Interpreting Prediction Market Prices as Probabilities

Justin Wolfers
Wharton – U.Penn,
CEPR, IZA & NBER
jwolfers@wharton.upenn.edu
www.nber.org/~jwolfers

Eric Zitzewitz
Stanford GSB
ericz@stanford.edu
www.stanford.edu/~ericz

LBS Prediction Markets Workshop

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Research Question

- "What is the logical basis for interpreting the price of an all-or-nothing futures contract as a market probability that the event will occur?"
  - Charles Manski
    "Interpreting the Predictions of Prediction Markets", NBER WP #10359
- Further observations:
  - "Researchers engaged in empirical study of prediction markets have been uncomfortably vague."
  - "Recent papers on prediction markets provide no formal analysis showing how such markets aggregate information or opinions"
- Objective: Explore the relationship between prediction market prices and the mean of the distribution of beliefs
  - Rationale: Individual traders receive noisy private signals of the true probability of the event occurring:

$$q_j = \mu + \varepsilon_j$$ ⇒Mean of q fully aggregates info

Model Ingredients

- Commodity: Contract pays $1 if event occurs
- Many traders, each characterized by:
  - *q: S*ubjective beliefs
  - *y:* Wealth
  - *U:* Utility function

  ⇒Implies: Individual demand: $x(\pi; q, y, U)$
- Aggregates to demand/supply functions:
  - π: Price of contract
  - Aggregate Demand and Supply of contracts
- Equilibrium condition: $\sum x(\pi)=0$

Model Ingredients: Manski

- Commodity: Contract pays $1 if event occurs
- Many traders, each characterized by:
  - *q: S*ubjective beliefs — General distribution: ~F(q,y)
  - *y:* Wealth — Orthogonal to beliefs: E[q,y]=0
  - *U:* Utility function — Always bet exactly $y

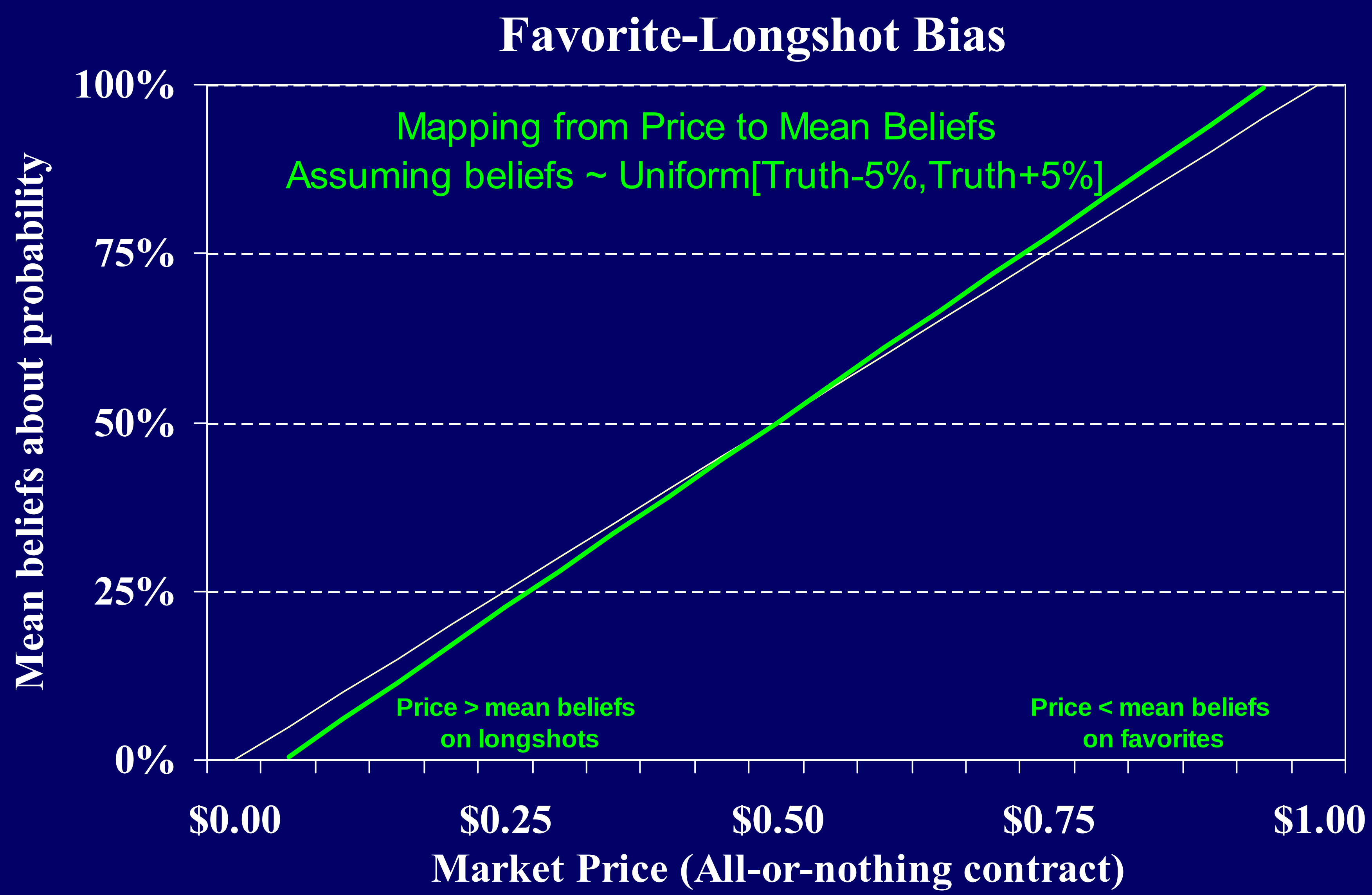
  ⇒Implies: Individual demand: $x(\pi; q, y, U)$

$$x = \begin{cases} y/\pi & \text{if } q > \pi \\ -y/(1-\pi) & \text{if } q < \pi \end{cases}$$

- Aggregates to demand/supply functions:
  - π: Price of contract
  - Aggregate Demand and Supply of contracts
    $1[q>\pi]\ (y/\pi) = 1[q<\pi]\ (y/(1-\pi))$
- Equilibrium condition: $\sum x(\pi)=0$
  - $[1-F(\pi)]/\pi = F(\pi)/(1-\pi)$

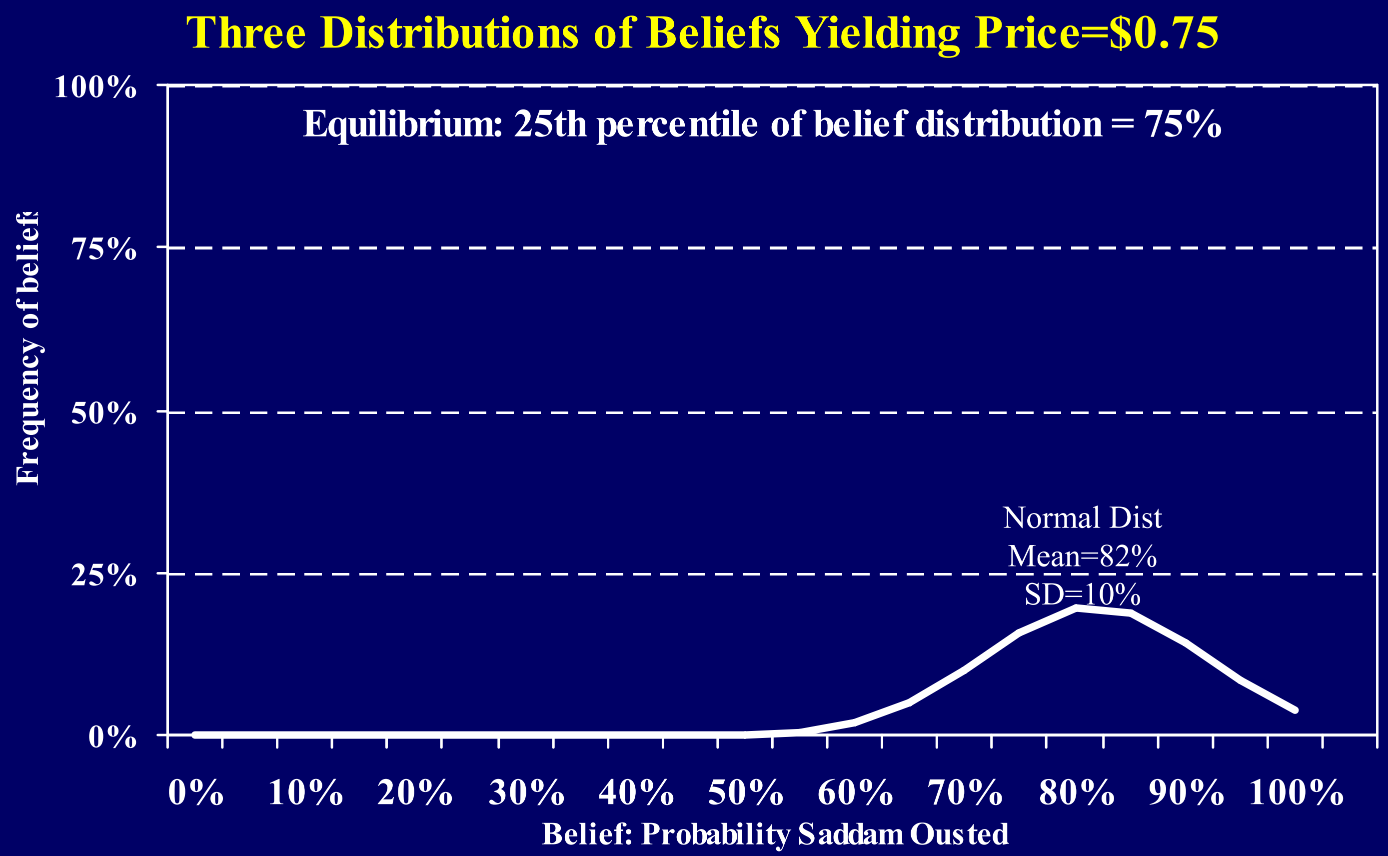
    $\Rightarrow \pi = F(1-\pi)$

What do Prices Say About Beliefs?



