Forecasting and Rank-Order Contests*

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Abstract

Forecasting is modeled as a rank-order contest with privately informed players. Rank-order contests are shown to be natural generalizations of Hotelling's classic location game. Positioning at the posterior mean is shown to be a Nash equilibrium in a perfectly symmetric setting with no prior information. In the presence of prior information the equilibrium is no longer at the posterior mean. Pure-strategy equilibria are shown to exist when the state space is discrete and the signal space continuous. A differential equation characterization of the symmetric equilibrium is provided for the winner-takes-all contest, in which only the forecaster with the lowest forecast error is rewarded. According to numerical simulations, in a winner-takes-all contest the amount of differentiation increases in the number of forecasters. If instead the forecaster with the highest forecast error is the only one to be punished, the amount of differentiation decreases in the number of players, and extreme conservatism results in the limit with an infinite number of forecasters. The more convex is the prize structure, the greater the incentive to differentiate.

Keywords: Forecasting, contest, winner take all, private information, Hotelling location game.

JEL Classification: D72, D82, D83, G20.

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1. Introduction

A number of forecasting contests are run regularly among meteorologists (see e.g. the National Collegiate Weather Forecasting Contest,[1] http://www.ems.psu.edu/NFC/) and economists (see the Wall Street Journal semi-annual forecasting survey).[2] More generally, markets as well as governments reward successful forecasters with implicit prizes, depending on the relative accuracy of their forecasts. For example, a number of prominent central bankers had distinguished careers as professional forecasters. Both Alan Greenspan[3] and Laurence Meyer[4] were economic forecasters before becoming respectively the current Chairman of the Board of Governors of the Federal Reserve Bank and one of its other recent members.

This paper develops the theory of forecasting and rank-order contests. In a general forecasting contest, each of a number of forecasters possess some private information about the variable to be predicted and issues a forecast. Prizes are then assigned to the forecasters depending on how the forecasts compare with the realization of the forecasted variable. In a rank-order forecasting contest, the forecaster whose forecast is closest to the realization obtains the first prize, and so on: Forecasts are compared ex post with the realization of the variable and ranked by accuracy, with the first prize being awarded to the forecaster with

[1] The National Collegiate Weather Forecasting Contest is held each year by the Department of Meteorology of Pennsylvania State University. Hundreds of entrants ranging from undergraduates to professors come from dozens of universities to participate in the National Collegiate Weather Forecasting Contest. For example, in 1996-7 there were 383 entrants, who prepared daily forecasts over 13 fortnight periods of the daily extreme temperatures and 24-hours rainfall (i.e. either zero, 0-1.3mm, 1.4-6.1mm, ..., up to 25 mm and above) at 13 places across the USA. A different place is chosen for each period.

According to Vislocky and Fritsch (1997), in 1996-7 the consensus (i.e. average) of all the forecasts had a score which was better than those of 94% of the entrants, but sophisticated objective numerical weather predictions (NWP) were slightly better. This was the first year that NWP forecasting proved better than the consensus from all entrants, in a competition biased in favor of human forecasting because plenty of time was available. About 13% of the university staff and 5% of the graduate students performed better than the NWP forecasts.

[2] As reported by Constance Mitchell Ford in the July 3, 2000 Wall Street Journal article "Economists Split On Fed Strategy To Curb Inflation": "But while economists are in agreement on the downward direction of GDP, they provided vastly different forecasts for inflation and interest rates. Some forecasters see long- and short-term interest rates falling as low as 5%, while others see interest rates rising to more than 7%. Inflation forecasts ranged from falling to 2.1% from current levels to rising as high as 4.1%."

[3] "From 1954 to 1974 and from 1977 to 1987 Dr. Greenspan was Chairman and President of Townsend-Greenspan & Co., Inc., an economic consulting firm in New York City." (see www.federalreserve.gov/bios/Greenspan.htm). The firm offered forecasts and research to large businesses and financial institutions until 1987, when Greenspan dissolved it to pursue his career at the Fed.

