On aggregation of information in competitive markets: The dynamic case

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Abstract

This article examines, in the spirit of Hellwig (1980), how a dynamic competitive security market aggregates and communicates information between various market participants at every instant. It employs a continuous time economy consisting of a large number of heterogeneous agents who are endowed with noisy private signals about the risky asset payoff of differing precision. It yields a stationary rational expectations equilibrium in which at every instant the agents use information contained in the market prices without rendering their private information redundant. It shows that the market price reflects only that part of private information which is common to many signals and, thus, extends Hellwig's single-period results to a multi-period model.

Keywords: Dynamically incomplete markets; Kalman filters

JEL classification: D82; G10

1. Introduction

In the spirit of Hellwig (1980), this article examines how a dynamic competitive security market aggregates and communicates the diverse information held by various market participants. It characterizes the evolution of a stationary rational expectations equilibrium in which at every instant agents use information...
contained in the market prices without rendering their private information redundant. Thus, it extends Hellwig’s work in a single-period model to a multi-period framework and provides a dynamic model to address several issues in information economics.

This article is a first step towards examining issues relating to aggregation of information in dynamic competitive markets. When compared with the single payoff—two trading period models in literature, it attempts to solve the information aggregation problem in a more general setting. More importantly, it contributes to the existing literature on information economics by providing us with a platform to address issues of insider trading and sale of information in an intertemporal framework. Before proceeding with the details of the model, I review below some recent developments in information economics.

Ever since the introduction of the notion of efficient markets, financial economists have been studying how a competitive security market aggregates and communicates the different pieces of information available to the various market participants. The meaning of efficient markets is unambiguous when there is only a single piece of information. However, problems arise when there are many agents with different pieces of information and the vector of market prices is of a smaller dimension than the vector of information available to different agents. In such a case, the equilibrium price vector corresponds to some aggregate of the individual pieces of information. Naturally, the question arises as to how this aggregate is formed.

Grossman (1976, 1978) proposed a solution to this problem by demonstrating a fully revealing equilibrium wherein the risky asset price aggregated all the available information perfectly. He used a static model of capital market with a single risky asset and with identically distributed signals. Grossman showed that the equilibrium price provided statistically superior information to any private signal which made the private information redundant given the price. Grossman’s fully revealing equilibrium was identical to the Walrasian equilibrium of an artificial economy where agents shared all their information before trading. That is, if the agents took their information from the market price, then the economy achieved the same allocation as if each agent knew the information available to all the market participants. Furthermore, any aspect of the information that was not reflected in the market price was simply noise and therefore was not worth communicating.

Unfortunately, Grossman’s solution had some conceptual difficulties. For example, the agents’ equilibrium demand functions were constant, independent of the agents’ initial wealth, private information and market prices. The independence of wealth was because of the assumption of constant absolute risk aversion preferences. The sufficient statistic property of the market prices was the cause of the agents’ private information becoming redundant. Finally, independence of prices was due to exact cancellation of the substitution effect by the information effect. This meant that at every price the agents had the same demand for the risky
asset (see Admati, 1989 for a detailed discussion). With constant demands, it was difficult to see how private information ever came to be reflected in the market prices in the first place.

Hellwig (1980) overcame these difficulties by analysing a single-period model with a single risky asset which involved a noisy transmission of information by price and aggregation of diverse information held by different agents. He allowed the agents to vary in their risk attitudes as well as the precision of their information. For a finite agent economy he examined how an agent’s characteristics affected his weight in the equilibrium price and proved the existence of a linear noisy rational expectations equilibrium. He took the limit as the number of agents approached infinity and showed that the effect of the noise in an agent’s private signal on the equilibrium price became negligible. Since an individual agent did not affect the market price, his private information entered the price only to the extent that it was shared by other agents. Thus, the price reflected only that part of information which was common to many private signals. Since the market price acted as a noisy signal of the risky asset payoff, agents used it in addition to their private signal to form their expectations about the payoff.

Over the last decade, the intuition behind Hellwig’s single-period model has been widely used to examine several important issues in information economics. In this article, I try to generalize Hellwig’s single-period model to a multi-period economy with infinitely lived agents. In particular, I examine how a dynamic competitive security market aggregates and communicates information between the various market participants at every instant. I study an economy consisting of a continuum of heterogeneous agents who are endowed with (noisy) private signals of differing precision (the signals provide some information about the risky asset payoff in the future). For the reasons of tractability, I model the state variables as linear mean reverting stochastic processes and assume that agents exhibit constant absolute risk aversion preferences. I solve for the steady-state condition of this economy which corresponds to its stationary rational expectations equilibrium and show that the stock price reflects only that part of information which is common to many signals. Thus, I extend Hellwig’s result from a static setting to a dynamic scenario.

Although the two trading period models of Brown and Jennings (1989) and Grundy and McNichols (1989) have some similarities with my model, the evolution of the risky asset payoff and the private information in my infinite horizon economy is considerably different. Unlike the two period models where the stock price function depends on the history of the economy explicitly, the equilibrium stock price function in my model depends on the history implicitly. The reason being in the two trading period models, there is only one payoff and the agents learn by looking at prices at the end of the first period. As a result, the precision of the agents’ conditional estimates of the payoff increases over time. My model, on the other hand, describes an economy where the fundamentals determining the risky asset payoff are continuously being subjected to exogenous shocks. Thus,
the agents face a problem, so to speak, of keeping up with a moving target. Moreover, due to the stationary nature of the equilibrium, the precision of the agents’ estimates does not change over time and *current changes* in the dividends and stock prices summarize all information about the new shocks hitting the economy.

Although Hellwig (1982) and Singleton (1987) also develop multi-period models to address similar issues, my model is relatively less restrictive. For instance, Hellwig (1982) does not allow his agents to condition their demand on current price while Singleton assumes that agents behave myopically and all private information becomes public after a certain time. On the other hand, I allow the agents to condition their demand on the current price and permit them to hedge against adverse changes in the investment opportunity set as well. Furthermore, to avoid the so-called *infinite regress* problem (discussed in detail later), I use the insights from Sargent (1991) and allow the private information to remain private, thereby enabling the agent’s learning problem to retain its dynamic flavour.

Since my model involves a continuum of agents facing individual uncertainty, but without aggregate uncertainty, some remarks are in order. There is a well-known problem associated with one’s ability to assert, with probability one, that the distribution of outcomes of a continuum of independent identically distributed random variables equals the theoretical distribution. A variety of constructions have been suggested to alleviate this so-called continuum of random variables problem. Since the focus of this article is on studying the aggregation of diverse information held by various agents through the market prices, I assume that the necessary regularity conditions are met and directly invoke the generalized law of large numbers for a continuum of agents. Furthermore, in order to keep the set up simple, I also assume that the *usual* measurability and integrability conditions are satisfied and request readers to see Karatzas and Xue (1991) for a discussion of the general problem of utility maximization under partial observations in dynamically incomplete financial markets.

The remaining article is organized as follows. Section 2 describes the economy which is along the lines of Campbell and Kyle (1993), and Wang (1993). Section 3 solves for the noisy rational expectations equilibrium and proves the main results of this article. Section 4 concludes and offers suggestions for further research.

2. Model

In order to generalize Hellwig’s noisy rational expectations equilibrium in a static framework to a dynamic setting, I require the economy to exhibit three critical features. First, in order to obtain a linear equilibrium representing the

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steady-state condition of the economy, I need the stochastic processes characterizing the underlying state variables to be linear and of a mean reverting nature. (I assume that all state variables evolve as Ornstein–Uhlenbeck processes with constant conditional variance.)

Second, in a single-period model, the stochastic nature of the risky asset supply and the risky asset payoff introduces uncertainty from two sources. This prevents the market price of stock (the single source of public information) from being fully revealing and one obtains a noisy rational expectations equilibrium. On the other hand, in a continuous time setting the risky asset payoff takes the form of an dividend stream. Since the dividend and the stock price evolution is publically observable, the public information becomes two-dimensional. Therefore, I need the economy to have at least three sources of uncertainty (this is achieved by assuming that dividends, growth rate of dividends and risky asset supply evolve stochastically over time).

Finally, to ensure that the distribution of agents’ wealth does not affect the stock prices in the economy, the agents must exhibit constant absolute risk aversion preferences which implies that the agents’ optimal stock demand functions are linear in their estimates of the state variables. (Just as Grossman and Hellwig, I assume agents with negative exponential utility.)

Having identified the three necessary features to obtain a noisy rational expectations equilibrium under a dynamic scenario, I enumerate below the specific assumptions describing the economy. Consider an economy with a single physical good (which may be consumed or invested) described by the following assumptions.

**Assumption 1.** The economy is endowed with a certain quantity of perfectly divisible risky capital. Each unit of risky capital generates an output (dividend) at an instantaneous rate $D$ which evolves according to the diffusion process

$$dD_t = a_1(\mu_t - kD_t)dt + \sigma_D dz_D,$$

where $\mu_t$ is a state variable following an Ornstein–Uhlenbeck process

$$d\mu_t = a_2(b - \mu_t)dt + \sigma_\mu dz_\mu,$$

$a_1, a_2, b, k, \sigma_D$ and $\sigma_\mu$ are positive constants and $z_D, z_\mu$ are two independent standard Wiener processes.