Manski Model: Implications

- Example: Saddam Security Price=\$0.75
  ⇒25% of traders believe probability<75%

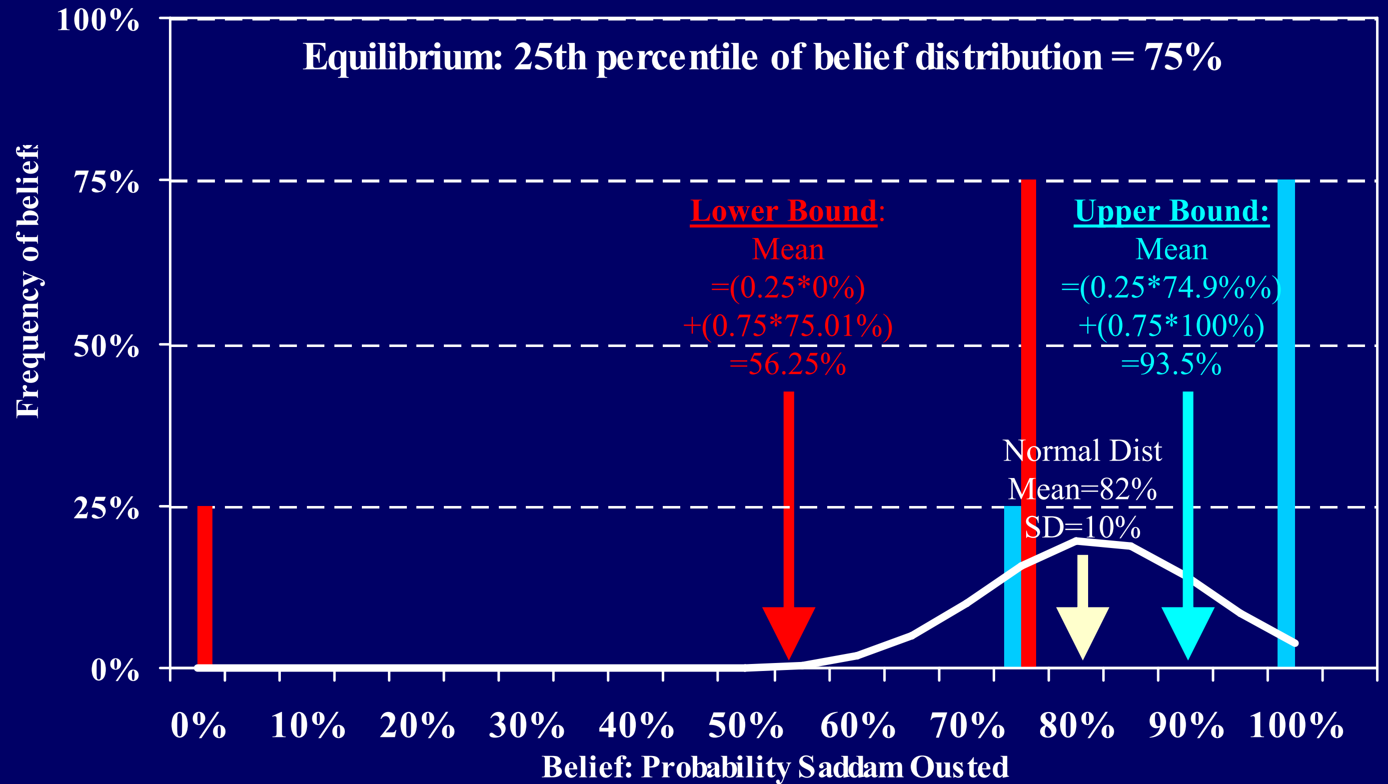


- Note:
  - "Price does not generally equal the mean belief of traders"
  - "π is the midpoint of an interval of width $2(\pi - \pi^2)$ that contains E(q)"

Manski Model: Implications

- Example: Saddam Security Price=$0.75
  ⇒75% of traders believe probability>75%

Three Distributions of Beliefs Yielding Price=$0.75



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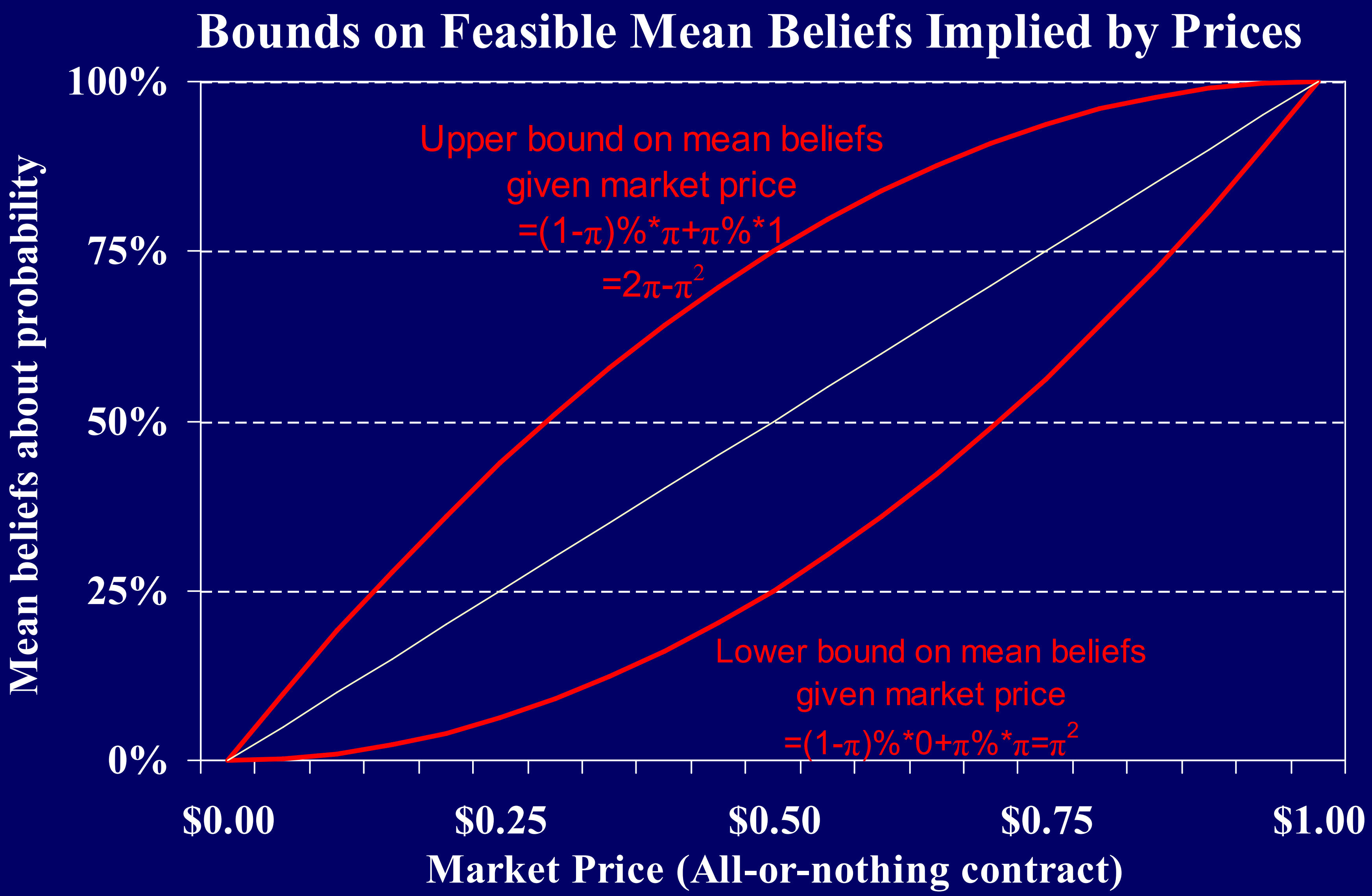


Implications (According to Manski)

- "Refutes the notion that prices in prediction markets are 'market probabilities'"
- "The price of an all-or-nothing futures contract does not equal the mean, median, or any other measure of the central tendency of traders' beliefs"
- "Prices near zero or one are very informative about the mean beliefs of traders, but prices near 0.5 are much less informative"

Model Ingredients: This Paper

- Commodity: Contract pays $1 if event occurs
- Many traders, each characterized by:
  - *q: S*ubjective beliefs — General distribution: ~F(q,y)
  - *y:* Wealth — Consider E[yq]≠0
  - *U:* Utility function — Consider many utility functions

  ⇒Implies: Individual demand: $x(\pi; q, y, U)$

  Endogenize choice of bet size
- Aggregates to demand/supply functions:
  - π: Price of contract
  - Aggregate Demand and Supply of contracts
- Equilibrium condition: $\sum x(\pi)=0$

Simple Model: Log Utility

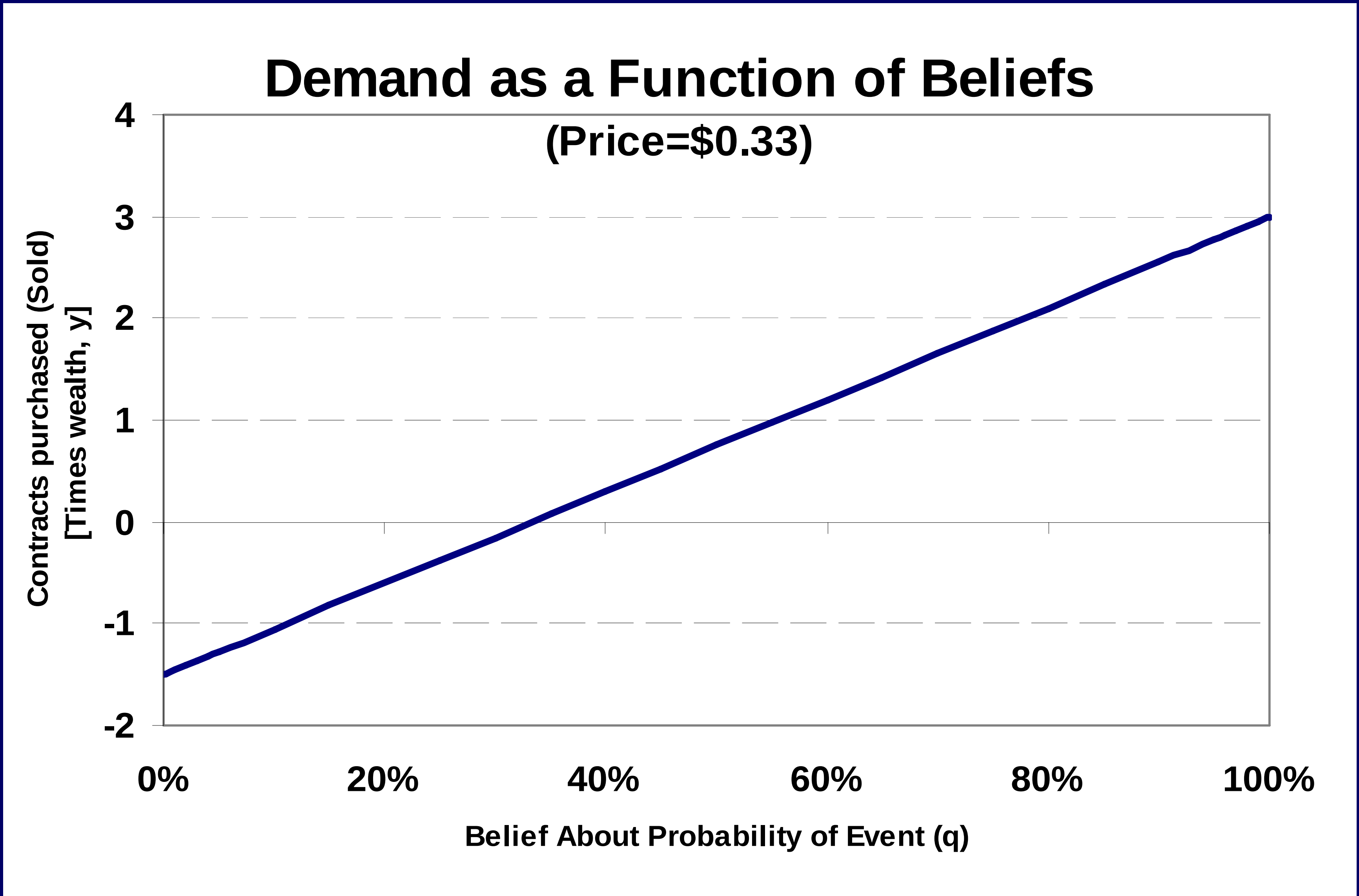
- Individual Demand:

$$\underset{\{x\}}{Max}\ EU_j = q_j Log(y + x_j(1-\pi)) + (1-q_j)Log(y - x_j\pi)$$

$$yielding: \quad x_j^* = y\frac{q_j - \pi}{\pi(1-\pi)}$$

- Individual demand is:
  - *Symmetric* in beliefs relative to price, *q-π*
  - *Linear* in beliefs, *q*
  - *Decreasing in risk*, *π(1-π)*
  - *Homothetic* in income, *y*
  - *Unique* for prices between 0 and 1