[4] "Before becoming a member of the Board, he was President of Laurence H. Meyer and Associates, a St. Louis-based economic consulting firm specializing in macroeconomic forecasting and policy analysis. [...] Dr. Meyer is widely recognized as one of the nation's leading economic forecasters. He was honored by Business Week in 1986 as the top forecaster for the year on its forecast panel. He was similarly honored in 1993 and 1996 with the prestigious Annual Forecast Award, presented to the most accurate forecaster on the panel for the Blue Chip Economic indicators." (see www.federalreserve.gov/bios/Meyer.htm)

lowest absolute deviation, the second prize to the forecaster with second lowest absolute deviation, and so on.

To gain some insights into this problem it is useful to focus on the case of a winner-takes-all simultaneous contest. Contrary to naive intuition, reporting the best prediction on the state (the posterior mean conditional on the signal observed) is not necessarily an equilibrium. There are two forces at play. On the one hand, each forecaster would want to report an accurate forecast, which is most likely to be close to the state. On the other hand, a forecaster has an incentive to differentiate one's forecast from those of the other forecasters. The second effect is due to the fact that the competing forecasters might be even closer to the realized state. The equilibrium strikes a balance between these two contrasting effects.

When the forecasters have no private information, or equivalently when their signals are perfectly correlated, this game is identical to Hotelling's (1929) location game. The pure Hotelling-Downs model considers candidates who are only interested in seeking office.[5] With more than two politicians, it makes a difference whether one considers proportional or plurality voting. With the *proportional rule*, politicians aim at maximizing (the expected value of an increasing and possibly concave function of) the fraction of votes they obtain. Under the *plurality rule*, the politician who collects the highest fraction of votes wins and in case of tie at the top the prize are shared.[6] With $N = 2$ politicians, the plurality and proportional objectives coincide, but with $N > 2$ the two objectives typically result in different location equilibria.[7] Economists (e.g. Palfrey (1984) and Weber (1992)) have typically studied the Hotelling model with the proportional rule, as this is natural in applications to industrial organization (e.g. to product differentiation in broadcasting). From the theoretical point of view, our paper extends Hotelling's location game with proportional rule to a private information environment.[8]

Our winner-takes-all contest then generalizes Hotelling's location game with proportional rule to allow for private information of the players on the distribution. Surprisingly, there is very little related work in the political science literature. Chan (2001) and Bernhardt, Duggan and Squintani (2002) are independently investigating a location game between two politicians with a discrete number of signals on the location of the median voter. We also analyze cases with more than two players, and turn our focus on the case of continuous signal distributions for analytical simplicity. Before these papers, the political science literature does not seem to have considered the case in which candidates have

[5] More generally, politicians might also have a partisan preference for certain policies.

[6] Winning results in a payoff N, tying for the top position with K other players the payoff is $N - K$, and losing gives a payoff of 0.

[7] For example, with $N = 3$ there is no pure strategy with proportional representation (Shaked (1982)) but there are pure strategy equilibria with plurality rule (Chisik and Lemke (2001)).

[8] The political science literature has instead focused mostly on the Hotelling model with plurality voting.

private information about the distribution of the voters' preferences.[9]

Consider what happens in the absence of private information, i.e. in the classic Hotelling location game. The equilibrium with two players is at the median of the distribution both in the simultaneous and the sequential case. More generally, the equilibrium depends on the number of players and often involve mixed strategies.[10] Osborne and Pitchik (1986) showed that with infinitely many players the distribution of equilibrium locations replicates the distribution of the state. Two recent papers have proposed versions of symmetric information forecasting contests without drawing the connection with Hotelling's location game. Laster, Bennett and Geoum (1999) derived the same infinite-player result in a winner-takes-all simultaneous forecasting contest in which forecasters have perfectly correlated information.[11] Laux and Probst (2000) study a simple simultaneous forecasting contest in which informed forecasters share the same signal on a binary state. They assume that the forecaster's report can be binary and that the rewards depend on whether the forecaster's report concords with the realized state.