With this form of the dividend process, $D$ can take negative values. However, by suitable selection of the parameter values, one can make the probability of the occurrence of such an event arbitrarily small (see, e.g., Campbell and Kyle, 1993). Wang (1993) suggests an interpretation that negative values of $D$ correspond to situations where investors have to plough back some investment in order to maintain future cash flows.
Note that \((D_t, \mu_t)\) is a Gaussian Markov system. Since \(k > 0\), \(\mu/k\) can be interpreted as the short run steady-state level of dividend rate \(D\) which fluctuates around the long run steady-state level \(b/k\).

**Assumption 2.** In real life, the financial markets consist of stocks with a certain number of outstanding shares. This makes the risky capital supply non-stochastic. However, the financial economics literature has generally modelled rational investors as facing a supply uncertainty due to trading by liquidity and noise traders. The liquidity traders are thought of as investors who buy or sell their assets due to life cycle or other reasons. For instance, in case of an individual investor, a life cycle reason to sell stock to raise cash could be a need to undergo a major operation like a by-pass heart surgery. The liquidity reasons can also force institutional investors to sell or buy stocks. For instance, after a natural disaster like a hurricane or an earthquake, an insurance company may face a number of claims which could call for sale of some of its stock holdings. Alternatively, a pension fund manager may receive cash from the company and employees at the end of the month on a regular basis. Since the fund manager may not be allowed to hold on to cash for too long, he may be forced to invest it in the stock market and therefore may end up buying the stock.

Apart from liquidity traders there could be noise traders in the market as well. The noise traders are typically modelled (see Shleifer and Summers, 1990) as individuals who are not fully rational in that their demand for risky assets is affected by beliefs or sentiments that are not fully justified by fundamental news. The sentiments of the noise traders oscillate between optimism and pessimism which affect stock returns (Lee et al., 1991) and can contribute to market over-reaction observed by researchers (DeBondt and Thaler, 1985).

In short, the liquidity or noise traders introduce uncertainty in the aggregate supply of risky asset available to the rational investors to invest in. Given the notion that noise traders’ sentiments oscillate between optimism and pessimism, one expects the noise traders’ stock demand to exhibit some mean reversion over time. This implies that the total amount of risky asset available to the rational traders would also exhibit a mean reverting process around the total number of outstanding shares in the economy.

This notion is captured by assuming that the aggregate quantity of the risky capital available to the rational agents, \(\phi_t\), evolves as

\[
d\phi_t = a_3(1 - \phi_t) dt + \sigma_\phi \, dz_\phi,
\]

where \(a_3, \sigma_\phi\) are positive constants and \(z_\phi\) is a standard Wiener process. Without loss of generality, the long-run stationary level of the risky capital \(\phi\) is normalized to unity and for simplicity, it is assumed that \(z_\phi\) is independent of \(z_D\) and \(z_\mu\).²

²Less than perfect correlation between risky asset supply and traders’ information ensures that private information is valuable in equilibrium. See Grossman and Stiglitz (1980) and De Long et al. (1987).
Note that the evolution of risky asset supply captures the mean reversion in
the noise traders demand described above in the following way. Define the noise
traders' stock demand as, say, \( \theta_t \). Then, the total risky asset supply available to
rational agents \( \phi_t \) would equal \( (1 - \theta_t) \) as the total risky asset supply is normalized to
unity. Now, if \( \theta_t \) evolves as \( d\theta_t = -a_\theta \theta_t - a_\phi \phi_t d\phi \), then the total risky asset
supply available to the rational agents will evolve as per Eq. (3) above.\(^3\)

Assumptions 1 and 2 satisfy the first necessary requirement (described at the
beginning of this section) for obtaining a noisy rational expectations equilibrium.

Assumption 3. There exists a perfect capital market wherein the shares of the
risky security, i.e., the stock, is the only risky asset traded. Each unit of risky
capital is represented by one perfectly divisible share of stock held by the agents
in the economy. Let \( S_t \) be the equilibrium price of stock. All agents also have
access to a risk free storage technology with the constant rate of return \( r \).

In order to study issues relating to aggregation of diverse information, it is
assumed that each agent is endowed with a noisy signal of the true value of the
unobservable underlying state variable. Since in Hellwig's model the agents ob-
servede signals with pure noise, it is assumed that the noise component in different
agents' signals are uncorrelated. Clearly, if these noise terms were to be corre-
lated, then there would be common noise in the investors' signals. This could
happen in real life if many investors use the same newsletter or advisory service
to decide which stocks to buy or sell. Suppose the noise terms are correlated,
i.e., many investors observe signals with a common noise component, then ag-
gregation of information across all investors would not reveal the true value of
underlying unobservable state variable even when the number of investors goes
to infinity. Further, a persistence of this common noise over time would lead to
continued mispricing of stocks.

These stock mispricings may get corrected with public release of informa-
tion by companies, like the earnings announcements. However, if the earnings
announcements are not frequent or for some reason the stock mispricing persists
in spite of the earnings announcements (due to, say, common misinterpretation
of the public information), then it is likely to give rise to panics and crashes in
the stock market. Indeed, specifying correlated noise terms in investors' signals
would be one way of capturing some interesting aspects of real life stock market
observations like mispricings and corrections in the form of panics and crashes.\(^4\)

However, in this article, keeping to the spirit of Hellwig's model where agents
observe signals with uncorrelated noise component, I restrict my attention to the
scenario of private signals containing pure noise component and leave the issue
of correlated signals for future research.

\(^3\) See Diamond and Verrecchia (1981) for explicit modelling of randomness in the endowments.

\(^4\) Persistent mispricing can be incorporated in the current framework; however, corrections in the form
of crashes would need a different and perhaps less tractable model.
Assumption 4. In the spirit of Hellwig (1980), each agent (denoted by superscript \( v \)) is assumed to be endowed with some private information (the growth rate of dividends perturbed by some idiosyncratic noise) which evolves according to the process

\[
d\pi^v_t = d\mu_t + \sigma^v_t \cdot dz^v_t, \tag{6}
\]

where the noise in the private signal \( \sigma^v_t \cdot dz^v_t \) consists of a product of two terms: a scaling factor \( \sigma^v_t \) and an individual agent specific pure noise component \( dz^v_t \). The term \( dz^v_t \) can be thought of as a fresh realization of a given standard Wiener process \( z_t \). In order to capture the notion that the idiosyncratic noise in the private signals does not contain any information, \( z_t \) is taken to be independent of Wiener processes \( z_D, z_D^0 \) and \( z_D^0 \).\(^5\) Further, to represent the idea that different agents are endowed with information of differing precision, the scaling factor \( \sigma^v_t (0 < \sigma^v_t < \Sigma < \infty) \) is defined to be a finite constant specific to an agent.

Note that this set up is quite general. Any combination of \( \sigma^v_t \in (0, \Sigma) \) and risk aversion \( \gamma \in (0, \Gamma) \) is permitted: a highly risk tolerant agent can have a very precise signal and vice versa. The signals are kept noisy to study how a competitive security market aggregates at every instant the diverse information available to different agents.\(^6\)

In the spirit of Dothan and Feldman (1986), Gennette (1986) and Detemple (1986), the agents' information about the underlying state variables is allowed to be imperfect, i.e., the agents are permitted to differ in their information about the current values of the underlying state variables. The restriction on the number of signals in Assumption 3 together with Assumptions 1 and 2 prevent the equilibrium price from being fully revealing and ensure that the capital market remains dynamically incomplete in the sense of Harrison and Kreps (1979). With incomplete markets there exist reasons other than the arrival of new information to trade. Therefore, the notions of no trade of Milgrom and Stokey (1982) and full revelation of Grossman (1981) do not apply in this economy.

Recall that, the second necessary requirement was that the vector of public information should be of a lower dimension than the sources of the uncertainty. This has been achieved by specifying three sources of innovations in the processes governing the economy (\( dz_D, dz_D^0 \) and \( dz_D^0 \)) and two sources of public information observable by each agent at every instant: the dividend flow \( dD_t \) and the stock price evolution \( dS_t \). Furthermore, due to the non-zero realization of the noise in the private signal, an agent's private information does not fully reveal the

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\(^5\) The assumption of independence is without loss of generality and is made only for computational convenience; less than perfect correlation is all that is necessary for the results to hold.

\(^6\) The results of this article hold even if there exists a fraction of agents with perfect information (see Wang, 1993). The reason being in a multi-period model, due to risk aversion the optimal risky asset demand of price taking agents with perfect information remains bounded. This prevents the market price from being fully revealing.
current values of the underlying state variables. Therefore, every agent uses the information contained in the market prices as well as in the private signal while determining the optimal consumption and investment strategies.

**Assumption 5.** The economy consists of a continuum of price taking risk averse agents denoted by \( v \) where \( v \in [0, 1] \). The preferences of the agents are given by

\[
u^*(c_t^v) = -\frac{e^{-\gamma^v c_t^v}}{\gamma^v},
\]

where \( \gamma^v (0 < \gamma^v < \Gamma < \infty) \) is the risk aversion coefficient and \( c_t^v \) is the rate of consumption of agent \( v \) at time \( t \). Each infinitely lived agent \( v \) maximizes the expected utility of lifetime consumption given by

\[
E \int_t^\infty e^{-\rho^v(t-\tau)} u^v(c^v_T) \, d\tau \mid \mathcal{F}^v(t),
\]

where \( \rho^v \) is the time impatience parameter and the expectations operator is based on the agent’s information set at time \( t \), \( \mathcal{F}^v(t) \equiv (D_t, S_t, \pi_t^v : \tau \leq t) \).

With negative exponential utility, the agents’ optimal stock demand functions are linear in their estimates of the state variables and the distribution of agents’ wealth does not affect the stock prices in the economy. This assumption is in the same spirit as Grossman (1978), Hellwig (1980) and Admati (1985) and fulfills the third necessary requirement to obtain the noisy rational expectations equilibrium in this dynamic economy.