Individual Demand (Log Utility)



Equilibrium

- Supply = Demand (Assuming y⊥q)

$$\int_{-\infty}^{\pi} y \frac{q-\pi}{\pi(1-\pi)} f(q)dq = \int_{\pi}^{\infty} y \frac{\pi-q}{\pi(1-\pi)} f(q)dq$$

- Implies: *Price = Mean belief*

$$\pi = \int_{-\infty}^{\infty} q f(q)dq = \overline{q}$$

- And if beliefs (*q*) are correlated with wealth (*y*)

$$\int y \frac{q-\pi}{\pi(1-\pi)} dF(q \le \pi, y) = \int y \frac{\pi-q}{\pi(1-\pi)} dF(q \ge \pi, y)$$

$$\pi = \int q \frac{y}{\overline{y}} dF(q, y)$$

=Wealth-weighted mean belief

Beyond Log Utility

More generally, if

- Individual demand is symmetric for equal-sized deviations of beliefs from market prices:
  $x(q-\pi)=-x(\pi-q))$

- The distribution of beliefs is symmetric:
  $f(q-\overline{q}) = f(\overline{q}-q)$

Implies

- $f(q)x(q)$ is symmetric around $E[q]$

$$f(q-\overline{q})x(q-\pi) = -f(\overline{q}-q)x(\pi-q)$$

$$\int_{-\infty}^{\overline{q}} x(q-\pi)f(q-\overline{q})dq = -\int_{\overline{q}}^{\infty} x(q-\pi)f(q-\overline{q})dq$$

- Supply = Demand
  $\Leftrightarrow$ Price is the mean of beliefs.

General Set-Up

$$\underset{\{x\}}{Max}\ EU_j = q_j U(y + x_j(1-\pi)) + (1-q_j)U(y - x_j\pi)$$

$$FOC: \quad \frac{U'(y + x - \pi x)}{U'(y - \pi x)} = \frac{\pi}{1-\pi}\frac{(1\text{-}q)}{q}$$

$$Implies{:}\ x^* = X(q, \pi, y)\ and\ sign(\frac{\partial X}{\partial \pi}) = -sign(\frac{\partial X}{\partial q})$$

Specific Utility Functions

Table 1: Utility Functions and Demand for Prediction Securities

Utility Function	Utility	Demand
Log utility (CRRA with $\gamma = 1$)	$u(y) = \ln(y)$	$\frac{y}{\pi(1-\pi)}(q-\pi)$
Constant relative risk aversion (CRRA) ($\gamma > 0, \gamma \neq 1$)	$u(y) = \frac{y^{1-\gamma}}{1-\gamma}$	$\frac{y}{\pi} \cdot \left(\frac{\pi\{[\frac{q(1-\pi)}{\pi(1-q)}]^{\frac{1}{\gamma}} - 1\}}{1 + \pi\{[\frac{q(1-\pi)}{\pi(1-q)}]^{\frac{1}{\gamma}} - 1\}} \right)$
Constant absolute risk aversion (CARA)	$u(y) = -e^{-ry}$	$r^{-1} \cdot [\ln(\frac{q}{1-q}) - \ln(\frac{\pi}{1-\pi})]$
Quadratic utility	$u(y) = -\frac{1}{2}(y^{\max} - y)^2$	$(y^{\max} - y)\frac{q-\pi}{q(1-\pi)-\pi(q-\pi)}$
Hyperbolic absolute risk aversion (HARA)*	$u(y) = \frac{1-\gamma}{\gamma}(\frac{ay}{1-\gamma} + b)^{\gamma}$	$\frac{y + b(1-\gamma)a^{-1}}{\pi} \cdot \left(\frac{\pi\{[\frac{\pi(1-q)}{q(1-\pi)}]^{\frac{1}{\gamma-1}} - 1\}}{1 + \pi\{[\frac{\pi(1-q)}{q(1-\pi)}]^{\frac{1}{\gamma-1}} - 1\}} \right)$

* The HARA utility function nests the others as special cases. (For log utility $\gamma \to 0$; risk neutral: $\gamma \to 1$; quadratic: $\gamma = 2$; CRRA: $\gamma < 1$ and b=0; CARA: $\gamma \to -\infty$ and b > 0).

Are Prices ≈ Mean Beliefs?

Evidence:

1. Calibrate the relationship between prices and probabilities based on *plausible* distributions of beliefs
2. Calibrate the relationship between prices and probabilities for different utility functions based on *known* distributions of beliefs:
   - » Election 2004
   - » NFL Football
3. Explore relationship between prices and endogenous bet sizes
   - » Microdata on bet sizes
4. Analyze relationship between prices and average outcomes
   - » Sports: Baseball, horse racing
   - » Politics
   - » Economic Derivatives

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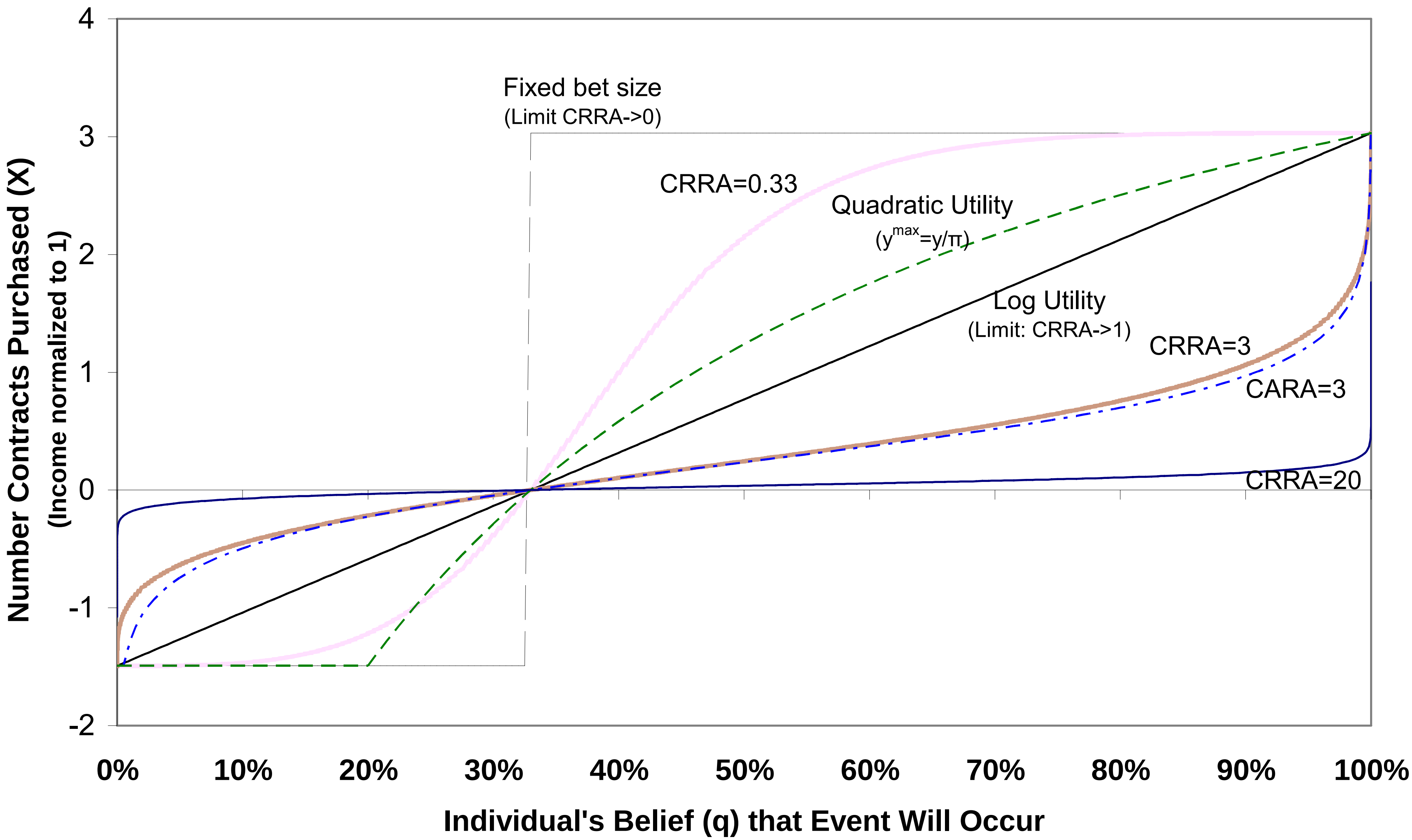
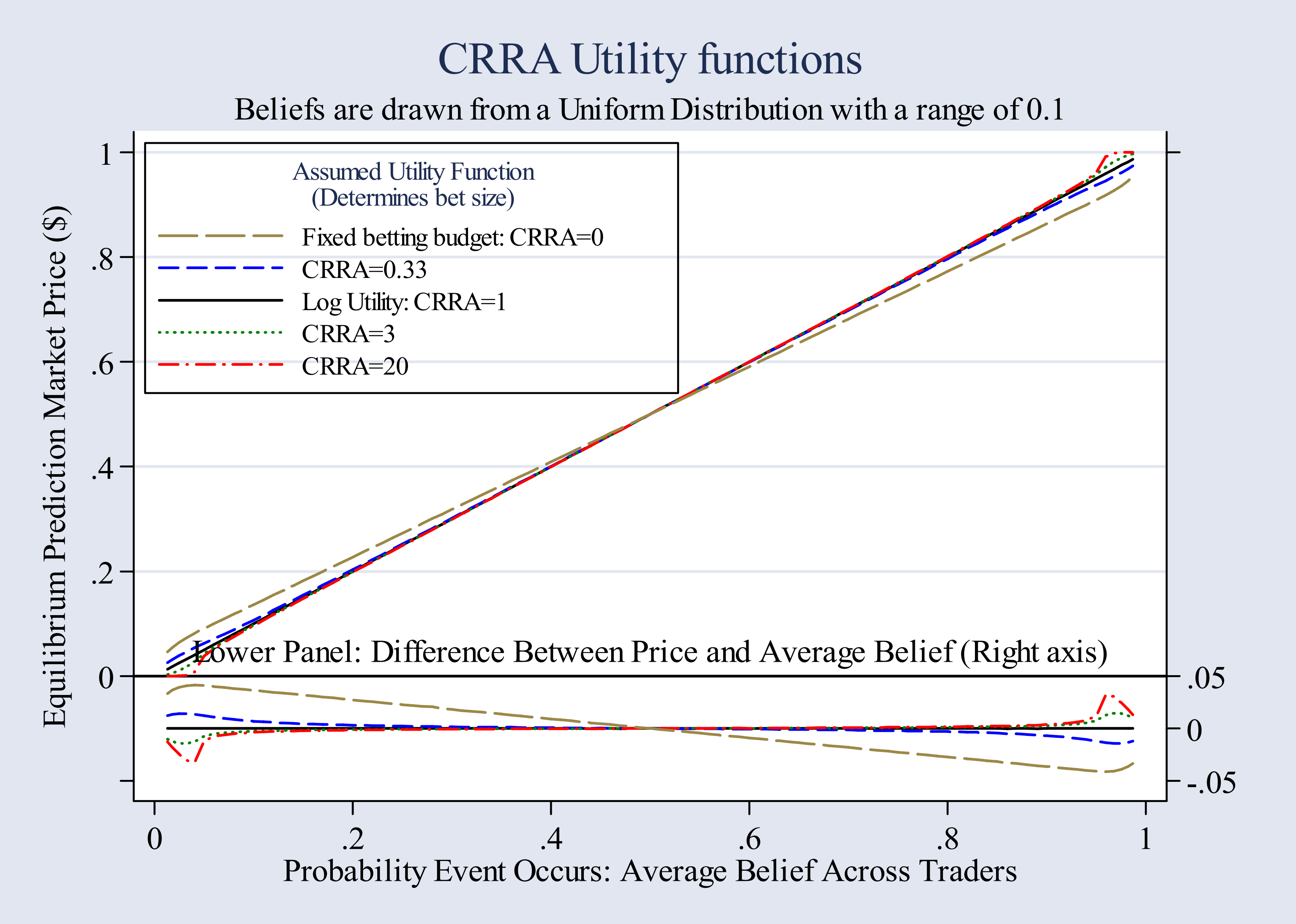
Demand as a Function of Beliefs