Despite their practical and theoretical interest, there is no systematic treatment of rank-order contests in the literature. There is a small literature in applied probability on the related problem of *second guessing*, starting with Steele and Zidek (1980). The following sequential forecasting contest with two privately informed players is considered. A privately informed second guesser makes a forecast after observing the posterior expectation that a first forecaster is assumed to truthfully report. Steele and Zidek show that under symmetry and other conditions the second guesser is three times as likely to win as the first. Some follow-up papers (Pittenger (1980) and Hwang and Zidek (1982)) study under more generality the probability that the second guesser wins the contest. All these papers assume away strategic interaction by allowing only the second guesser to make a decision.

The analysis of the strategic behavior of forecasters with private information is tech-

[9] The reader is referred to Osborne's (1995) review. Some papers have been looking at what happens under the proportional rule when the candidates have imperfect (but symmetric) information about the distributions of voters. A precursor is Wittman (1983), who assumed probabilistic voting so that each candidate faces an ex-ante exogenously given probability of winning the election as a function of the policy proposed by both candidates. Roemer (1994) endogenizes this probability as the result of voting uncertainty due to imperfect information of the candidates about the distribution of voters' preferences. Roemer shows that Hotelling's median result does not extend to that setting provided that the candidates cares also about the policy. Osborne (2000) obtains a similar differentiation result in a setting in which the candidates are uncertain about the voters' preferences and have the possibility of moving asynchronously.

[10] Dasgupta and Maskin (1986a,b) prove existence of mixed-strategy equilibrium in Hotelling's location game. Eaton and Lipsey (1975) attempted the first systematic analysis of pure strategy equilibria as a function of the number of players. See Shaked (1982) for the first characterization of the mixed-strategy equilibrium with three players. Osborne and Pitchik (1989) offer the most general analysis of equilibrium in the simultaneous location model. See Prescott and Visscher (1977) for an equilibrium analysis of the sequential version of the location game.

[11] This is clearly equivalent to say that forecasters have no private information.

nically quite challenging. Two recent papers have focused on tractable special cases. Bernhardt and Kutsoati (2001) analyze the strategic behavior of the last forecaster. They find that the last forecaster strategically selects a forecast that overshoots the consensus forecast in the direction of her private information and provide evidence of bias in the financial analysts' earnings forecasts. In a companion paper, Ottaviani and Sørensen (2002) develop a tractable winner-takes-all forecasting contest model with a continuum of forecasters and compare its predictions to those of alternative theories. The addition of private information has the desirable effect of inducing a symmetric location equilibrium in pure rather than mixed strategies. Forecasters differentiate themselves from their competitors by putting excessive weight on their signals. In this paper instead, we perform a general analysis of forecasting contests with a finite number of players.

Forecasting contests may be a useful metaphor for financial markets. While a forecasting contest rewards those who forecast better than others the true state, a *beauty contest* rewards those whose forecast is close to the forecasts of others. A beauty contest might capture better situations where information on the state is not revealed, so that the reward can only be based on the forecasts released by the other forecasters.[12]

The paper proceeds as follows. The model is introduced in Section 2. In Section 3 we show that in a perfectly symmetric world there is a truthtelling equilibrium in the forecasting contest provided that there is no prior information. We then turn to illustrate the incentive to bias the forecast away from the ex ante expected in a simple two-state model in Section 4. There we also provide a first result on equilibrium existence. In Section 5 we derive an equation to characterize the symmetric pure strategy equilibrium with n players. We also show that the equilibrium in the two-person simultaneous forecasting contest with normal signals is *not* at the posterior median. The median voter's theorem fails to extend to situations in which the politicians are privately informed about the location of voters.

In Section 6, we simulate the normal location model in order to perform some comparative statics exercises. In the winner-take all contest, the weight put by forecasters on their own signal increases with the number of players. The equilibrium of the limit winner-takes-all contest derived by Ottaviani and Sørensen (2002) is a good approximation of the equilibria with a large number of forecasters. The comparative statics is reversed in the loser-loses-all contest: the greater is the number of forecasters, the more conservative is the equilibrium. As the prize structure changes from greatly rewarding the winner towards greatly punishing the loser, the forecasters respond by being more conservative.

[12] O'Flaherty (1987) analyzes an interesting dynamic model of a beauty contest. He considers a situation where the price of an asset at any period is proportional to the number of agents who choose it. The payoff to each agent is then assumed to be decreasing in the fraction of people who contemporaneously choose the same asset, but increasing in the number of followers who pick the same asset in the next period.