**Assumption 6.** Structure of the economy is common knowledge to all agents.\(^7\)

There exists a body of literature where the exogenous variables driving the market are allowed to be serially correlated so that past prices help the (imperfectly informed) agents in their signal extraction problem. Some of these papers examine the trade of a risky asset (Hellwig, 1982; Singleton, 1987; Hussman, 1992; Wang, 1993) while others model production with an uncertain output price (Townsend, 1983; Sargent, 1991).\(^8\) It is well known that if the exogenous variables driving the market forces are serially correlated, then, solving for equilibria can become technically difficult. The fundamental problem being that the infinite sequence of equilibrium prices is determined endogenously within the system. Townsend (1983) points out that in such a scenario agents try to predict each others’ expectations which leads to the so-called infinite regress or the forecasting the forecasts of others problem.

\(^7\) As will be seen later, the fact that each agent is of measure zero and knows the structure of the economy (especially that private signals contain uncorrelated noise) makes the problem tractable.

\(^8\) Interestingly, both these strands of literature involve learning – optimization and have close counterparts in terms of linearity of underlying state variables, linear pricing and demand functions, etc.
Several methods are proposed to circumvent this problem. The one closest to the scenario under consideration is the novel method proposed by Sargent (1991). It builds on the insights from Marce and Sargent's (1989) work on the convergence of least squares learning to rational expectations equilibria in environments with private information. The notable difference in Sargent's approach vis-à-vis earlier approaches being the distinct formulation of the forecasting problem of the agents.

In particular, Sargent specifies agents as fitting vector autoregressive moving average (arma) models to whatever information they have available while arriving at their conditional forecasts. He then defines the notion of linear rational expectations equilibrium and computes the fixed point of the mapping from the perceived arma processes to the actual arma processes that is induced by agents' behaviour and market clearing. It is the inclusion of the moving-average components in agents' perceptions and of lagged innovations to agents' information in the state vector that permits the formulation of the equilibrium as a fixed point of a finite-dimensional operator and circumvent the infinite regress problem.

Our model is similar to the one discussed in Sargent (1991, Section 5). Sargent's model has a continuum of product markets. Individual markets face uncertain output prices and consist of firms with quadratic adjustment costs. The prices in different markets are determined by a downward sloping demand schedules containing both economy-wide and idiosyncratic shocks. All firms observe the economy-wide average of prices with noise (public information) and use it along with their individual demand shock (private information) while deciding their optimal production plans.

Every aspect of Sargent's model has a counterpart in my economy. Sargent's continuum of markets is like the continuum of agents. The common and idiosyncratic components in the demand schedule in each market is like the common and idiosyncratic component in each agent's private signal. Each firm in Sargent's model observing the average price perturbed by some noise is like each agent in my economy observing the market price (which contains the average of the signals) perturbed by noise due to liquidity traders. Finally, firms in Sargent's model (like the agents in my model) optimally use the public as well as the private information to forecast the unobserved state variables. Therefore, using

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10 Sargent's method draws upon Muth's (1960) observation that the space spanned by a pure infinite-order autoregression may, in some cases, be replicated by a finite-order arma process. Sargent starts with specifying a limited information (reduced order) equilibrium and then verifies if it is a full information (full order) equilibrium or not. Numerically, Sargent shows that his method offers a very good approximation to the full information equilibrium.
a methodology similar to Sargent (1991), one can side-step the infinite regress problem in the economy under consideration.\footnote{The procedure would involve defining the discrete time counterpart of the continuous time economy under consideration, applying Sargent’s technique, examining the limiting case and showing that the economic intuition behind Sargent’s technique carries over to the continuous time case as well.}

In essence, the issue boils down to whether the agents forecast the forecast of others or not. It is well known that if they do, then in all but a few cases described above there may not exist an equilibrium. Alternatively, if one takes the view that due to the continuum of agents, zero mean nature of the signals and the structure of the economy being common knowledge, agents behave as price takers and do not forecast the forecast of others, then one can examine the nature of equilibrium, how the aggregation of information takes place and so on. This work takes the latter view and explores the aggregation issue in dynamic markets.

Before solving for the stationary linear rational expectations equilibrium, I first review a benchmark case which demonstrates how a risky asset is valued in an economy defined by Assumptions 1–6 above.

2.1. Benchmark case: Identical agents with perfect information

Consider a scenario where all agents in this economy are identical and have perfect knowledge of the current values of the underlying state variables (this is similar to the case studied by Campbell and Kyle, 1993). The equilibrium stock price under this scenario provides a measure of the fundamental value of the stock. The corresponding excess return and stock price volatility indicate the risk inherent in investing in the stock in the presence of noise or liquidity traders. In this framework of a representative investor with perfect information, the stock price depends only on $D_t$, $\mu_t$ and $\phi_t$. Let $S_t^b$ be the price of stock at time $t$; then it can be shown that

$$ S_t^b = s_0^b + s_1 D_t + s_2 \mu_t + s_3^b \phi_t, $$

(7)

where $s_1 = (r + a_1 k)^{-1}$, $s_2 = a_1 s_1 (r + a_2)^{-1}$, $s_0^b = a_2 s_2 b (r)^{-1} - s_1^b \sigma_{D_t}^2 - s_2^2 \sigma_{\mu}^2$ and $s_3^b < 0$ are constants (see Wang, 1993, Theorem 3.1 for proof).\footnote{The superscript $b$ is used to denote that this scenario is a special case of the generalized economy analysed in the next section.}

This benchmark economy has three basic features resulting from the linearity of the underlying processes and the nature of agents’ preferences which imply linear stock demand functions. First, the equilibrium stock price is a linear function of the state variables and is independent of aggregate wealth as well as its distribution. Second, the expected future dividends are discounted at the risk free rate (note $s_1$ and $s_2$ above). Third, risk aversion leads to subtraction of a term from the equilibrium stock price instead of increasing the discount rate to provide
for excess returns ($s_3$ is negative). These basic features continue to hold in the
economy defined by Assumptions 1–6 as well.

Intuitively, my model generalizes the benchmark case in the following way. In
the benchmark case, each agent has perfect knowledge of the state of the world
(so there is no question of aggregation of information), while in this economy,
individually the agents have imperfect knowledge; however, collectively they hold
perfect information. This difference in the information structure enables me to
address issues relating to aggregation of information in this dynamic economy.

3. Rational expectations equilibrium

I use the rational expectations equilibrium notion of Sargent (1991) and demon-
strate its existence in three steps. First, I conjecture the form of the equilibrium
stock price function. Then based on the assumed price function, I solve each
agent’s learning and optimization problems. Finally, I impose the market clear-
ing condition to verify the conjectured stock price function and demonstrate that
each agent’s expectations are fulfilled.

Step 1. This consists of conjecturing the form of the equilibrium stock price
function. Although the agents differ in their risk attitudes and private informa-
tion, given the agents’ preferences and the nature of underlying processes, the
equilibrium stock price function turns out to be similar to the benchmark case
described above.

3.1. Conjecturing the stock price function

Proposition 1. For the economy defined by Assumptions 1–6, there exists
a stationary rational expectations equilibrium. Further, the equilibrium stock
price function has the following linear form:

$$S_t = s_0 + s_1 D_t + s_2 \mu_t + s_3 \phi_t,$$

(8)

where $s_1 = (r + a_1 k)^{-1}, s_2 = a_1 s_1 (r + a_2)^{-1}$, and $s_0$ and $s_3 < 0$ are constants.\(^{13}\) Note that since the actual cash flows are not affected by the information
structure, $s_1$ and $s_2$ are same as the benchmark case.

Before proceeding with the proof, some comments about the proposed price
function are in order. First, one would expect the equilibrium stock price to
depend on the whole history of dividends and prices up to and including time $t$.

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\(^{13}\) The conjectured form of the equilibrium stock price function, in principle, could be of a general
form in which case there may or may not exist a rational expectations equilibrium. Since the economic
set-up in this article is similar to that of Campbell and Kyle (1993) and Wang (1993), I have focused
attention on the stock price function which is similar to theirs.
This is indeed true, however as is shown later, this dependence is linear and the agents’ filters provide a sufficient statistic for the dividends, prices and private signals. Therefore, Eq. (8) can be thought of as an implicit or condensed form of the price function. Second, the knowledge of current dividend $D_t$ and equilibrium price $S_t$ provides every agent perfect information about the sum of $\mu_t$ and $\phi_t$, namely, $\xi_t = s_2 \mu_t + s_3 \phi_t$. Hence, observing $D$ and $S$ is equivalent to observing $D$ and $\xi$ or $\mathcal{F}^{D,S} = \mathcal{F}^{D,\xi}$.

**Step 2.** This consists of solving the agents’ learning and optimization problems based on the conjectured stock price function. Note that the agent’s learning problem can be solved independent of the agent’s optimization problem because of the separation principle which holds in this economy given the linearity of the underlying processes (see Fleming and Rishel, 1975 for details). So, I first solve each agent’s learning problem. Using the agent’s estimates of the current values of the state variables, I solve the agent’s Bellman equation to arrive at his optimal stock demand function.

