). In other words, to find \( \hat{\delta}_{L,t} \) we optimize over the curvature parameter in \( H_+^s(\epsilon_t, \delta_{L,t}) \) and the liquidation boundary in presence of the shock of size \( \tilde{\epsilon} \). To find \( \hat{\delta}_{B,t} \), we optimize over the curvature parameter in \( H_+^s(\epsilon_t, \delta_{B,t}) \), the constant default boundary in absence of the temporary shock, and the default boundary in presence of the shock of size \( \tilde{\epsilon} \). It is easy to establish one-to-one correspondence between the original and the new set of parameters.

The final set of inequality constraints that we put on reparametrized sets \( \hat{\Omega}_L \) and \( \hat{\Omega}_B \) allow us to significantly shrink the domain of admissible parameter values. For \( \hat{\Omega}_L \),

\[
\begin{align*}
\frac{\partial \delta_{L,t}(\epsilon_t)}{\partial \epsilon_t} &< 0, \forall \epsilon < \epsilon_L, \\
-\infty < \frac{\partial^2 \delta_{L,t}(\epsilon_t)}{\partial \epsilon_t^2} &< 0, \forall \epsilon < \epsilon_L, \\
\delta_{L,t}(\epsilon_t) &< \delta_{L,t}^{1J}(\epsilon_t), \forall \epsilon < 0,
\end{align*}
\]

(S.B11–B14)

and for \( \hat{\Omega}_B \),

\[
\begin{align*}
\frac{\partial \delta_{B,t}(c, \epsilon_t)}{\partial \epsilon_t} &< 0, \forall \epsilon < \epsilon_B, \forall c, \\
-\infty < \frac{\partial^2 \delta_{B,t}(c, \epsilon_t)}{\partial \epsilon_t^2} &< 0, \forall \epsilon < \epsilon_B, \forall c, \\
\delta_{B,t}(c, \epsilon_t) &< \delta_{B,t}^{1J}(c, \epsilon_t), \forall \epsilon < c, \forall c \text{ (in presence of shock)}, \\
\delta_{B,t}(c) &< \delta_{B,t}^{1J}(c), \forall c \text{ (in absence of shock)}.
\end{align*}
\]

(S.B15–B18)

These constraints ensure that we only search over parameters that shape decreasing, concave and negatively sloped boundaries, and that boundaries in the model with multiple shocks lie strictly below their counterparts in the model with the single shock for every \( \epsilon_t \) and \( c \) (1.J in expressions above stands for “one-jump boundary”).

For the set of benchmark parameters (and any other set of realistic parameters) and reasonably low \( c \) we find that \( \delta_{B,t}^{1J}(c) = 0 \) in absence of the temporary shock, which means that \( d_r = 0 \) in \( \Omega_B \), so \( \hat{\Omega}_B \) further reduces to a two-dimensional parameter set.

S.B.II Simulation

We find optimal parameters in \( \hat{\Omega}_L, \hat{\Omega}_B \) by checking that the derivative with respect to \( \delta_t \) of, correspondingly, simulated value of post-bankruptcy debt and simulated equity, calculated at \( \delta_t = \delta_{L,t}(\epsilon_t) \) or \( \delta_t = \delta_{B,t}(c, \epsilon_t) \), is as close to zero as possible for any \( \epsilon_t \) and \( c \). This approach emulates theoretical solution to the smooth-pasting condition in a two-dimensional cash flow space \( (\delta_t, \epsilon_t) \). Below we give the step-by-step maximization algorithm.
1. We set the value of all exogenous parameters \((\delta_0, \mu, \sigma, r, \lambda_s, \lambda_r, \gamma, \zeta, \tau_e, \tau_r, \alpha)\).

2. We simulate \(N_{\text{sim}} (= 500,000)\) paths \((\delta_t, \epsilon_t)\) of the geometric Brownian motion and the exponential mean-reverting temporary shock for \(T_{\text{sim}} = 120\) years with the frequency of \(\Delta_{\text{sim}} = \frac{1}{12}\) years. \(N_{\text{sim}}\) of paths \((\delta_t, \epsilon_t)\) are original; the rest are antithetic to achieve better convergence of simulated values of equity, debt, etc. to their theoretical values. Specifically, Brownian and exponential shocks, that are used to obtain antithetic paths, mirror those of original paths \(((W_t - W_{t-\Delta_{\text{sim}}})/\Delta_{\text{sim}}, \epsilon_t)\) in different combinations \((-(W_t - W_{t-\Delta_{\text{sim}}}), \epsilon_t), (W_t - W_{t-\Delta_{\text{sim}}}, -\epsilon_t), -(W_t - W_{t-\Delta_{\text{sim}}}, -\epsilon_t))\), starting either at time \(t_1 = 0\) years or at time \(t_2 = 20\) years.

3. We fix \(I = 10\) values of \(\epsilon_{t,i} = \epsilon_L e^{(i-1)k}\), where \(k\) is the scaling parameter that ensures a sufficient coverage of different \(\epsilon_t\) (for the benchmark case we set \(k = 0.2\)). We do an exhaustive search over parameter space \(\hat{\Omega}_L(a_{s,0}, \delta_{L,t}(\tilde{\epsilon}))\) for optimal parameters by simulating value of post-bankruptcy debt at \(\delta_{L,t}(\epsilon_{t,i})\) and \(\delta_{L,t}(\epsilon_{t,i}) + \nu\), where \(\nu\) is small, and then minimizing the following criterion
\[
\hat{\Omega}_L = \arg\min_{\Omega_L} \left\{ \frac{1}{I} \sum_{i=1}^{I} \left( V \left( (1-\alpha) \delta_{L,t}(\epsilon_{t,i}; \hat{\Omega}_L) + \nu, \epsilon_{t,i} \right) - V \left( (1-\alpha) \delta_{L,t}(\epsilon_{t,i}; \hat{\Omega}_L), \epsilon_{t,i} \right) \right)^2 w_i \right\}, \quad w_i = \frac{1}{I},
\]
(S.B19)

where \(w\) is the vector of weights. Intuitively, we find the boundary that, for any given \(\epsilon_t\), most closely satisfies the standard smooth-pasting condition (47) for \(V ((1-\alpha)\delta_t, \epsilon_t)\).