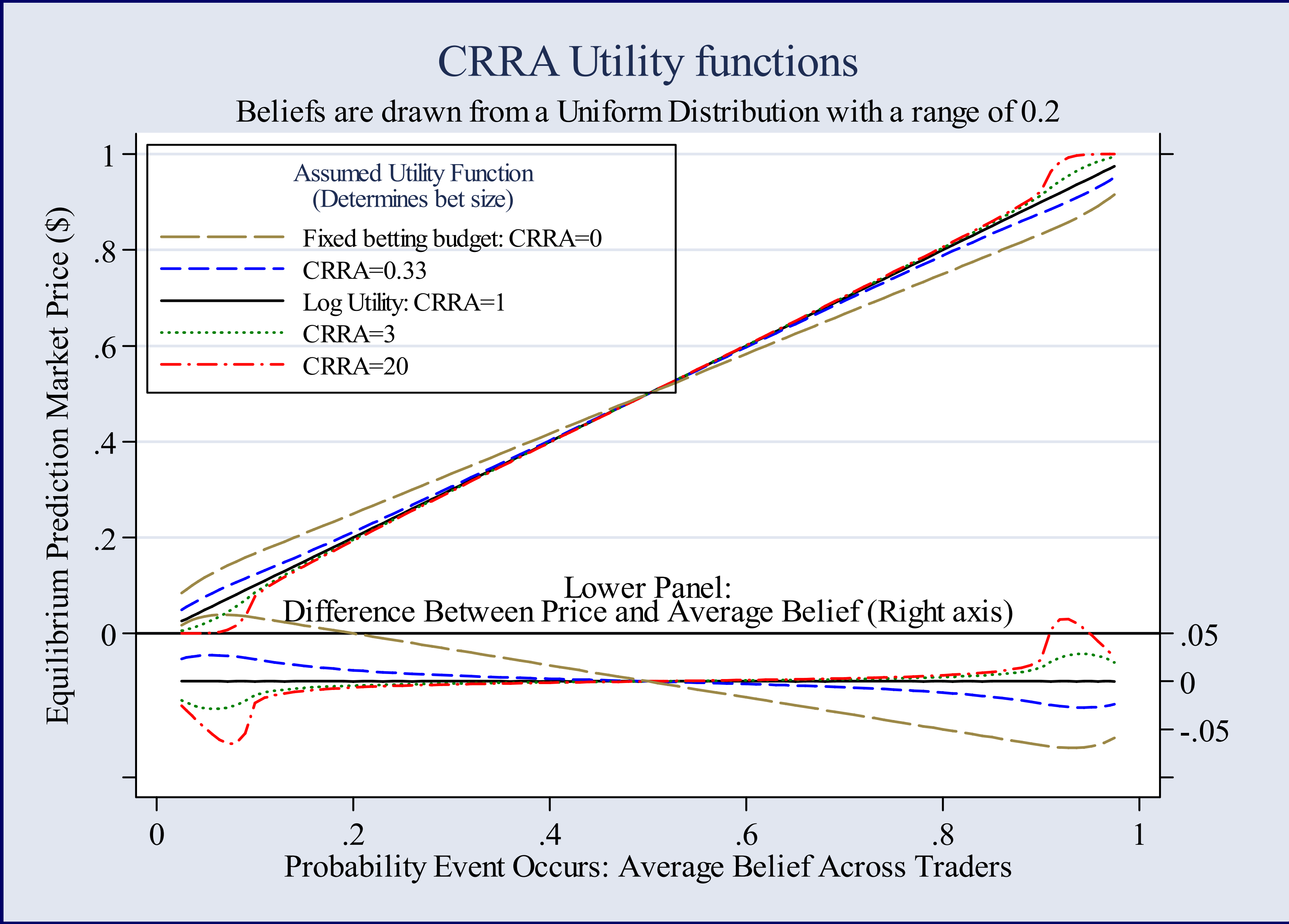
Price=$0.33; Individual demand depends on beliefs and utility function