### 3.2. Agent’s learning problem

Recall that although each agent has less than perfect knowledge of the current values of $\mu_t$ and $\phi_t$, a linear combination of the two is in the information set of each agent. To see why this is so, define $\xi_t = S_t - s_0 - s_1 D_t = s_2 \mu_t + s_3 \phi_t$. Then $d\xi_t = dS_t - s_1 dD_t = s_2 d\mu_t + s_3 d\phi_t$. Since every agent observes the evolution of dividends and stock prices (since $\mathcal{F}^\tau(t) = D_{\tau}, S_{\tau}, \pi^\tau_{\tau}, T \leq \tau \leq t$), $d\xi_t$ belongs to every agent’s information set.

Based on the information set $\mathcal{F}^\tau(t)$, each agent $v$ solves for his conditional estimates $E[\mu_t | \mathcal{F}^\tau(t)] = \hat{\mu}_t^v$ and $E[\phi_t | \mathcal{F}^\tau(t)] = \hat{\phi}_t^v$ of the underlying state variables which evolve as per Theorem 1 below.

Clearly, rationality requires that as each agent is dividing $\xi_t$ into two parts, the linear combination of the two estimates must add up to $\xi_t$ and the following equation must hold for every agent at every instant:

$$
\begin{align*}
\frac{d\xi_t}{dt} &= dS_t - s_1 dD_t \\
&= s_2 d\mu_t + s_3 d\phi_t
\end{align*}
$$

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14 Under certain regularity conditions, the separation principle permits solving of the stochastic control problem in case of partially observed state variables in two steps. The first step involves obtaining the best estimates of the unobserved state variables. The second step involves solving the control problem using the estimates as realizations of state variables.

15 Instead of using an ARMA process for the agents’ signal extraction problem, for expositional convenience, I work with a pure autoregressive process. Clearly, the conditional estimates arrived at by both these methods is identical. Although this allows the equilibria solved for to be of a reduced order type in the sense of Sargent (1991), it helps a great deal in making the economic intuition behind the aggregation process more transparent.
\[ = s_2 \, d\hat{\mu}_t^v + s_3 \, d\hat{\phi}_t^v \]
\[ = \mathbb{E}[d\tilde{\xi}_t \mid \mathcal{F}^v(t)]. \] (9)

**Theorem 1.** Given the agent’s information set \( \mathcal{F}^v(t) \), the Kalman filters \( \hat{\mu}_t^v \) and \( \hat{\phi}_t^v \) satisfy the stochastic differential equations
\[
\begin{pmatrix}
    d\hat{\mu}_t^v \\
    d\hat{\phi}_t^v
\end{pmatrix} = \begin{pmatrix}
    a_2[b - \hat{\mu}_t^v] \\
    a_3[1 - \hat{\phi}_t^v]
\end{pmatrix} dt + \begin{pmatrix}
    h_{2\xi}^v & h_{2\pi}^v \\
    h_{3\xi}^v & h_{3\pi}^v
\end{pmatrix} d\tilde{\omega}_t^v(t),
\]
where
\[
d\tilde{\omega}_t^v(t) = \begin{pmatrix}
    d\tilde{\xi}_t - a_2 s_2 (b - \hat{\mu}_t^v) dt - a_3 s_3 (1 - \hat{\phi}_t^v) dt \\
    dD_t - a_1 (\hat{\mu}_t^v - kD_t) dt \\
    d\pi_t^v - a_2 (b - \hat{\mu}_t^v) dt
\end{pmatrix}
\]
and the innovation process of the filters \( d\tilde{\omega}_t^v(t) \) is a standard Wiener process with respect to \( \mathcal{F}^v(t) \). Furthermore, the information structure generated by \( \mathcal{F}^v \) (t) is equivalent to that generated by \( \mathcal{F}^{D,S,\pi}(t) \).

**Proof.** This is a generalization of Theorem 4.1 in Wang (1993). See Appendix A for the proof. \( \square \)

Note that due to the large number of market participants, an individual agent’s demand does not affect the equilibrium price and therefore the learning problem becomes independent of agent’s preferences. Furthermore, given an allocation of information in the economy (i.e., for a given level of risk aversion and the precision of the private signal of every agent), the precision of an agent’s estimates depends only on the precision of the agent’s private signal. Finally, given the linearity of stochastic processes governing the evolution of the state variables, the agent’s estimates depend linearly on the state variables.

This completes the first step, namely solving the agents’ learning problem. Before solving an agent’s optimization problem, I quantify below the investment opportunities which affect the agent’s budget constraint. This is for the sake of future convenience in interpreting the optimal stock demands of the agents.

### 3.3. Investment opportunities

From Proposition 1 and Eqs. (1)–(3), the stock price evolves as
\[
    dS_t = [s_1 a_1 (\mu_t - kD_t) + s_2 a_2 (b - \mu_t) + s_3 a_3 (1 - \phi_t)] dt \\
    + s_1 \sigma_D \, dz_D + s_2 \sigma_\mu \, dz_\mu + s_3 \sigma_\phi \, dz_\phi. \] (10)
Let the stochastic part of $dS_t$ be denoted by $\sigma_S dZ_S$; then from the independence of $z_D, z_\mu$ and $z_\phi$,
\[ \sigma_S^2 = \sigma_1^2 \sigma_D^2 + \sigma_2^2 \sigma_\mu^2 + \sigma_3^2 \sigma_\phi^2. \] (11)

Define a zero-wealth portfolio $\Omega_t$ consisting of one unit of the stock financed by borrowing at the risk-free rate. The instantaneous return on this zero-wealth portfolio is the excess return on one share of stock. Given the price process in Eq. (10), the cash flow from the zero-wealth portfolio is given by\(^{16}\)
\[ d\Omega_t = (D_t - rS_t) \, dt + dS_t \]
\[ = [(a_2 b s_2 - r s_3 - r s_0) + (r + a_3) s_3 (1 - \phi_t)] \, dt + \sigma_S \, dz_S. \] (12)

Note that the excess return on the risky asset depends only on the level of the aggregate stock supply — a measure of the collective risk in the economy. Furthermore, depending on the realization of the noise term in the private signal, each agent perceives the investment opportunities differently (expects a different return on the stock). This affects the agent’s perception of the attractiveness of the risky asset relative to the risk free asset, and in turn the agent’s risky asset demand.

I now proceed to the second step of solving each agent’s optimization problem and arrive at his optimal controls.

### 3.4. Agent’s optimization problem

Let $W^v_t$ denote an agent’s wealth, $x^v_t$ his holding of stock financed by borrowing at the risk free rate and $c^v_t$ his consumption. Then the agent’s optimization problem can be stated as
\[
\text{Maximize} \quad E \left[ \int_t^\infty e^{-\rho(t-\tau)} \mu'(c^v_\tau) \, d\tau \mid \mathcal{F}^v(t) \right] \] (13)
subject to
\[ dW^v_t = (r W^v_t - c^v_t) \, dt + x^v_t \, d\Omega_t. \] (14)

In general, solving this optimization problem can be quite complicated. The agent’s consumption–investment policy is a function of his information set, which contains the entire history of dividends and prices. However, given the underlying stochastic processes and the equilibrium stock price function, the information structure generated by $\mathcal{F}^v(t)$ has an equivalent representation which is the one generated by $\tilde{\omega}^v(t)$ (where, from Theorem 1, $\tilde{\omega}^v(t)$ is the innovation process of the filters which is a Wiener process with respect to $\mathcal{F}^v(t)$). Thus, the filters provide a sufficient statistic for $\mathcal{F}^v(t)$. Using this equivalent representation of

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\(^{16}\)Use from Section 3.1 the expressions for $s_1$ and $s_2$. Add and subtract $r s_3$ from the drift term and simplify and use to obtain (12).
the information structure, one can restate the agent’s optimization problem as a standard Markovian one with the agent’s filters being the effective state variables and the innovation process \(\hat{\phi}_i(t)\) generating the dynamics.\(^{17}\)

Thus, every agent \(v\) solves the Bellman equation

\[
\text{Maximize} \quad C(c^v_i, t) + \mathbb{E} \left[ \frac{d J^v(W^v; D, \hat{\mu}^v, \hat{\phi}^v; t)}{dt} \bigg| \mathcal{F}^v(t) \right] = 0
\]

subject to the budget constraint \(dW^v = rW^v dt + \lambda^v_i d\Omega_t - c^v_i dt\) and the transversality condition \(\lim_{t \to \infty} [e^{-r t} J^v(W^v; D, \hat{\mu}^v, \hat{\phi}^v; t)] = 0\) where \(J^v(W^v; D, \hat{\mu}^v, \hat{\phi}^v; t)\) denotes the agent’s value function.

**Theorem 2.** The Bellman equation given by Eq. (15) has a solution of the form

\[
J^v(W^v; D, \hat{\mu}^v, \hat{\phi}^v; t) = \frac{1}{\gamma r} \exp[-\gamma^v r W^v_t - M^v(\hat{\phi}^v_t)],
\]

where \(M^v(\hat{\phi}^v_t)\) is a quadratic in the argument \(\hat{\phi}^v_t\), namely

\[
M^v(\hat{\phi}^v_t) = \frac{1}{2} m_{1}^v (\hat{\phi}^v_t)^2 + m_{2}^v (\hat{\phi}^v_t) + m_{0}^v.
\]

The agent’s optimal stock demand is given by

\[
x^v_i = \frac{x_{0}^v}{\gamma^v r \tilde{s}^2} + \frac{x_{1}^v}{\gamma^v r \tilde{s}^2} (\hat{\phi}^v_t),
\]

where \(x_{0}^v = (a_2 s_2 - r s_3 - r s_0) + (r + a_3) s_3 - \sigma_a m_{1}^v\) and \(x_{1}^v = -(r + a_3) s_3 - \sigma_a m_{1}^v\).