To make sure that optimal parameters are interior, we constrain the domain of \(a_{s,0}\) by (S.B11)–(S.B12) and that of \(\delta_{L,t}(\tilde{\epsilon})\) by \([0, 2\delta_{L,t}(\tilde{\epsilon})]\) in numerical calculations this upper boundary, obtained from the model with only one temporary shock, is always substantially above the optimal value \(\delta_{L,t}(\tilde{\epsilon})\). Additionally, for each set of parameters we check that (S.B14) holds. For all parameters that pass these checks we use Nelder-Mead (1967) optimization routine to minimize (S.B19).

4. The analogue of Step 3 is performed for any given coupon \(c\) to find the set of parameters \(\hat{\Omega}_B(a_{s,0}, \delta_{B,t}(\tilde{\epsilon}))\) that minimizes the criterion
\[
\hat{\Omega}_B = \arg\min_{\Omega_B} \left\{ \left( \frac{E \left( \delta_{B,t}(\Omega_B) + \nu, \epsilon_{t,i} \right) - E \left( \delta_{B,t}(\hat{\Omega}_B) \right) }{\nu} \right)^2 w_0 \right. \\
+ \left. \sum_{i=1}^{I} \left( \frac{E \left( \delta_{B,t}(\epsilon_{t,i}; \Omega_B) + \nu, \epsilon_{t,i} \right) - E \left( \delta_{B,t}(\epsilon_{t,i}; \hat{\Omega}_B) \right) }{\nu} \right)^2 w_i \right\},
\]
where \(w_0 = \frac{\lambda_s}{\lambda_s + \lambda_r}\), \(w_i = \frac{1}{I} \frac{\lambda_s}{\lambda_s + \lambda_r}\),
(S.B20)

where weights \(w_0\) and \(w_i\), \(i \geq 1\) are based on the relative expected amount of time of shock presence and absence. In our benchmark case, \(\delta_{B,t}(c) = 0\) for reasonably low \(c\); so \(\hat{\delta}_{B,t}(c; \hat{\Omega}_B) = 0\) and the first term of (S.B20) can be excluded from the minimization program. We fix \(I = 10\) values of \(\epsilon_{t,i}(c) = \epsilon_B - k_1 ce^{(i-1)k_2}\), where \(k_1, k_2\) are scaling parameters that ensure a sufficient coverage of different \(\epsilon_t\) (for the benchmark case we set \(k_1 = 0.03, k_2 = \frac{2}{3}\)).

To make sure that optimal parameters are interior, we constrain the domain of \(a_{s,0}\) by (S.B15)–(S.B16).
and that of $\delta_{B,t}(\hat{c})$ by $[0, 2\delta_{J,t}^B(0)]$ (in numerical calculations this upper boundary, obtained for the same coupon $c$ from the model with only one temporary shock, is always substantially above the optimal value $\delta_{B,t}(\hat{c})$). Additionally, for each set of parameters we check that (S.B17)–(S.B18) hold. For all parameters that pass these checks and for every chosen $c$ we use Nelder-Mead optimization routine to minimize (S.B20).

5. To find the optimal coupon $c^*$, on the first step we take 50 points uniformly from the set $[\xi, \tau]$, where $\xi = 0$ and $\tau = c^{*,0J}$ ($0J$ stands for the model with only permanent cash flows, and so $\tau$ must always be above $c^*$). For each $c$ we solve for the optimal $\hat{\delta}_{B,t}(c)$, and then simulate optimal equity and debt values with permanent cash flows starting at $\delta_0$. The value $c_1^*$ that maximizes the levered firm value on the first step is the starting point of the second step. Because strict convexity of levered firm value with respect to $c$ implies that true $c^*$ lies between the grid points directly below and above $c_1^*$, we redefine the set $[\xi, \tau]$: $\xi = c_1^* - step_1$, $\tau = c_1^* + step_1$, where $step_1$ is the first-step distance between any two adjacent coupon values. $c_2^*$ that maximizes the levered firm value on the second step is then chosen as the optimal coupon $c^*$.

In order to check the quality of the approximation, we add more terms from the theoretical functional form of the boundary (S.B1) (with coefficients $a_{s,1}^+, a_{s,1}^-, b_{s,1}^+, b_{s,1}^-$, one at a time) to the approximate functional form (S.B3), solve for the extended optimal set of parameters $\hat{\Omega}_L$ and $\hat{\Omega}_B$, and compare the optimal values of post-bankruptcy debt, equity, and leverage under the new approximation to those under the original approximation. While the increase of dimensionality of the parameter set significantly slows down the search for optimal boundaries, the difference between the optimal values of debt and equity is always smaller than 0.1%, and the difference between the optimal leverage is even more negligible. We also check the measurement error induced by the simulation bias and confirm that our method allows us to find the optimal coupon $c^*$ and optimal leverage with an error of at most 1%.

Simulation of the benchmark model with multiple transitory shocks, in which the manager values securities in expectation of only one shock, is easier. Specifically, steps 3. and 4. in the above procedure are substituted with finding analytical functional forms for $\delta_{L,t}$ and $\delta_{B,t}$, while the rest of the simulation proceeds as before.