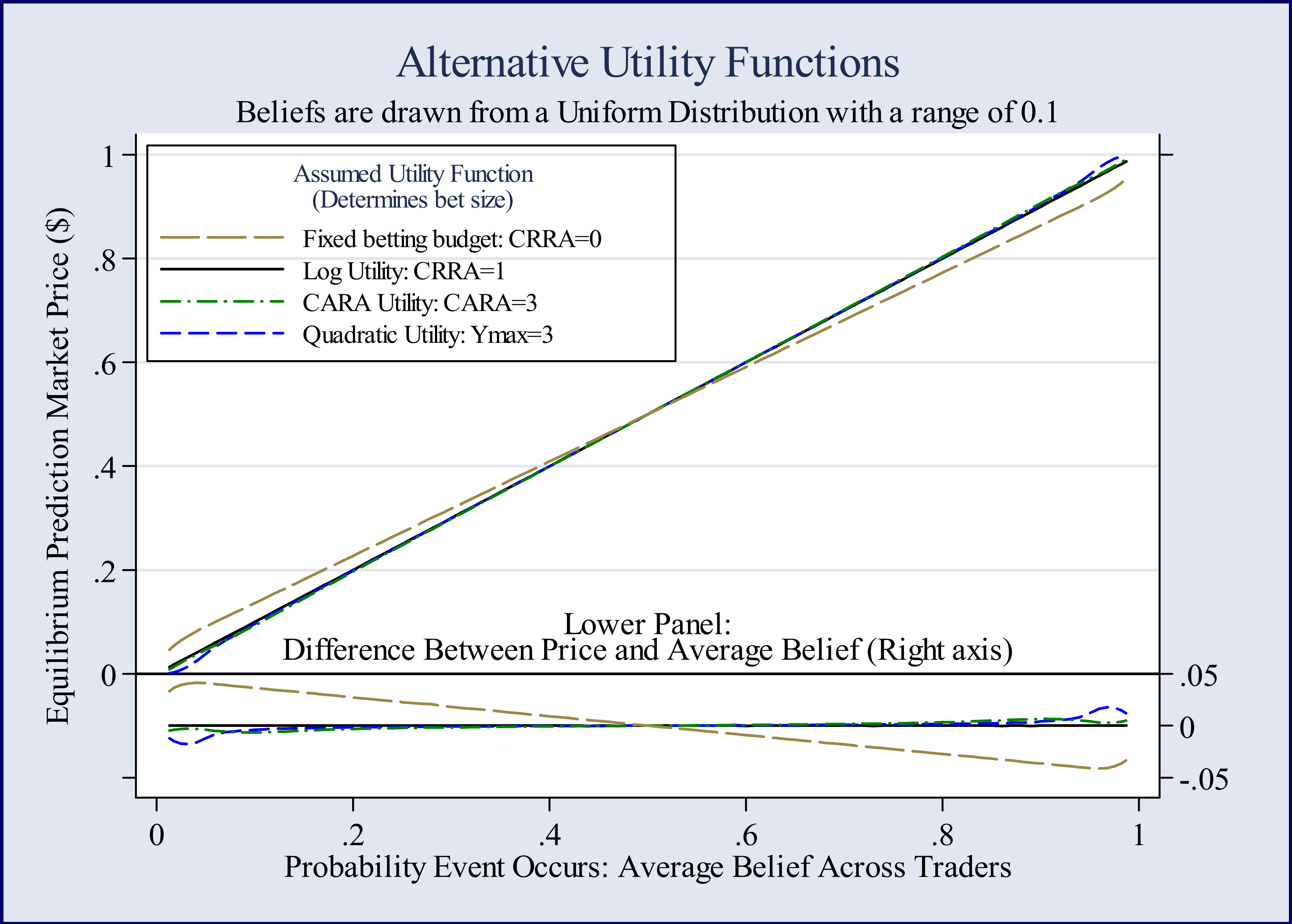
Solving for Equilibrium Prices



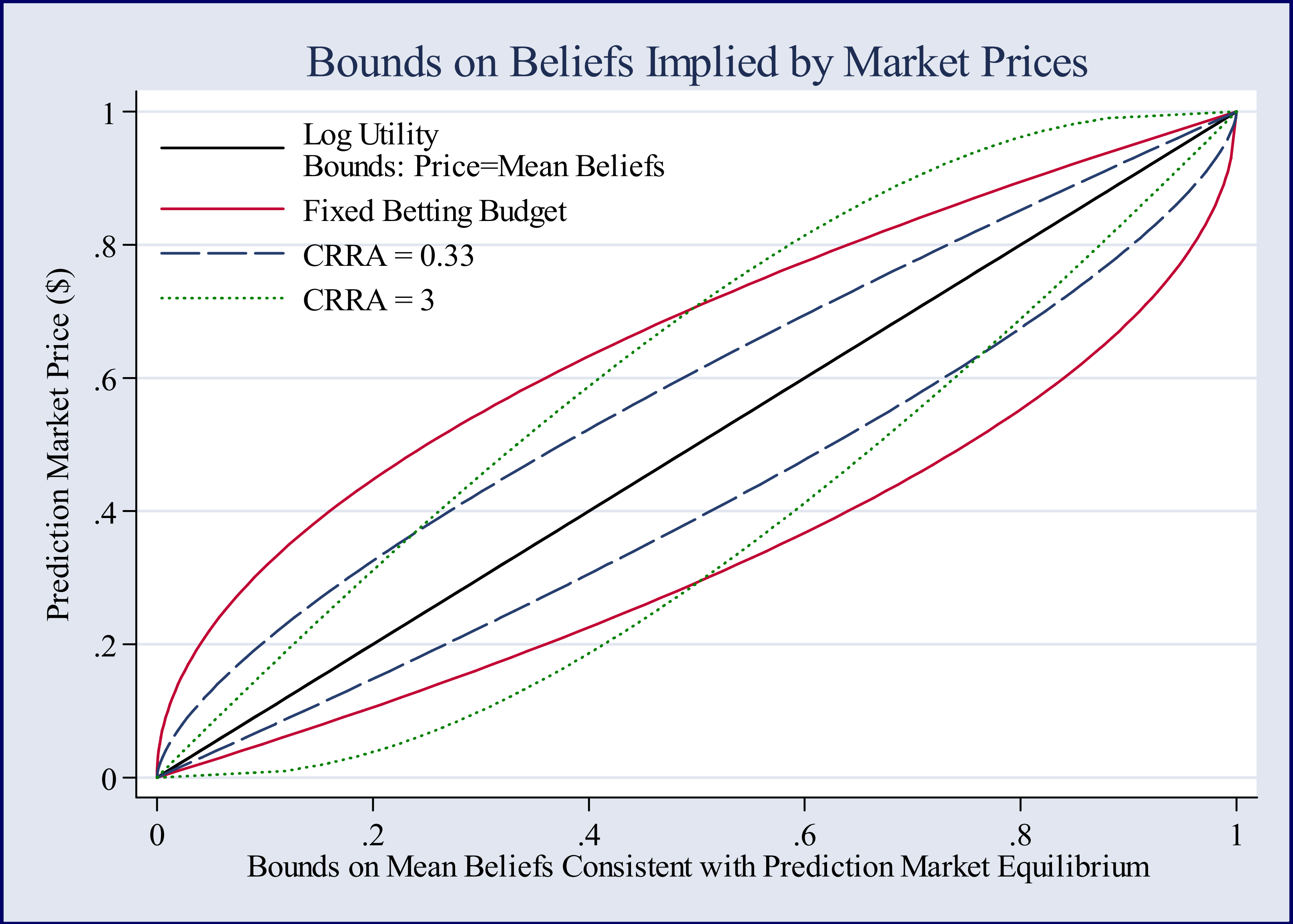
Increasing Dispersion of Beliefs



Robustness: Alternative Utility Functions



Bounds Analysis: Different Models



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Are Prediction Market Prices Close to Probs?

- Election 2004: Two Facts
  1. 62% of Americans thought Bush would win

     $$\int_{0.5}^{1} f(q)dq = 0.62$$

  2. Tradesports contract was trading at $0.55

     $$\int_{0}^{0.55} x(q)dF_{\mu,\sigma}(q) = -\int_{0.55}^{1} x(q)dF_{\mu,\sigma}(q)$$

- For plausible distributions of beliefs (*F*) and utility functions (*U*), how close is the market price to the mean belief?

Mean Beliefs Implied by Price=$0.55

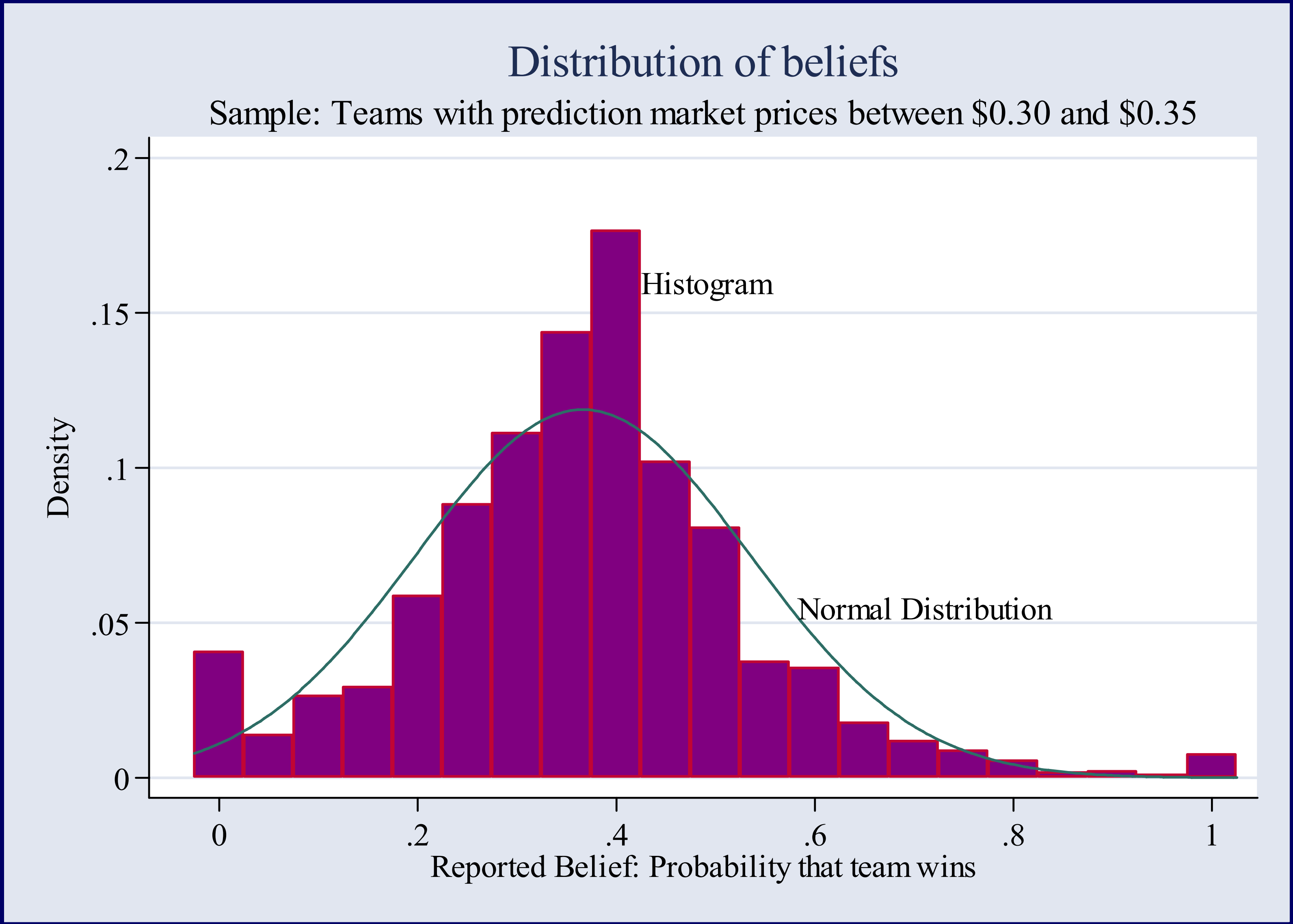
	Normal $[\mu,\sigma]$	Beta (α,β)	Uniform (q_L,q_H)
Implied Distribution of Beliefs			
Fixed bet size (Limit; $\gamma\to 0$)	57.8% [0.584,0.278]	57.1% [2.112,1.589]	58.6% [0.229,0.942]
CRRA; γ=⅓	56.0% [0.561,0.201]	55.8% [3.370,2.675]	57.5% [0.252,0.897]
Log Utility (γ=1)	55.0% [0.550, 0.163]	55.0% [4.640, 3.804]	55.0% [0.342,0.758]
CRRA; γ=3	54.6% [0.546,0.149]	54.7% [5.337,4.432]	54.9% [0.343,0.755]
CRRA; γ=20	54.4% [0.544,0.144]	54.6% [5.640,4.707]	54.8% [0.345,0.752]
CARA; ρ=3	54.4% [0.544,.0144]	54.5% [5.692,4.754]	54.7% [0.351,0.743]
Quadratic; y^{max}=3	54.2% [0.542,0.138]	54.2% [6.568,5.553]	54.6% [0.351,0.742]

Notes: Table shows mean of distribution. [Parameters of the belief distribution shown in parentheses]
Source: Authors' calculations. Note that beliefs outside (0,1) were treated as lim. $q\to 0$ or 1, respectively.

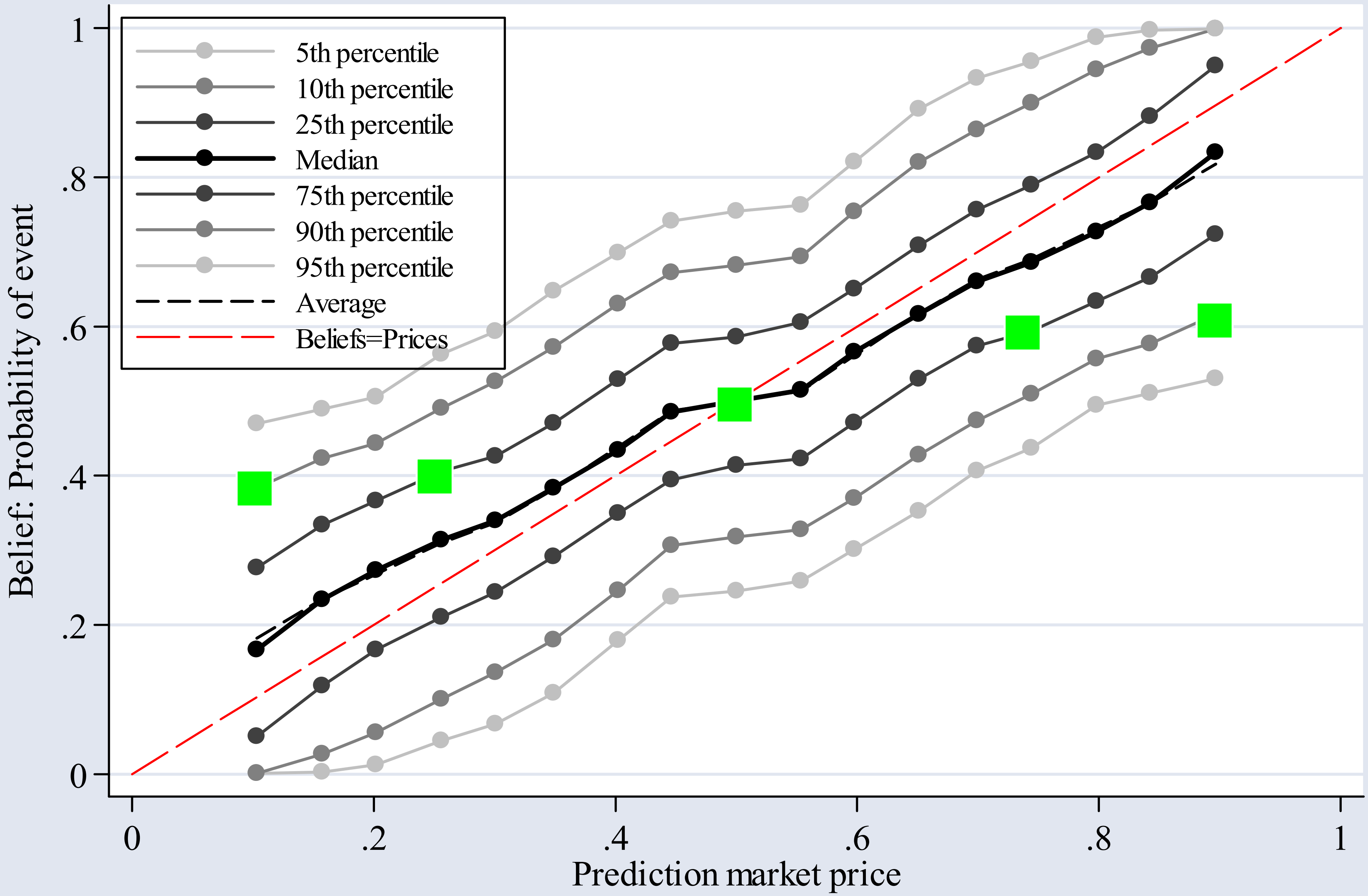
Distribution of Beliefs: NFL Football

- Probability Football Data
  - Online competition: Cash prizes for top 3 scores
  - Elicit belief about probability that team will win
  - Quadratic scoring rule: $100\text{-}400(w\text{-}q)^2$
- Data
  - 4 seasons (2000-2003)
  - Around 260 games per season
  - Average of 1320 players per game
  - Yields 1.4 million observations
  - We drop players who report 0 or 1 in >10% of games (sub-optimal strategy)