**Proof:** This is along the lines of Merton (1971) and Theorem 3.2 of Campbell and Kyle (1993). Appendix B provides the formal proof. \(\Box\)

The agent’s optimal stock demand consists of two parts. The first part \(x_{0}^v/\gamma^v r \tilde{s}^2\) is a constant and depends only on the risk aversion and the precision of the private signal. On the other hand, the second part \((x_{1}^v/\gamma^v r \tilde{s}^2) \hat{\phi}^v_t\) depends not only on the risk aversion and the precision of private information, but also on the agent’s estimate of the investment opportunities, \(\mathbb{E} [\Omega_t | \mathcal{F}^v(t)]\), through his estimate \(\hat{\phi}^v_t\).

In general, at any time \(t\), because of the non-zero realization of the noise in the private signal, every agent is either optimistic (\(\hat{\mu}^v > \mu_t\)) or pessimistic (\(\hat{\mu}^v < \mu_t\)) about the investment opportunities. Intuitively, one expects the attractiveness of the risky asset to be greater and consequently the agent’s stock demand to be greater, the greater the agent’s estimate of the growth rate of dividends \(\hat{\mu}^v\). The agent’s stock demand function given by Eq. (18) is consistent with this intuition in the following way. Recall from Eq. (9) that \(d\xi_t = s_2 d\mu_t + s_3 d\phi_t = s_2 d\hat{\mu}^v_t + \)

\(^{17}\) See Theorem 4.4 of Wang (1993).
s_3 \, d\phi_t^\nu, \ \forall t \ \text{and} \ s_3 \ \text{is negative. Therefore, whenever an agent's estimate} \ \hat{\mu}_t^\nu \ \text{is greater (less) than the true value of} \ \mu_t, \ \text{the agent's estimate of the aggregate quantity of stock} \ \hat{\phi}_t^\nu \ \text{is also greater (less) than the true value of} \ \phi_t. \ \text{Thus, an agent's optimal stock demand given by Eq. (18) is directly proportional to his perception of the attractiveness of the stock relative to the risk free rate.}

The agent's optimal stock demand function has three noteworthy properties. First, it is linear in the estimate of the state variable \( \hat{\phi}_t^\nu \). This is due to the assumption of constant absolute risk aversion preferences. Second, for a given allocation of information in the economy, the value of the coefficients \( x_0^\nu \) and \( x_1^\nu \) are individual-agent-specific constants and therefore independent of the realizations of the estimates. Finally, because of risk aversion an agent's optimal stock demand remains bounded even with infinitely precise private information.\(^\text{18}\)

Having solved the learning and optimization problems of the individual agents, I now need to impose the market clearing condition which will enable me to determine the equilibrium stock price function coefficients \( s_0 \) and \( s_3 \).

(Step 3). This step involves aggregation of stock demands of all agents in this economy. Before doing this, I need to introduce some notation which by now is standard in the information economics literature (see Admati, 1985, pp. 635–636) and which primarily has to do with the so-called continuum of independently and identically distributed random variables problem referred to in the introduction.

Recall from Assumption 4 that, in this economy the pure noise component in an agent's private signal \( dz^\nu_t \) is a fresh realization of the same Wiener process \( z_t \). Therefore, when one looks at the noise realizations in the private signals of all the agents, one gets a continuum of independent random variables \( dz^\nu_t \).

In general, if \((\hat{Y}^\nu)_t \in [0,1]\) is a process of independent random variables, then realizations of the process (as a function of \( v \)) need not be measurable and, thus, the Lebesgue integral \( \int_0^1 \hat{Y}^\nu \, dv \) is not well defined. Suppose, however, that \( E(\hat{Y}^\nu) = 0 \) for all \( v \), and that \( \text{Var}(\hat{Y}^\nu) \) is uniformly bounded. Then for every sequence \( \{v_i\} \) of different indices from \([0,1]\), the strong law of large numbers applied to the sequence \((\hat{Y}^\nu)\) yields that \((1/N)\sum_{i=1}^N \hat{Y}^\nu_t \to 0\), almost surely. Therefore, the natural counterpart of this notion in my economy with a continuum of i.i.d. random variables is to define \( \int_0^1 \hat{Y}^\nu \, dv \equiv 0 \).

In the light of this notion, I will adopt a slightly more general convention: If \((\hat{Y}^\nu)_v \in [0,1]\) are independent, with zero mean and bounded variance and \((\hat{Z}^\nu)_v \in [0,1]\) is almost surely integrable, then \( \int_0^1 (\hat{Y}^\nu + \hat{Z}^\nu) \, dv = \int_0^1 \hat{Z}^\nu \, dv \). In my economy, for example, this convention implies that \( \int_0^1 d\pi_t^\nu \, dv = d\mu_t \), almost surely, as one would intuitively expect.

\(^{18}\)In this economy perfect knowledge of the current values of state variables does not induce the risk averse agents to take large enough positions in the stock to cause fully revealing prices. This happens because the agents are risk averse and the future payoff on the stock is stochastic.
Having introduced the notation needed for aggregating the stock demand of a continuum of agents, I now proceed with the final step of market clearing.

3.5. Stock market clearing

Under the assumed form of the price function, each agent’s stock demand is given by Eq. (18). In order for the market to clear at every instant, the sum total of all agents’ stock demands must equal \( \phi_t \), i.e.,

\[
\phi_t = \int_0^1 x_t^{*r} \, dv
\]

\[
= \int_0^1 x_0^{*r} + x_t^{*r} \hat{\phi}_t^{*r} \, dv
\]

\[
= \int_0^1 x_0^{*r} + x_t^{*r} \left[ \phi_t + (\hat{\phi}_t^{*r} - \phi_t) \right] \, dv
\]

\[
= \int_0^1 \frac{x_0^{*r}}{\gamma^{*r} \sigma_\delta^2} \, dv + \int_0^1 \frac{x_t^{*r} \phi_t}{\gamma^{*r} \sigma_\delta^2} \, dv + \int_0^1 \frac{x_t^{*r} (\hat{\phi}_t^{*r} - \phi_t)}{\gamma^{*r} \sigma_\delta^2} \, dv.
\]  

(19)

Notice that the agent’s estimate \( \mathbb{E}[\phi_t | \mathcal{F}^t(t)] = \hat{\phi}_t^{*r} \) is written as a sum of two components: one being the true current value of the state variable \( \phi_t \) and the other being an estimation error \( (\hat{\phi}_t^{*r} - \phi_t) \). This break up enables me to segregate the impact of the information and the noise in the private signal on the agent’s stock demand.

In a single-period model, Hellwig has shown that the part of agents’ demand caused by the estimation error gets washed out during the aggregation process. To generalize this result to my dynamic economy, I need to show that the third integral on the right-hand side of Eq. (19) equals zero. Since this is not obvious, in Theorem 3 below I prove that, indeed, the impact of the individual specific noise in the agents’ private signals gets washed out during the aggregation process.

**Theorem 3.** The net impact of the realizations of the individual specific noise on the sum of the stock demands of all the agents equals zero, almost surely.

**Proof.** For simplicity, only a heuristic argument is provided here. Readers are requested to see Anderson (1990) for the applicability of the law of large numbers in case of a continuum of agents.

Recall that given an allocation of information in the economy, \( x_t^{*r}, \gamma^{*r}, r \) and \( \sigma_\delta^2 \) are finite constants which are independent of the realizations of the noise component in the private signals. From Section 3.2, note that every agent’s estimation error \( (\hat{\phi}_t^{*r} - \phi_t) \) is a finite linear multiple of the realization of the noise component in that agent’s private signal, namely, \( \sigma_\delta^{*r} dz_\delta^{*r} \). Thus, \( x_t^{*r} (\hat{\phi}_t^{*r} - \phi_t) / \gamma^{*r} \sigma_\delta^2 \) can be written as \( \kappa^{*r} \sigma_\delta^{*r} dz_\delta^{*r} \) where \( \kappa^{*r} \in (0, \mathcal{H}) \) is some finite constant which is
specific to agent \( v \). Clearly, \( \kappa^v \) is also independent of the realization of the noise in the private signal. Therefore, the values of \( \kappa^v \sigma^v_x \ dz^v_{x_t} \) for different agents at any time \( t \) are simply mutually independent realizations of a random variable with zero mean and a finite variance.

Since \( x_t^v(\hat{\phi}_t^v - \phi_t) / \gamma^v \sigma^v_x \ dz^v_{x_t} \) equals \( \kappa^v \sigma^v_x \ dz^v_{x_t} \), in order to prove that \( \int_0^1 \ k_t^v(\hat{\phi}_t^v - \phi_t) / \gamma^v \sigma^v_x \ dz^v_{x_t} \ dv \) equals zero, it suffices to show that \( \int_0^1 \ k_t^v \sigma^v_x \ dz^v_{x_t} \ dv = E_v [k^v \sigma^v_x \ dz^v_{x_t}] \) equals zero. But in order for \( E_v [k^v \sigma^v_x \ dz^v_{x_t}] \) to equal zero, the following three conditions must be satisfied.

(i) the scalar multiples of the realizations of noise components in the agents’ private signals, \( \kappa^v \sigma^v_x \ dz^v_{x_t} \), must be mutually independent of \( v \in [0, 1] \).

(ii) these scalar multiples must have an expected value of zero, i.e., \( E(\kappa^v \sigma^v_x \ dz^v_{x_t}) \) must equal zero for all \( v \).

(iii) the variance of these scalar multiples must be bounded, i.e., \( \kappa^v \sigma^v_x \) must be finite for all \( v \).