2. Model

In a general forecasting contest there are n forecasters. Each forecaster i has a noisy signal $s_i \in S$ informative about the state to be predicted $x \in X$. The prior joint distribution of $(x, s_1, \ldots, s_n)$ is common knowledge. The forecasters simultaneously report forecasts $m_i \in X$. The prizes to the forecasters depend on how the forecasts made $m_1, ..., m_n$ compare with the realized state x. In a rank-order forecasting contest, the forecaster whose forecast is closest to the realization obtains the first prize, and so on: Forecasts are compared ex post with the realization of the state and ranked by accuracy, with the first prize Z_1 being awarded to forecaster i with lowest absolute deviation $\arg\min_i |m_i - x|$, the second prize Z_2 to forecaster j with second lowest absolute deviation $\arg\min_{j \neq i} |m_i - x|$, and so on with $Z_n \leq ... \leq Z_2 \leq Z_1$. The objective of each forecaster is to maximize the expected value of the prize won.

3. Symmetric Location Experiment without Prior Information

We first identify an important class of statistical models under which the most basic intuition is verified: the optimal strategy for any forecaster is to issue his best forecast of the state. The statistical model is a symmetric location experiment with no prior information. To allow for perfect symmetry, assume that the spaces X and S are both the unit circle, corresponding to the circumference of the unit ball in $\mathbb{R}^2$. A real number z indicates a point on the circle in the usual way, giving the anti-clockwise distance along the circumference from $(1, 0)$, the circle's origin in the plane.[13] To build our location family, we start from any p.d.f. $g(s)$ over the unit circle, with these two properties:

(i) *Symmetry*: s is distributed on the circle symmetrically around 0, i.e. $g(s) = g(-s)$ for all $s \in [0, \pi]$.

(ii) *Unimodality*: s is distributed unimodally around 0, i.e. $g(s)$ is a decreasing function of $s \in [0, \pi]$.

A member of our location family then has p.d.f. given by $f(s|x) = g(s - x)$. It is simple to see that f inherits the symmetry and unimodality properties such that $f(x + s|x) = f(x - s|x)$ for all $s \in [0, \pi]$ and $f(x + s|x)$ is decreasing in $s \in [0, \pi]$.

If $q(x)$ is the uniform distribution, the posterior belief on x is described by the p.d.f. $q(x|s) = f(s|x)/f(s)$. By symmetry and unimodality of $f(s|x)$, this distribution of x is symmetric and unimodal around s. The truthtelling strategy is then to report the mode of this distribution, namely $m = s$.

[13] For instance, the numbers $-2\pi, 0, 2\pi$ all indicate the origin, while $\pi/2$ indicates the point $(0, 1)$ of the plane.

Proposition 1 (Truthtelling as Best Reply). *Assume that forecaster i's signal is drawn from a member of the location family $f_i(s|x) = g_i(s-x)$ with g_i satisfying symmetry and unimodality. Assume that the common prior $q(x)$ is the uniform distribution, and that signals are independent, conditionally on x. Assume that the strategies of all players j other than i satisfy that the distribution of $m_j - x$ is independent of x. Then truthtelling is a best reply of player i.*

Proof. See the Appendix. □

Proposition 2 (Truthtelling Equilibrium). *Assume that each forecaster's signal is drawn from a member of the location family $f_i(s|x) = g_i(s-x)$ with g_i satisfying symmetry and unimodality. Assume that signals are independent, conditionally on x. If the common prior $q(x)$ is the uniform distribution, truthtelling is a Bayes-Nash equilibrium.*

Proof. This is a direct consequence of Proposition 1. Assuming that every player $j \neq i$ applies the honest strategy, we have satisfied the assumption that $m_j - x = s_j - x$ follows a distribution independent of x. Then truthtelling is a best reply for player i, as needed.□

For truthtelling in this example it is crucial that the prior on the state $q(x)$ is uniform, so that there is essentially no prior information.[14] Intuitively, truthfully reporting $m_i = s_i$ is then equivalent to reporting the mode of the posterior distribution $q_i(x|s_i)$. A particular important instance of this arises for the classical statistician who can be modeled as having an improper uniform prior belief on $\mathbb{R}$. The posterior $q_i(x|s_i)$ is then $f_i(s_i|x)$. If f_i is a symmetric unimodal location experiment, the proposition applies to show that truthtelling is an equilibrium. But truthtelling is incompatible with equilibrium as soon as we give a proper prior belief on the state, as illustrated below.