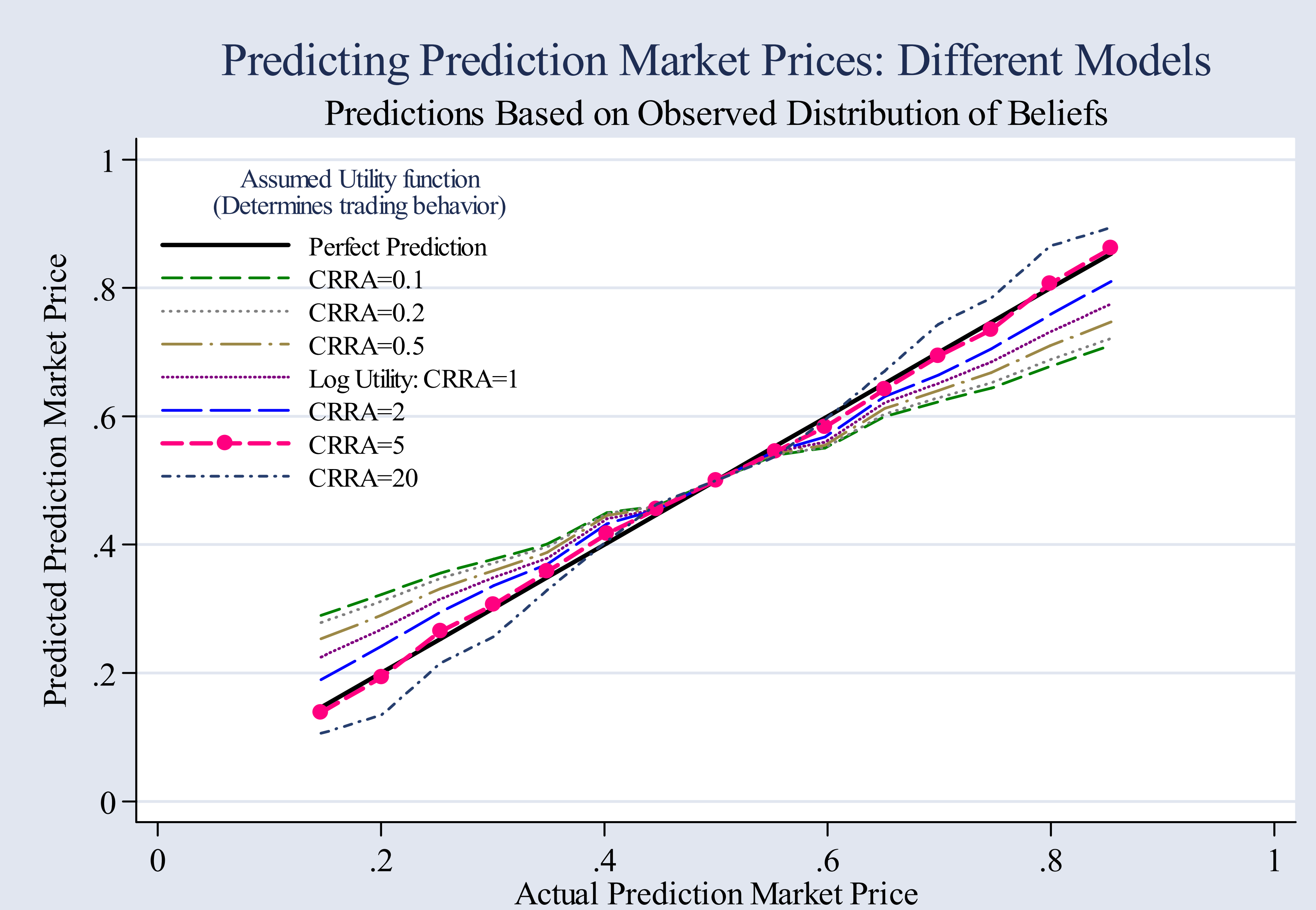
Distribution of Beliefs: NFL Football



Distribution of Beliefs and Prediction Market Prices



Predicting Prices Based on Beliefs



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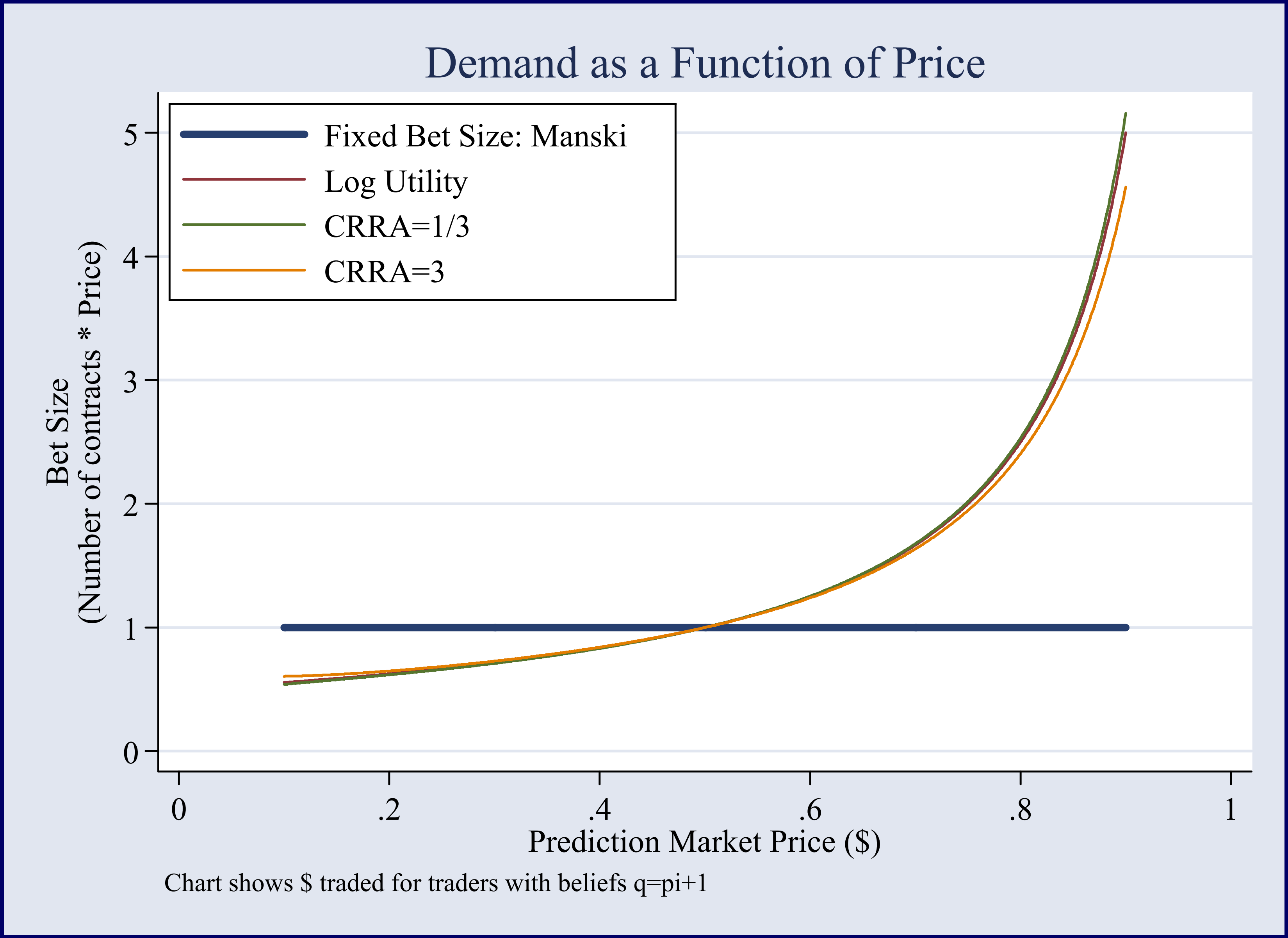
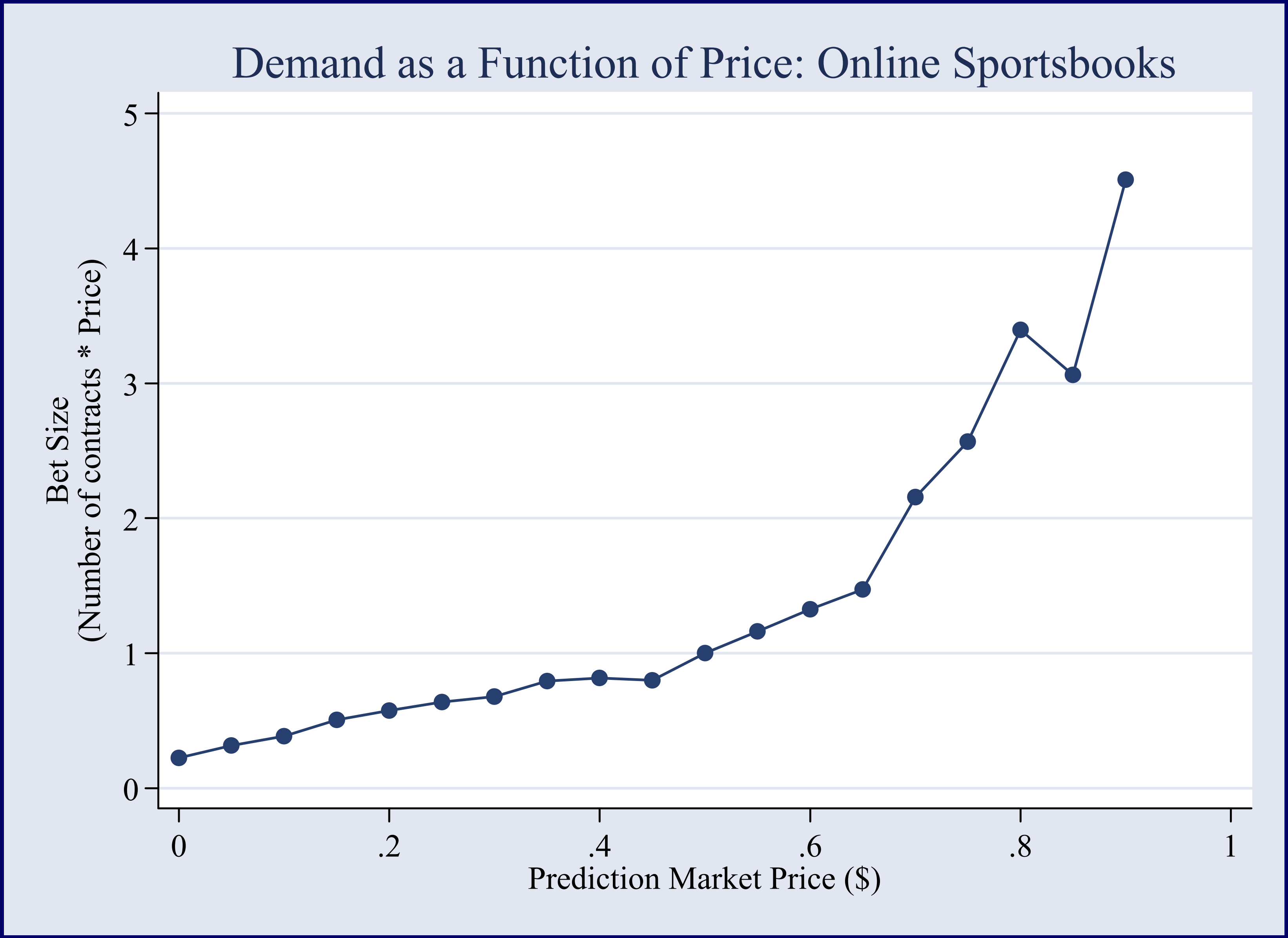
Utility Functions Determine Bet Size