Since all these three conditions are satisfied, \( \int_0^1 \ k_t^v \sigma^v_x \ dz^v_{x_t} \ dv \) equals zero, almost surely, i.e., the sum total of the scalar multiples of the realizations of noise components in the private signals of all the agents equals zero at every instant.

Intuitively, the argument behind the proof is as follows. Every agent observes the dividend \( D_t \) and the market clearing stock price \( S_t \). From Eq. (9), this is equivalent to the agent observing the linear combination \( \tilde{\xi}_t = s_2 \mu_t + s_3 \phi_t \). Using the private information \( \pi^v_t \) (which is a noisy signal of \( \mu_t \)), every agent rationally decomposes the observed linear combination \( \tilde{\xi}_t \) into two parts. One due to the growth rate of dividends \( \mu_t \) and other due to the aggregate quantity of stock \( \phi_t \). In general, the noise in every agent’s private signal introduces an error in the agent’s estimates \( E[\mu_t | \mathcal{F}^t(t)] = \hat{\mu}_t^v \) and \( E[\phi_t | \mathcal{F}^t(t)] = \hat{\phi}_t^v \) which in turn affects the agent’s optimal stock demand. However, due to the zero mean nature of the noise, it gets washed out during aggregation process and only the part that is common to many signals continues to exist.

Note that the intuition behind this result remains valid even if the Wiener process causing the noise \( z^v_x \) were to be correlated with \( z_D, z_{\mu} \) or \( z_{\phi} \). In such a case, the Kalman filter would take into account the correlation through the matrix \( C \) of Eq. (A.3) and one would still observe that an agent’s estimation error is an innovation with respect to his information set \( \tilde{\mathcal{F}}^t(t) \). This is similar to orthogonalizing the stochastic part of the signal into two components: one correlated to the underlying processes and the other a white noise component. Since it is the latter which causes the estimation error, one obtains the same result as Theorem 3 above.

\[10\] This follows since \( E_v (dz^v_x) = 0 \).
Notice that Theorem 3 has enabled me to get rid of the pure noise in the aggregation process at any given instant. However, in order to make sure that the market clearing condition does hold at every instant during the dynamic evolution of this economy, I need to show that the averages of the estimates of all the agents do indeed evolve as the true underlying state variables. I show this in Corollary 1.

**Corollary 1.** d \( \int_0^1 \hat{\mu}_t \, dv \) evolves as d\(\mu_t\) and d \( \int_0^1 \hat{\phi}_t \, dv \) evolves as d\(\phi_t\).

The intuition behind the corollary is as follows.\(^{20}\) Consider the above economy but with a large but finite number of agents, say N, who behave competitively as price takers.\(^{21}\) Given that all the agents observe the same public information, their estimates of the state variables differ only due to the different realizations of the noise in their private information. The expected values of the estimates of the N agents equal, by definition, \((1/N) \sum_{i=1}^{N} \hat{\mu}_t^i\) and \((1/N) \sum_{i=1}^{N} \hat{\phi}_t^i\). These can be obtained by solving a Kalman filtering problem in which one has N observations (based on \(\mathcal{F}^1(t), \mathcal{F}^2(t), \ldots, \mathcal{F}^N(t)\)) which are linear functions of a common component perturbed by N realizations of zero mean i.i.d. noise. For this problem, it is easy to show that the estimation error covariance matrix (Eq. (A.5)) approaches zero as \(N\) grows large. This implies that for large \(N\), \((1/N) \sum_{i=1}^{N} \hat{\mu}_t^i \rightarrow \mu_t\) and \((1/N) \sum_{i=1}^{N} \hat{\phi}_t^i \rightarrow \phi_t\). Using the notation for the case of a continuum of agents, this implies that \(\int_0^1 \hat{\mu}_t \, dv \rightarrow \mu_t\), and \(\int_0^1 \hat{\phi}_t \, dv \rightarrow \phi_t\) or (d \(\int_0^1 \hat{\mu}_t \, dv\), d \(\int_0^1 \hat{\phi}_t \, dv\)) evolves as (d\(\mu_t\), d\(\phi_t\)). \(\square\)

Notice that by Theorem 3, I have managed to get rid of the effect of the noise in the private signals. By Corollary 1, I have shown that this noise elimination will hold at every instant during the dynamic evolution of my economy. Although I still need to obtain the coefficients \(s_0\) and \(s_3\) in the stock function, I can write the simplified market clearing condition as

\[
\phi_t = \int_0^1 \frac{x_0^r}{\gamma^r r \sigma_S^2} \, dv + \int_0^1 \frac{x_1^r \phi_t}{\gamma^r r \sigma_S^2} \, dv.
\]

(20)

Note that the market clearing condition involves integrating over all the agents. So, along the lines of Hellwig (1980, Proposition 5.2) and Admati (1985, Eqs. (4) and (5)) I express the market clearing condition in terms of the risk attitudes and quality of learning of a representative agent. In particular, using Eq. (18),

\(^{20}\) I am grateful to Mark H.A. Davis for suggesting this simpler approach.

\(^{21}\) As \(N\) gets large, there will be a continuum of agents who, being infinitesimally small, will not explicitly take into account the effect of their demand on prices.
I define the following aggregates for the economy:

$$
\int_0^1 \frac{1}{\gamma^y} \, dv = \Theta, \quad \int_0^1 \frac{m_1^y}{\gamma^y} \, dv = \Psi, \quad \int_0^1 \frac{m_2^y}{\gamma^y} \, dv = \bar{\gamma}, \quad \int_0^1 \sigma_{\bar{\gamma}} \, dv = \bar{\Sigma}_{\bar{\gamma}}.
$$  (21)

Clearly, $\Theta, \Psi$, and $\bar{\gamma}$ are functions of the agents’ risk attitudes and $\bar{\Sigma}_{\bar{\gamma}}$ is a function of the precisions of the private signals. Since the precision of the private signal and the agent’s risk attitude are independent, it follows that $\Theta, \Psi$ and $\bar{\gamma}$ are independent of $\bar{\Sigma}_{\bar{\gamma}}$.

Note that the market clearing condition (Eq. (20)) holds for all realizations of $\phi_t$. Since the right-hand side of Eq. (20) consists of a constant and a multiple of $\phi_t$, in order for the market to clear at every instant the constant must equal zero and the coefficient of $\phi_t$ must equal unity. Substituting $\bar{z}_0^t$ and $\bar{z}_1^t$ from Eq. (18) and letting $\bar{\epsilon} = a_2 b s_2 - s_3 - r s_0$, the market clearing condition can be written as

$$  \bar{\epsilon} \Theta - \bar{\Sigma}_{\bar{\epsilon}} \bar{\gamma} = 0,  \tag{22} $$

$$  - (r + a_3) s_3 \Theta - \bar{\Sigma}_{\bar{\epsilon}} \bar{\gamma} = r \sigma^2_{s_3}.  \tag{23} $$

Now that I have expressed the market clearing condition in terms of the single representative agent of this economy, I need to solve for the coefficients of the value function of the representative agent. In order to do this, it is straightforward to repeat the learning and optimization exercise for the representative agent along the lines of Appendices A and B. Let $\bar{m}_2, \bar{m}_1$ and $\bar{m}_0$ be the coefficients of the value function and let $\bar{\rho}$ be the rate of time preference of the representative agent. Let $\sigma^2_{\bar{\phi}}$ be the variance of the representative agent’s estimate of the aggregate stock and let $\sigma_{\bar{\phi} \bar{m}}$ be the covariance of the estimate with stock price. Then the following equations which are counterparts of Eqs. (B.8)–(B.10) are satisfied:

$$ 0 = \sigma^2_{s_3} \{(r + 2 a_3) \bar{m}_2 + \sigma^2_{\bar{\epsilon}} \bar{m}_2^2\} - \{r s_3 + a_3 s_3 + \sigma_{\bar{\epsilon} \bar{\gamma}} \bar{m}_2\}^2,  \tag{24} $$

$$ 0 = \sigma^2_{s_3} \{(r + a_3) \bar{m}_1 + \sigma^2_{\bar{\epsilon}} \bar{m}_1^2 - a_3 \bar{m}_2\} + \{r s_3 + a_3 s_3 + \sigma_{\bar{\epsilon} \bar{\gamma}} \bar{m}_2\} \{\bar{\epsilon} + (r + a_3) s_3 - \sigma_{\bar{\epsilon} \bar{\gamma}} \bar{m}_1\},  \tag{25} $$

$$ 0 = \sigma^2_{s_3} [2(r - \bar{\rho} + r \bar{m}_0 - a_3 \bar{m}_1) + \sigma^2_{\bar{\epsilon}} (\bar{m}_1^2 - \bar{m}_2^2)] - \{\bar{\epsilon} + (r + a_3) s_3 - \sigma_{\bar{\epsilon} \bar{\gamma}} \bar{m}_1\}^2.  \tag{26} $$

Eqs. (22)–(26) comprise a system of five equations in five unknowns, solution to which provides $s_0$ and $s_3$ the coefficients in the price function as well as $\bar{m}_2, \bar{m}_1$ and $\bar{m}_0$ the coefficients of the representative agent’s value function. From the works of Campbell and Kyle (1993), one knows that there exists a solution
to this system of equations. This completes the final step and along with it the proof of Proposition 1.