Note that truthtelling in this result does not at all depend on the prize structure, other than the weak monotonicity that higher prizes are rewarded to forecasters who rank higher. This means that there is no relevance of the design of the prize structure in this case. Whether the reward structure is convex or not, truthtelling is the outcome in this benchmark model of a perfectly symmetric location model. When we consider failures of truthtelling in alternative statistical models, those failures cannot be ascribed directly to convexity or concavity properties of the reward structure.

[14] Notice that Propositions 1 and 2 apply to more signal structures than those on the unit circle presented there. Assume that φ is a one-to-one mapping of $X = S$ into some other space $X' = S'$. Using φ we can transform $q(x)$ into a distribution on X', transform $g(s)$ into a distribution on S', and construct f as before. Then it is clear that the analysis carries over. For instance, with φ we could cut the circle open and straighten it out to an interval. The resulting family of signal distributions is no longer a proper location family, since it is wrapped at the ends of the interval, but it has $X, S \subseteq \mathbb{R}$.

Likewise, the truthtelling result does not depend on the number of forecasters. When we later find that the number of forecasters has a systematic influence on the equilibrium, the statistical model is no longer the perfectly symmetric location model.

The assumption of conditional independence is clearly used in the proof of the proposition, but we can also give a counterexample to the proposition based on a failure of conditional independence. Assume that there are two forecasters, and let g_1 be a continuous p.d.f. over the unit circle, symmetric and unimodal around 0, defining forecaster 1's signal distribution as $f_1(s_1|x) = g_1(s_1 - x)$. Let $\varepsilon_1 = s_1 - x$ be the error term of 1's signal, and note that ε_1 has p.d.f. g_1. Now define $\varepsilon_2 = \varepsilon_1 \left(1 + (\varepsilon_1/\pi)^2\right)/2$ for any $\varepsilon_1 \in [-\pi, \pi]$. Note that $|\varepsilon_2| < |\varepsilon_1|$ for all $\varepsilon_1 \in (0, \pi)$ and that the continuous p.d.f. g_2 of ε_2 inherits the properties of being symmetric and unimodal around 0. Let the signal of forecaster 2 be $s_2 = x + \varepsilon_2$. For forecaster 1, truthtelling is not the optimal response when forecaster 2 employs truthtelling. To see this point, note that when both forecasters are truthtelling, then almost surely $|m_2 - x| = |s_2 - x| = |\varepsilon_2| < |\varepsilon_1| = |s_1 - x| = |m_1 - x|$. Thus there is probability one that forecaster 2 wins the contest. Since the signal of forecaster 2 is imperfect, forecaster 1 could employ some other strategy with positive chance of winning. For instance, forecaster 1 could guess $m_1 = 0$ regardless of his signal.

4. Discrete State Distribution

4.1. Equilibrium Existence

Proposition 3. *Assume that the state space X is finite. Assume that conditionally on x, the private signals are independently and continuously distributed (no atoms) on a complete, separable metric space. Then there exists a pure strategy equilibrium in the contest.*

Proof. We apply results from Milgrom and Weber (1985). For notation, their state space T_0 is our X, and also their actions spaces A_i are our X. Our payoff functions are bounded between zero and the maximum prize. By their Proposition 1 (a), their assumption R1 is satisfied. By conditional independence and finiteness of X, their R2 is satisfied. In our game, a player's payoff function depends only on the state x and the vector of forecasts. Their Theorem 4 then states that there exists an equilibrium in pure strategies. □

4.2. Binary State Example

This example serves to illustrate the incentive to bias the forecast when the prior is not uninformative, while also providing simple comparative statics results on variations in the number of players and in the precision of their signal.