Bookmaker Data

- Account-level sports betting data from 6 online sportsbooks
- $40 million gambled in 700,000 bets from 500 clients across many bet types
- Regression:
  - *Ln(BetSize) = Prediction market price*
    + individual gambler fixed effects

Relationship Between Bet Size and Price



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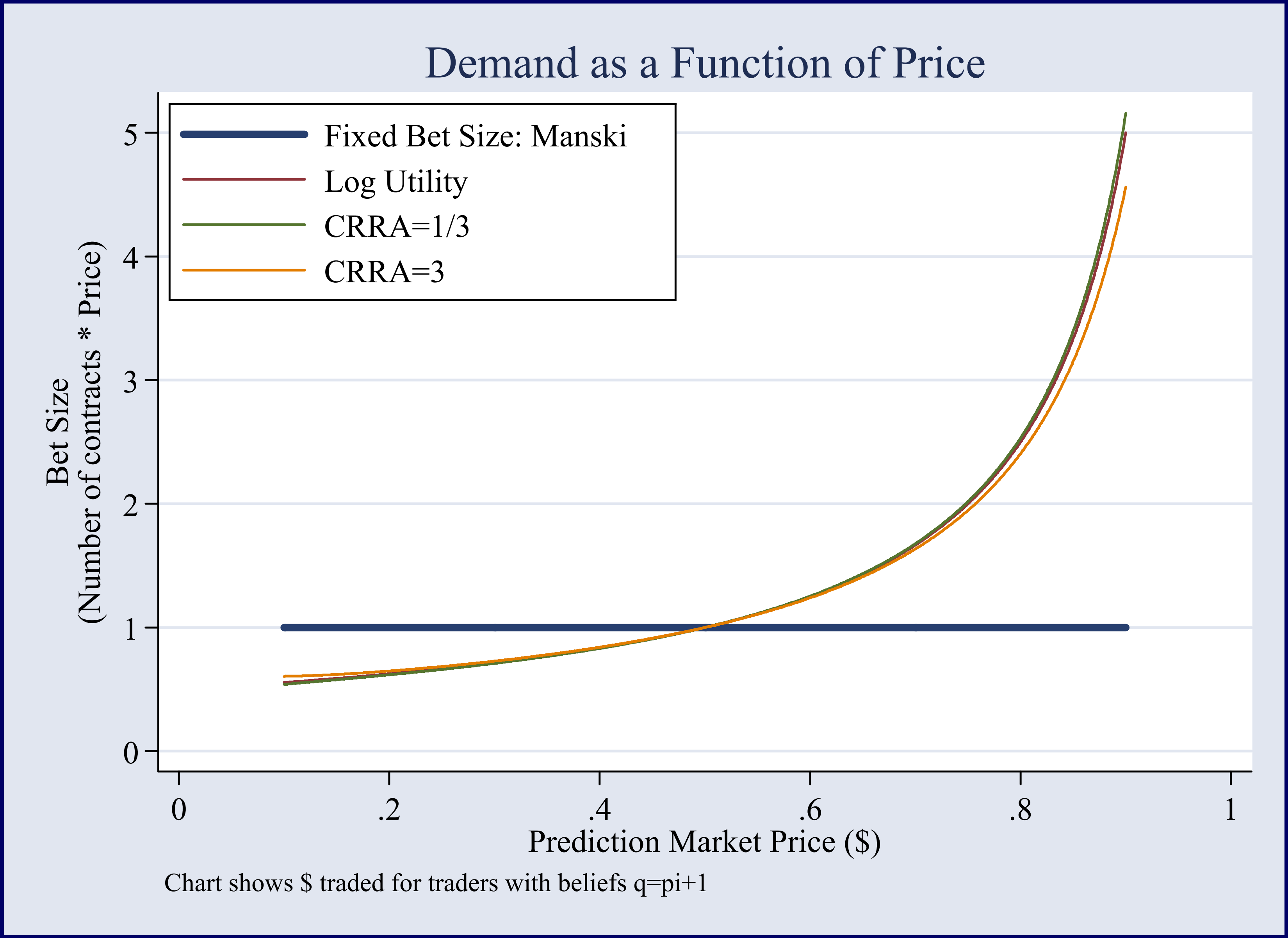

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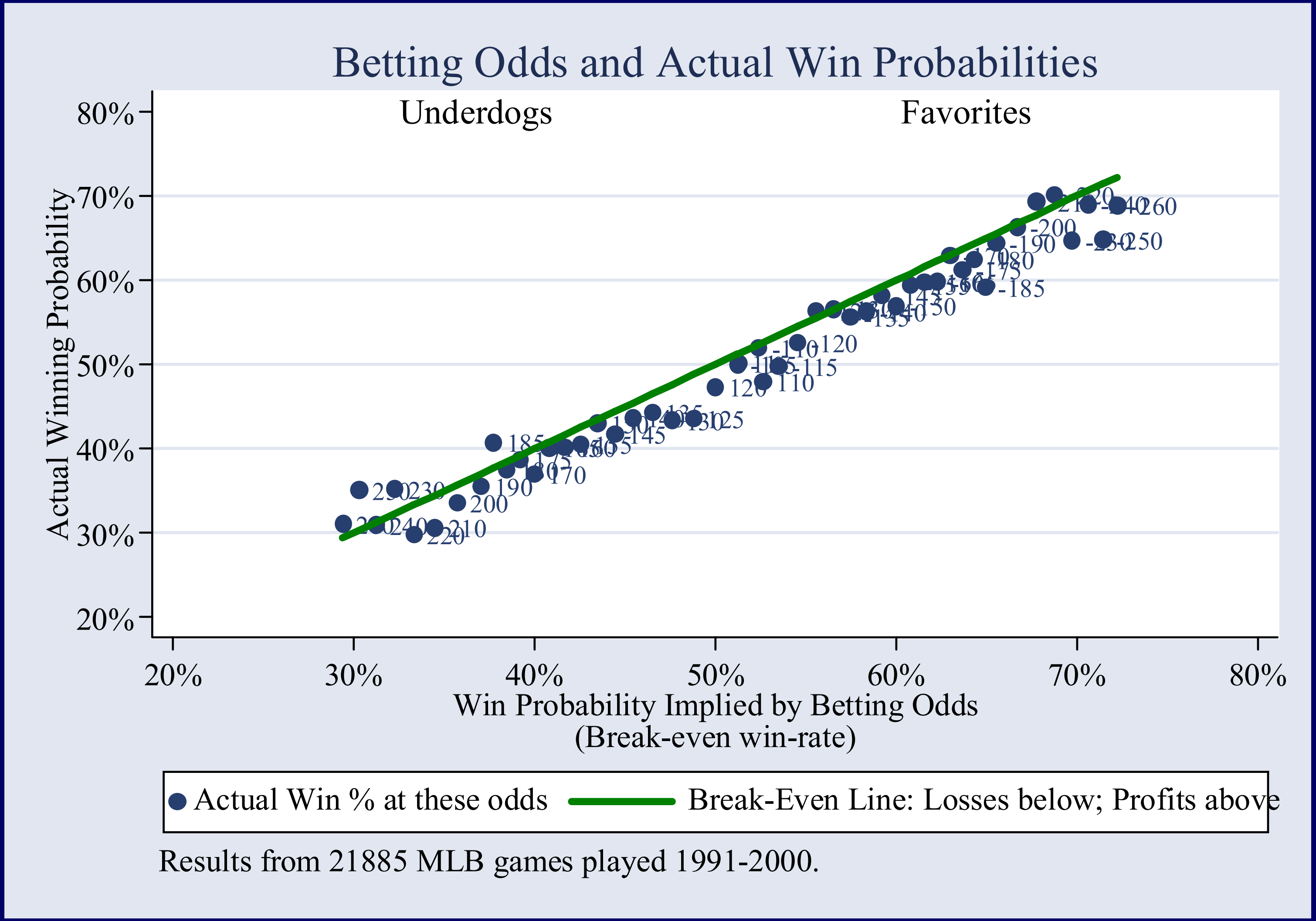
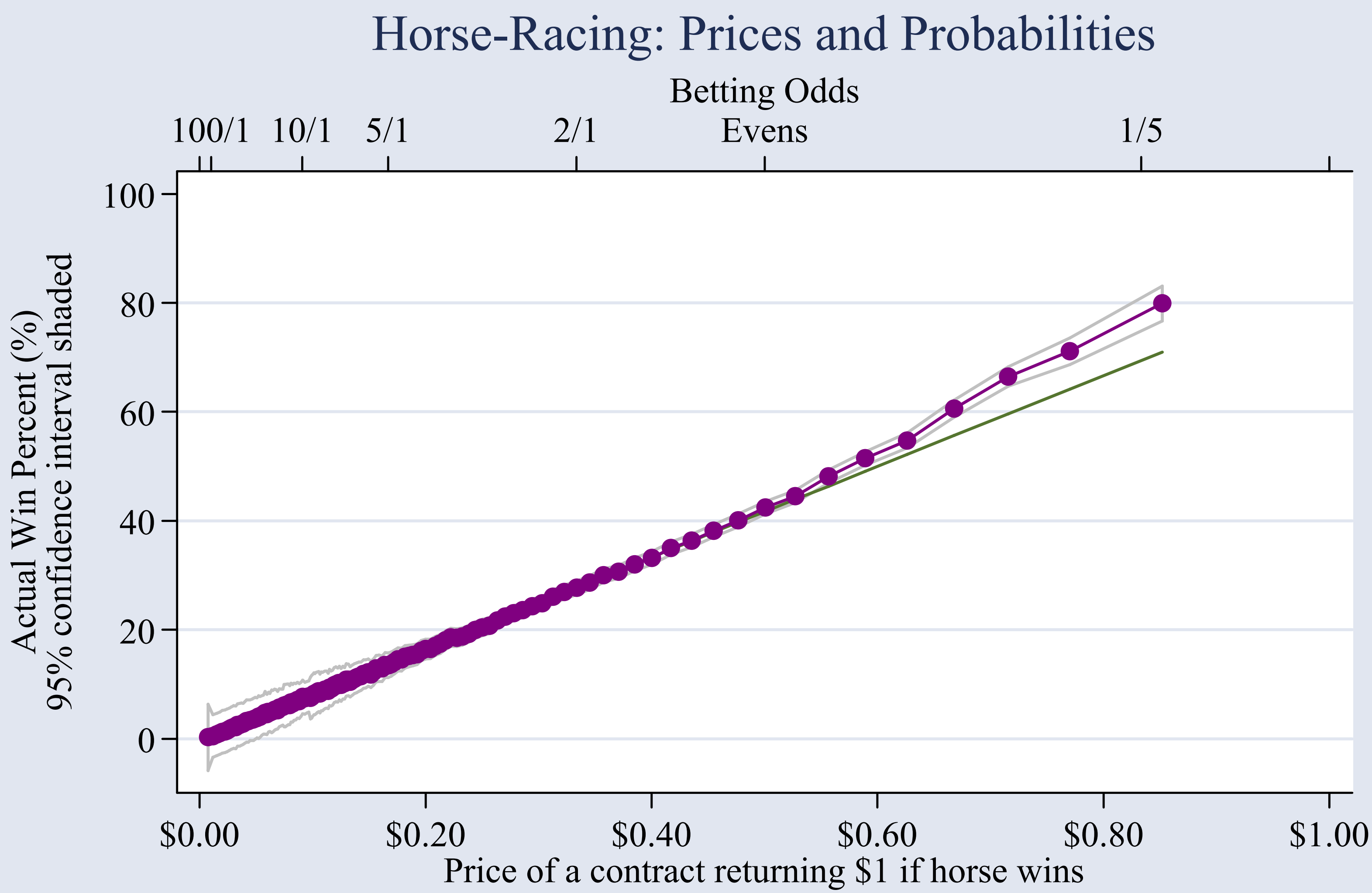
Prices and Probabilites: Baseball