To summarize, in this section, I generalized Hellwig’s results in a static world to a dynamic set up by showing the existence of a stationary rational expectations equilibrium in the following way. First, I conjectured the equilibrium stock price function (Proposition 1). Based on the conjectured price function, I solved the learning problem of each agent (Theorem 1) and arrived at the best estimates of the underlying state variables given an agents information set. Then, using these estimates, I solved the optimal control problem of each agent (Theorem 2). Finally, I aggregated the optimal stock demands of all agents and imposed the market clearing condition. In Theorems 3, I showed that the impact of the noise realizations in the private signals gets washed out in the aggregation process at every instant. Finally, by restating the problem in terms of a representative agent of the economy, I solved for the equilibrium stock price coefficients.

4. Concluding remarks

This article, in the spirit of Hellwig (1980), examined how a dynamic competitive security market communicated the diverse information held by various market participants at every instant. It demonstrated the existence of a noisy rational expectations equilibrium in which the agents used the information contained in the market prices without rendering their private information redundant and thus extended Hellwig’s single period insights to a multi-period model. In particular, it showed that the market price reflects only those elements of information which are common to the private signals of a large number of agents.

This article is a first step in the study of issues relating to aggregation of information in dynamic competitive markets. It provides a platform to address many issues in information economics. For example, in the spirit of Admati (1985) it enables one to specify many risky assets and examine the complex interactions which arise only in case of many risky assets. Alternately, along the lines of Admati and Pfleiderer (1986), it permits the study of the optimal strategy of an information seller having superior information in a multi-period economy (see, for example, Naik, 1996). Furthermore, by allowing for correlation among the signals of several investors, it allows one to model issues relating to mispricing of stocks, panics and crashes. Finally, it also enables one to address several other information-related issues like insider trading, volume of trade, and direct versus indirect sale of information in a dynamic yet tractable framework.

This work can be generalized in many directions. For tractability reasons, the attention was focused on the equilibrium stock price function along the lines of Campbell and Kyle (1993) and Wang (1993). It is possible that this conjectured form although intuitively appealing may only be a subset of admissible price functions in the ‘guess and solve’ method of solving for rational expectations
equilibria. It would be interesting to specify the stock price process in general terms and examine conditions required for the markets to clear. It would also be interesting to specify an economy consisting of few agents who 'forecast the forecast of others' and study the stock price behaviour. Insights provided by these generalizations would be of great help in improving our understanding of information economics issues in a dynamic context.

Appendix A

A.1. Optimal filtering specifications

The agents use the Kalman filter to obtain the best conditional estimates of the current values of underlying state variables (see Jazwinski, 1970).

Let the continuous time linear system of unobservable state variables be described by the vector (Itô) stochastic differential equation

\[ dz_t = (a + Fz_t) \, dt + G \, dz_t, \]

where \( z_t \) is \( n \times 1 \) vector of state variables, \( dz_t \) is a \( k \times 1 \) vector Wiener process, and \( a \) is an \( n \)-vector of constants, while \( F \) and \( G \) are conformable matrices of constants. The agents observe signals given by the vector (Itô) equation

\[ dy_t = (b + Mz_t) \, dt + d\psi_t, \]

where \( y_t \) is \( m \times 1 \) vector of observable signals, \( d\psi_t \) is a \( m \times 1 \) vector Wiener process, and \( b \) is an \( m \)-vector of constants, while \( M \) is a conformable matrix of constants.\(^{22}\) Also let

\[ \mathbb{E}(d\xi_t d\xi_t^T) = Q \, dt, \quad \mathbb{E}(d\psi_t d\psi_t^T) = R \, dt, \quad \mathbb{E}(d\xi_t d\psi_t^T) = C \, dt, \]

where \( Q, R \) and \( C \) are matrices of constants of conformable dimensions.

The estimates of the unobservable underlying state variables (\( \hat{z}_t \)) is given by the vector (Itô) stochastic differential equation

\[ d\hat{z}_t = (a + F\hat{z}_t) \, dt + [P_t M^T + G G^T] R^{-1} [dy_t - (b + M\hat{z}_t) \, dt], \]

where \( P_t = \mathbb{E}[(z_t - \hat{z}_t)(z_t - \hat{z}_t)^T] \) is the estimation error covariance matrix and is a solution of the Riccati equation

\[ \dot{P}_t = FP_t + P_t F^T + G G^T - [P_t M^T + G C] R^{-1} [M P_t + C C^T G^T], \]

where \( \dot{P}_t \) is the rate of change of \( P_t \) with respect to time.

In order for the Kalman filter to be asymptotically time invariant, certain regularity conditions (boundedness, Lipschitz, etc.) need to be satisfied by the input

\(^{22}\) This \( M \) is simply a matrix and is unrelated to the value function of the agent in Theorem 2.
variables. One can then use the Liapunov function approach to show that the economy will reach a steady state that is independent of the initial conditions. Given the stationary mean reverting nature of the Ornstein–Uhlenbeck processes and the constancy of parameters, I directly solve for the steady-state value of \( P_t \) (by setting \( \dot{P}_t \) to equal zero) and request the readers to Kushner (1967) for a more formal treatment of stability issues.

A.2. Agent’s learning problem

This is a generalization of Theorem 4.1 of Wang (1993). Unlike my economy, the uninformed investors in Wang’s model do not have a private signal which they can use to decompose optimally the linear combination of \( \mu_t \) and \( \phi_t \).

In this case \( z_t = (\mu_t, \phi_t)^T \) and \( y_t = (P_t, \tilde{z}_t, \tilde{z}_l)^T \). Given the equilibrium price function in this economy, \( \tilde{z}_t = s_2 \mu_t + s_3 \phi_t \) is in the information set of all agents. \( \tilde{z}_t \) evolves as

\[
d\tilde{z}_t = [s_2 a_2 (b - \mu_t) + s_3 a_3 (1 - \phi_t)] dt + \sigma_{\tilde{z}} d\tilde{z}_t,
\]

where \( \sigma_{\tilde{z}} d\tilde{z}_t = s_2 \sigma_{\mu} d\mu_t + s_3 \sigma_{\phi} d\phi_t \). \( \) From Eqs. (1)–(4), I have

\[
\begin{align*}
\alpha &= \begin{pmatrix} a_2 b \\ a_3 \end{pmatrix}, & \beta &= \begin{pmatrix} -a_1 kD \\ a_2 b s_2 + a_3 s_3 \end{pmatrix}, \\
F &= \begin{pmatrix} -a_2 & 0 \\ 0 & -a_3 \end{pmatrix}, & G &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, & M &= \begin{pmatrix} a_1 & 0 \\ -a_2 s_2 & -a_3 s_3 \end{pmatrix}, \\
\end{align*}
\]

\[
d\tilde{z}_t = \begin{pmatrix} \sigma_{\mu} d\mu_t \\ \sigma_{\phi} d\phi_t \end{pmatrix}, \quad d\psi_t = \begin{pmatrix} \sigma_D d\tilde{\psi}_t \\ \sigma_{\tilde{z}} d\tilde{z}_t \\ \sigma_{\tilde{z}} d\tilde{z}_t \end{pmatrix}
\]

where \( \sigma_{\tilde{z}} d\tilde{z}_t = \sigma_{\mu} d\mu_t + \sigma_{\phi} d\phi_t \) and \( (\sigma_{\tilde{z}})^2 = \sigma_{\mu}^2 + (\sigma_{\phi})^2 \). From Eq. (A.3) I get

\[
Q = \begin{pmatrix} \sigma_{\mu}^2 & 0 \\ 0 & \sigma_{\phi}^2 \end{pmatrix}, \quad R = \begin{pmatrix} \sigma_D^2 & 0 & 0 \\ 0 & \sigma_{\tilde{z}}^2 & 0 \\ 0 & 0 & (\sigma_{\tilde{z}})^2 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & s_2 \sigma_{\mu}^2 & \sigma_{\mu}^2 \\ 0 & s_3 \sigma_{\phi}^2 & 0 \end{pmatrix}.
\]

Let the estimation error covariance matrix of an agent \( v \) be (see (A.4) and (A.5))

\[
P_{v}^{\infty} = \lim_{t \to \infty} P_{v,t}^{\infty} = \lim_{t \to \infty} E_{t}^{v}[(z_t - \tilde{z}_t^v)^T, (z_t - \tilde{z}_l^v)^T] = \begin{pmatrix} p_{22}^v & \cdots & p_{23}^v \\ \cdots & \cdots & \cdots \\ p_{23}^v & \cdots & p_{33}^v \end{pmatrix}
\]
where

\[ P_{22}^v = \text{Variance}(\hat{\mu}_v - \mu), \quad P_{23}^v = \text{Covariance}\{(\hat{\mu}_v - \mu), (\hat{\phi}_v - \phi)\}, \]

\[ P_{33}^v = \text{Variance}(\hat{\phi}_v - \phi). \]

Then from Eq. (9), \( P_{22}^v = (s_3/s_2)^2 P_{33}^v \) and \( P_{23}^v = -(s_2/s_3)P_{22}^v \).