The state space is binary, $X = \{-1, 1\}$. The common prior probability of state $x = 1$ is $q \in (0, 1)$. Conditionally on x, the continuous private signals are conditionally i.i.d. over $S = [-1, 1]$ with p.d.f.

$$f(s|x, t) = \frac{1}{2}(1 + stx).$$

Notice that the widely used symmetric binary model has the same generalized p.d.f., with $S = X = \{-1, 1\}$. The parameter $t \in [0, 1]$ determines the Blackwell informativeness of the signal. To see this, consider the garbling of s into $\widetilde{s}$ whereby $\widetilde{s} = s$ with probability $\tau < 1$, and otherwise $\widetilde{s}$ is independently redrawn from the uniform distribution over $S = [-1, 1]$. Then the p.d.f. of $\widetilde{s}$ is $\tau f(\widetilde{s}|x, t) + (1 - \tau)/2 = f(\widetilde{s}|x, \tau t)$, so that the garbled signal is distributed as the ungarbled signal of informativeness $\tau t < t$. When $t = 0$ the signal is entirely uninformative.

The contest is a winner-takes-all contest with the following tie-break rules. The prize (normalized to equal the number of players, n) is shared equally among all players who correctly forecast the state. If all forecasts are incorrect, the prize is shared equally among all players.

Consider a forecaster with signal s. The expected payoff of forecast $m \in \{-1, 1\}$ is $V(m|s) = q(s) W(m|1) + (1 - q(s)) W(m|-1)$ where $W(m|x)$ is the expected payoff of forecast m when state x is realized and $q(s)$ is the posterior belief of $x = 1$ given signal s. Unless all other forecasters always issue the same constant forecast, there is a positive chance of earning more than 1 whenever $m = x$, and so $W(1|1), W(-1|-1) > 1 > W(1|-1), W(-1|1)$. It follows immediately that the best reply of each individual forecaster has a simple cutoff characterization. There exists a threshold posterior belief $q^* \in [0, 1]$ such that for $q(s) < q^*$ it is optimal to let $m = -1$, and for $q(s) > q^*$ it is optimal to let $m = 1$. The posterior

$$q(s) = \frac{q(1 + st)}{q(1 + st) + (1 - q)(1 - st)}$$

is easily seen to be strictly increasing in s, so this cutoff characterization directly translates into a cutoff $s^* \in S$.

We then restrict attention to cutoff strategies and aim for a characterization of a symmetric Bayesian Nash equilibrium where all forecasters use the same cutoff $s^* \in S$. The probability that any other forecaster forecasts 1 when 1 is the state is then

$$\pi(1|1) = \int_{s^*}^{1} \frac{(1 + st)}{2} \, ds = \frac{1}{2}\left(1 - s^* + \frac{(1 - s^{*2})\, t}{2}\right).$$

Likewise, the probability the other forecaster forecasts -1 when -1 is the state is

$$\pi(-1|-1) = \int_{-1}^{s^*} \frac{(1 - st)}{2} \, ds = \frac{1}{2}\left(1 + s^* + \frac{(1 - s^{*2})\, t}{2}\right).$$

Now, by the conditional independence of the private signals,

$$W(x|x) = \sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n-1}{k} \pi(x|x)^k (1-\pi(x|x))^{n-1-k}.$$

Note that

$$\sum_{k=0}^{n-1} \frac{n}{k+1} \binom{n-1}{k} \pi^k (1-\pi)^{n-1-k} = \frac{1}{\pi} \sum_{k=1}^{n} \binom{n}{k} \pi^k (1-\pi)^{n-k} = \frac{1-(1-\pi)^n}{\pi}.$$

Thus, we obtain finally

$$\begin{aligned} W(1|1) &= \frac{1-(1-\pi(1|1))^n}{\pi(1|1)}, \\ W(-1|-1) &= \frac{1-(1-\pi(-1|-1))^n}{\pi(-1|-1)}, \\ W(-1|1) &= (1-\pi(1|1))^{n-1}, \\ W(1|-1) &= (1-\pi(-1|-1))^{n-1}. \end{aligned}$$