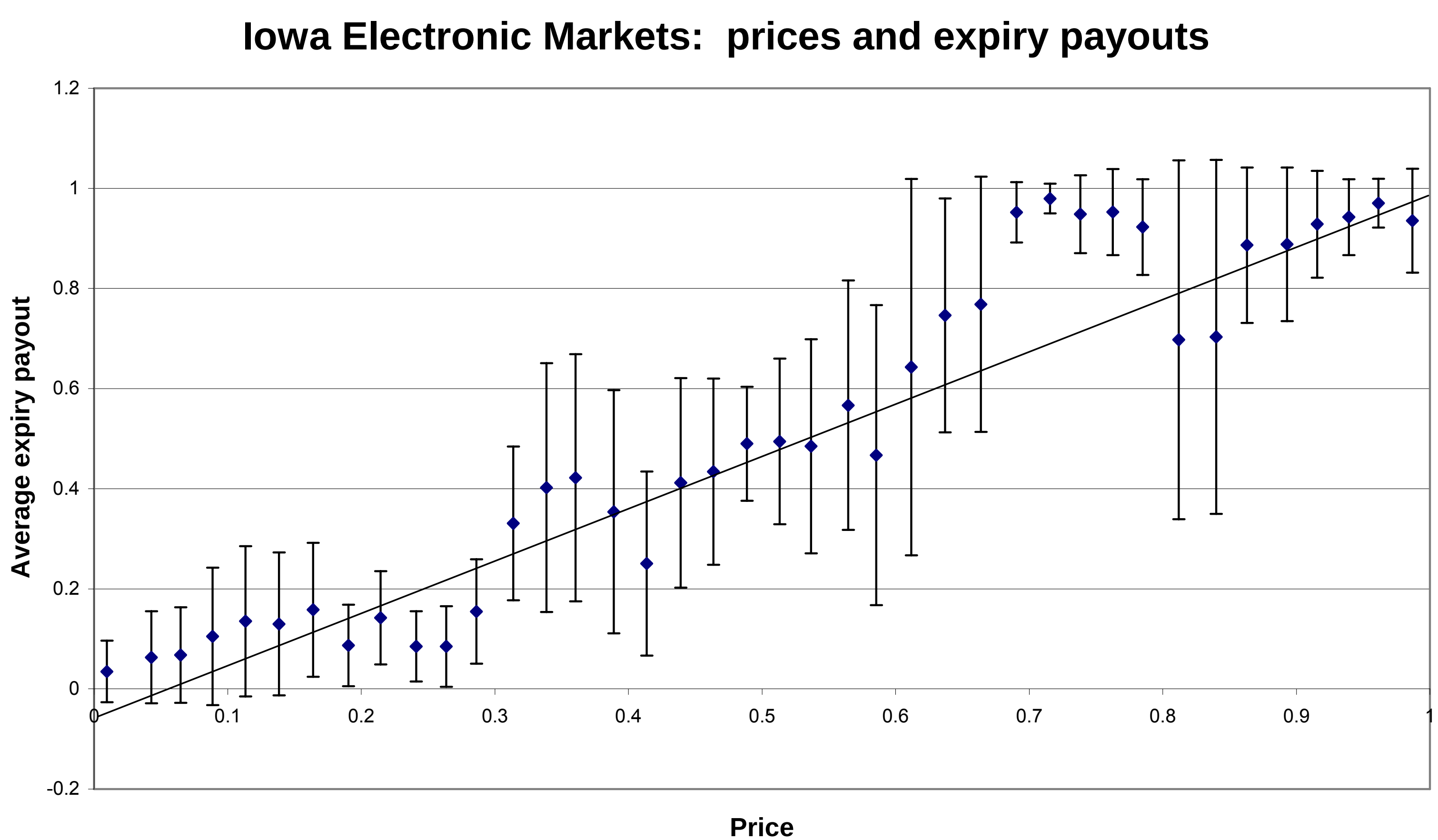


Prices and Probabilities: Horse Racing



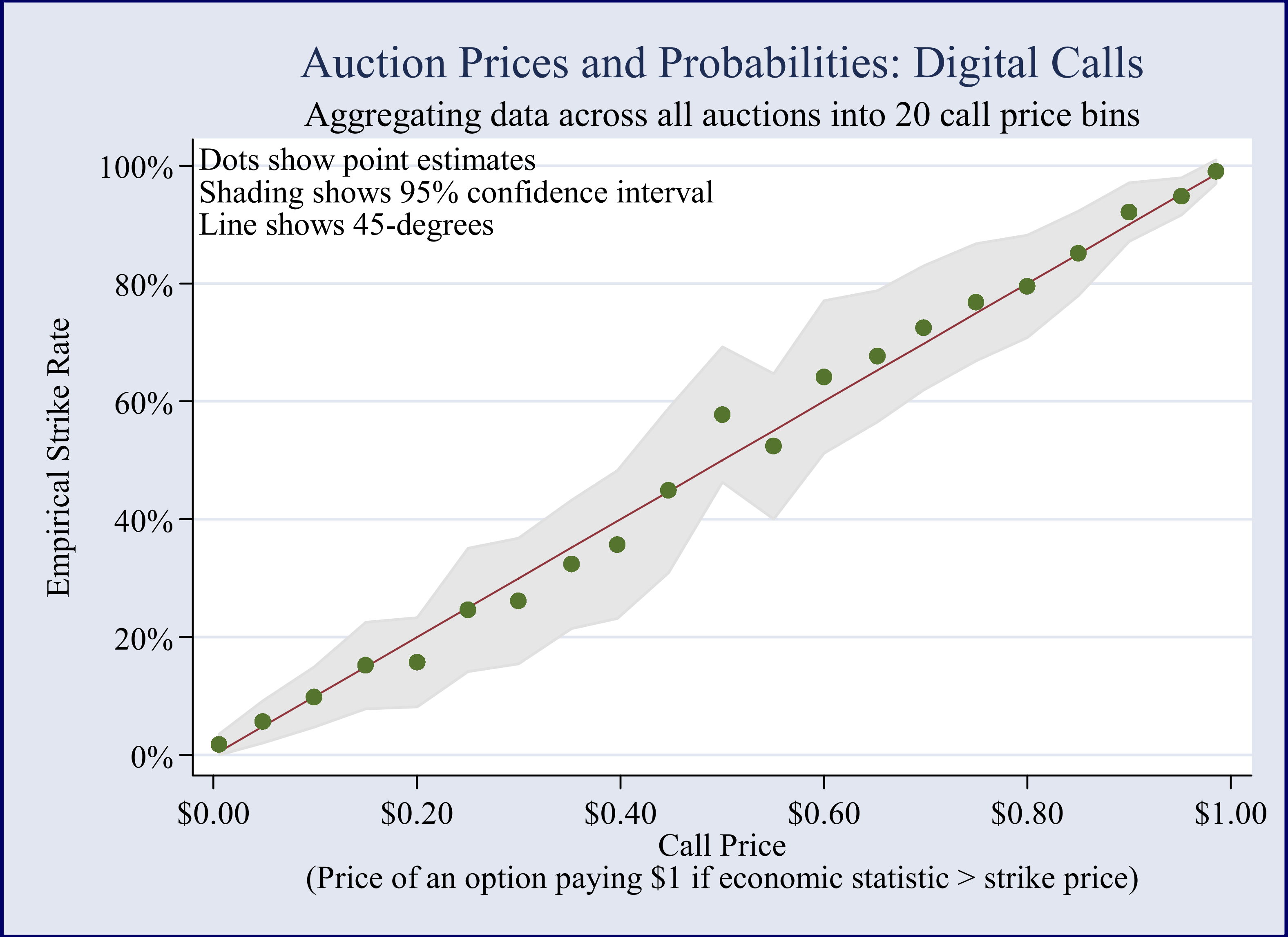
Line shows market expectation, assuming bettors lose a constant 17%
Source: Trackmaster, Inc. Sample is all horse races in the United States, 1992-2002. n=5,067,832 starts in 611,807 race

Prices and Probabilities: Politics



Graph plots the average expiry price of a winner-take-all contract on the Iowa market, conditional on its current price. Data is divided by current price into groups that are 2.5 percentage points wide. Error bars are 95 percent confidence intervals of the estimate of the mean expiry price, calculated from standard errors that are adjusted for sampling the same contract type multiple times.

Prices and Probabilities: Economic Derivatives



Reference: Refet Gurkaynak and Justin Wolfers (2005) "Economic Derivatives", *International Seminar on Macroeconomics.*

Conclusions

- Under what conditions to market prices aggregate all private info?
  - Grossman (1976): Stockmarket
    - » CARA utility and private signals ~N(μ,σ)
  - This paper: Prediction markets
    - » Log utility; any distribution of beliefs
- Manski:
  - Analytic results: Prices may fail to aggregate info
- This paper:
  - Calibration results: Prices are "close" to mean beliefs for plausible utility functions and distributions of beliefs (especially for $\pi \approx ½$)
  - Empirical results:
    - » Market prices are generally close to mean *beliefs*
    - » Market prices are generally close to actual *outcomes*
    - » Market prices are *fairly efficient* estimators of outcomes