Let \( \mathcal{D} = \sigma_\varepsilon^2 (\sigma_\mu^2 + \sigma_\mu^2)^2 - s_2^2 \sigma_\mu^4, \ a_4 = s_2 \sigma_\mu^2 - a_2 s_2 P_{22}^v, \ a_5 = s_3 \sigma_\phi^2 - a_2 s_2 P_{22}^v, \ a_3 = s_3 P_{23}^v \) and \( a_5 = s_3 \sigma_\phi^2 - a_2 s_2 P_{22}^v - a_3 s_3 P_{33}^v \). Define

\[ h_{21}^v = \frac{a_1 P_{22}^v}{\sigma_D^2}, \quad h_{2\pi}^v = \frac{a_4 (\sigma_\mu^2 + \sigma_\mu^2) - s_2 \sigma_\mu^2 (\sigma_\phi^2 - a_2 P_{22}^v)}{\mathcal{D}}, \]

\[ h_{2\pi}^v = \frac{-a_4 s_2 \sigma_\phi^2 + (\sigma_\mu^2 - a_2 P_{22}^v) \sigma_\phi^2}{\mathcal{D}}, \]

\[ h_{31}^v = \frac{a_1 P_{23}^v}{\sigma_D^2}, \quad h_{3\pi}^v = \frac{a_5 (\sigma_\mu^2 + \sigma_\mu^2) + a_2 s_2 P_{23}^v \sigma_\mu^2}{\mathcal{D}} \quad \text{and} \]

\[ h_{3\pi}^v = \frac{-a_5 s_2 \sigma_\mu^2 - a_2 P_{23}^v \sigma_\mu^2}{\mathcal{D}}. \]

Under steady-state conditions, the elements of matrix \( P_\infty^v \) satisfy the Riccati equation (A.5) and by definition the following holds:

\[ s_2 h_{21}^v + s_3 h_{31}^v = 0, \quad s_2 h_{2\pi}^v + s_3 h_{3\pi}^v = 1 \quad \text{and} \quad s_2 h_{2\pi}^v + s_3 h_{3\pi}^v = 0. \quad (A.7) \]

The estimates of an agent \( v \) with the information set \( \mathbb{F}^v(t) \) evolve as

\[ \begin{pmatrix} d\hat{\mu}_v^r \\ d\hat{\phi}_v^r \end{pmatrix} = \begin{pmatrix} a_2 [b - \hat{\mu}_v^r] \\ a_3 [1 - \hat{\phi}_v^r] \end{pmatrix} dt + \begin{pmatrix} h_{21}^v \\ h_{2\pi}^v \end{pmatrix} d\hat{\phi}_v^r(t), \quad (A.8) \]

where

\[ d\hat{\phi}_v^r(t) = \begin{pmatrix} d\xi_t - a_2 s_2 (b - \hat{\mu}_v^r) dt - a_3 s_3 (1 - \hat{\phi}_v^r) dt \\ dD_t - a_1 (\hat{\mu}_v^r - kD_t) dt \\ dp_t - a_2 (b - \hat{\mu}_v^r) dt \end{pmatrix}. \quad (A.9) \]

The innovation process of the Kalman filter \( \hat{\phi}_v^r(t) \) is a standard Wiener process with respect to the \( \sigma \)-algebra generated by the agent’s information set \( \mathbb{F}^v(t) \).

The estimates \( \hat{\mu}_v^r \) and \( \hat{\phi}_v^r \) of the agents with more precise private information are more precise compared to those who observe less precise private signals. The smaller is the value of \( \sigma_\varepsilon^2 \), the smaller is the estimation error covariance matrix \( P_\infty^v \). This follows from the property of the Kalman filter. Further, from the stochastic parts of Eqs. (10) and (A.9), the covariance of an agents estimate
with the stock price is given by

$$\sigma_{S,\phi} = h^{r}_{1} s_{1} \sigma_{D}^{2} + h^{r}_{2} \sigma_{x}^{2} + h^{r}_{3} s_{2} \sigma_{\mu}^{2}. \tag{A.10}$$

This term is negative since an increase in the value of aggregate supply decreases the stock price to provide for excess returns for bearing the increased risk in the economy. More precise the private signals, the more is the magnitude of $\sigma_{S,\phi}^{r}$ and the greater is aggressiveness of stock trading by that agent.

**Appendix B. Optimization problem of an individual agent**

This problem is along the lines of Merton (1971) and Campbell and Kyle (1993) Theorem 3.2.\textsuperscript{23} Let $W_{t}, c_{t}$, and $x_{t}$ be the wealth, consumption rate and stock holding (financed by borrowing at the risk-free rate) of an individual agent at time $t$. Let his derived utility of wealth be $J(W; D, \hat{\mu}, \hat{\phi}; t)$.

Every agent solves the Bellman equation

$$0 = \text{Maximize}_{c, x} \left[ u(c_{t}, t) + E_{t} \left\{ \frac{dJ(W; D, \hat{\mu}, \hat{\phi}; t)}{dt} \bigg| \mathcal{F}^{v}(t) \right\} \right] \tag{B.1}$$

subject to the budget constraint

$$dW_{t} = rW_{t} dt + x_{t} [\varepsilon + (r + a_{3}) s_{3} (1 - \hat{\phi}_{t})] dt - c_{t} dt + x_{t} \sigma_{S} dz_{S}, \tag{B.2}$$

where $\varepsilon = a_{2} s_{2} + r s_{3} - r s_{0}$ and the transversality condition

$$\lim_{t \to \infty} [e^{-\rho t} J(W; D, \hat{\mu}, \hat{\phi}; t)] = 0.$$  

Substitute $E_{t}(\phi) = \hat{\phi}_{t}$ in the Bellman equation and let the subscript denote the partial derivative with respect to that argument. Then an agent solves

$$0 = \text{Maximize}_{c, x} \left[ \frac{-e^{-\gamma c}}{\gamma} - \rho J + J_{W} \{rW - c + x[\varepsilon + (r + a_{3}) s_{3} (1 - \hat{\phi}_{t})]\} + a_{3} (1 - \hat{\phi}_{t}) J_{\hat{\phi}} + \frac{1}{2} \begin{pmatrix} J_{WW} & J_{W\phi} \\ J_{W\phi} & J_{\phi\phi} \end{pmatrix} \begin{pmatrix} \sigma_{x}^{2} x^{2} \\ \sigma_{S}^{2} x^{2} \end{pmatrix} \right] \tag{B.3}$$

where $\Box$ sign indicates that corresponding elements of the matrices are multiplied and then summed. Along the lines of Merton (1971), conjecture the form of the value function as

$$J(W; D, \hat{\mu}, \hat{\phi}; t) = -\frac{1}{\gamma r} \exp[-\gamma r W - M(\hat{\phi}_{t})], \tag{B.4}$$

\textsuperscript{23}Since this optimization problem pertains to an individual agent, to ease the notational burden the superscript $v$ is suppressed from $W, c, x, J, M, m_{2}, m_{1}, m_{0}, \hat{\mu}, \phi, \gamma$ and $\rho$ until Eq. (B.6).
where $M(\hat{\phi})$ is a quadratic in the argument $\hat{\phi}$. The first-order conditions are

$$c_t^* = -\frac{\log J_t}{\gamma} = rW_t + \frac{M(\hat{\phi})}{\gamma},$$

$$x_t^* = \frac{\varepsilon + (r + a_3)s_3\hat{\phi} - \sigma_{S\phi}M_\hat{\phi}}{\gamma r\sigma_S^2},$$

where the log function in the optimal consumption function is the natural logarithm to the base $e$. Substitute the optimal controls $c_t^*$ and $x_t^*$ obtained above in Eq. (B.3) and simplify to get

$$0 = r - \rho + rM(\hat{\phi}) + \frac{\sigma_\phi^2}{2} [M_\phi^2 - M_{\phi\hat{\phi}}] - a_3(1 - \hat{\phi})M_\phi$$

$$- \frac{[\varepsilon + (r + a_3)s_3(1 - \hat{\phi}) - \sigma_{S\phi}M_\hat{\phi}]^2}{2\sigma_S^2},$$

(B.5)

where $\varepsilon + (r + a_3)s_3(1 - \hat{\phi})$ is the excess return anticipated by that agent (see Section 3.3).

Conjecture the form of $M^r(\hat{\phi}^r)$ to be

$$M^r(\hat{\phi}^r) = \frac{1}{2}m_2^r(\hat{\phi}^r)^2 + m_1^r(\hat{\phi}^r) + m_0^r.$$  

(B.6)

Then the optimal stock demand of an individual trader is

$$x_t^{r*} = \left[ x_0^r + x_1^r(\hat{\phi}^r) \right] \frac{1}{\gamma r\sigma_S^2},$$

(B.7)

where $x_0^r = \varepsilon + (r + a_3)s_3 - \sigma_{S\phi}m_1^r$ and $x_1^r = -(r + a_3)s_3 - \sigma_{S\phi}m_2^r$.

Substitute (B.6) into (B.5). Since this must hold for all realizations of $\hat{\phi}^r$, equate to zero (which is the left-hand side) the coefficient of $(\hat{\phi}^r)^2$ to get (B.8), coefficient of $(\hat{\phi}^r)$ to get (B.9) and the coefficient of the constant to get (B.10):

$$0 = \sigma_S^2[(r + 2a_3)m_2 + \sigma_\phi^2m_1^2] - \{rs_3 + a_3s_3 + \sigma_{S\phi}m_2\}^2,$$

(B.8)

$$0 = \sigma_S^2[(r + a_3)m_1 + \sigma_\phi^2m_1m_2 - a_3m_2] + \{rs_3 + a_3s_3 + \sigma_{S\phi}m_2\} \times \{\varepsilon + (r + a_3)s_3 - \sigma_{S\phi}m_1\},$$

(B.9)

$$0 = \sigma_S^2[2(r - \rho + rm_0 - a_3m_1) + \sigma_\phi^2(m_1^2 - m_2)] - \{\varepsilon + (r + a_3)s_3 - \sigma_{S\phi}m_1\}^2.$$

(B.10)
Along the lines of Campbell and Kyle (1993), in case of a quadratic, one takes the larger of the two roots which gives the agent higher expected utility of life-time consumption.

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