If the cutoff is interior to S, it is determined by the indifference condition $q(s^*) W(1|1) + (1-q(s^*)) W(1|-1) = q(s^*) W(-1|1) + (1-q(s^*)) W(-1|-1)$. This can be rewritten as

$$\frac{q(s^*)}{1-q(s^*)} = \frac{W(-1|-1) - W(1|-1)}{W(1|1) - W(-1|1)},$$

or substituting the above derivations,

$$\frac{q}{1-q} = \frac{1-s^*t}{1+s^*t} \frac{1-s^*+\frac{(1-s^{*2})t}{2}}{1+s^*+\frac{(1-s^{*2})t}{2}} \frac{2^{n-1} - \left(1-s^* - \frac{(1-s^{*2})t}{2}\right)^{n-1}}{2^{n-1} - \left(1+s^* - \frac{(1-s^{*2})t}{2}\right)^{n-1}}.$$

It is worth comparing the equilibrium cutoff signal with the honest cutoff corresponding to the signal which results in a fair posterior belief: $q(\hat{s}) = 1/2$. This is solved by

$$\hat{s} = \frac{(1-2q)}{t}.$$

Holding fixed $t = 1$, in Figure 4.1 we compare the honest cutoff $\hat{s}$ and the equilibrium cutoff s^* for varying numbers of players. As soon as $n > 2$, for $q < 1/2$ equilibrium forecasting is biased in favor of the ex-ante unlikely outcome $x = 1$. By the logic of the winner's curse, conditionally on x the opponents receive information in favor of x, and so they are more likely to forecast x. There is thus a positive correlation of the true state and the number of forecasters that have taken the corresponding position. This creates an incentive to forecast the less likely alternative, in order to avoid sharing the prize.

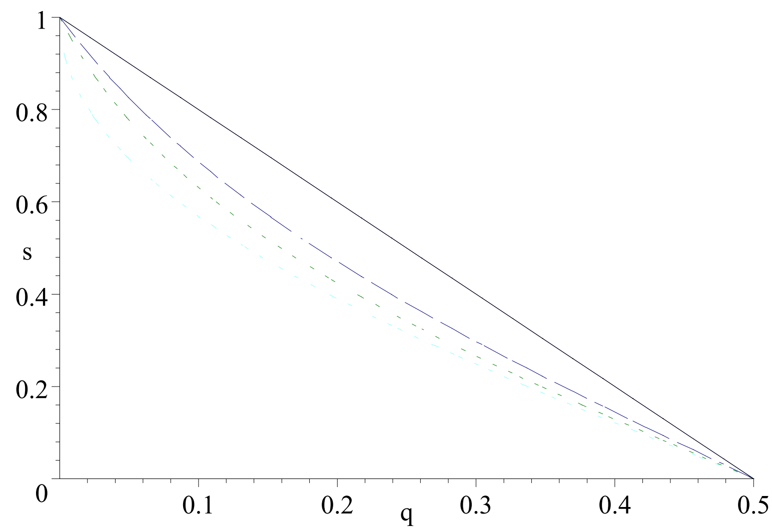

Figure 4.1: The equilibrium cutoff signal s^* is plotted against the prior belief $q \in [0, 1/2]$ for $n = 1, 2, 3, 4, 100000$, in progressively thinner shade. The downward sloping diagonal ($\hat{s} = 1 - 2q$) corresponds to the honest rule. As the number of players increases, the cutoff signal decreases closer to zero.

Holding fixed $n = 3$, in Figure 4.2 we next compare the honest cutoffs $\hat{s}$ for some decreasing values of the quality of information t. To understand why the cut-off is not always interior when $t < 1$, notice that if there is a very small prior chance q of outcome 1, and there is almost no private information, then $(1 - q) > qn$ — it is better to knowingly share the prize in outcome -1 than to gamble on winning alone in outcome 1. This intuition readily extends to any situation where $t < 1$, since there is then a uniform upper bound to how informative any signal realization can be. We observe that the weaker is the information, the smaller is the tendency to go against the ex-ante. Perhaps less obvious from the figure, there is still a bias towards the ex-ante unlikely state — the honest cutoff would also be changed. Indeed, in the limit as t tends to zero, the cutoff remains non-trivial in nature. In this case, the signal represents pure noise, so we are essentially back in the world of symmetric information. Without the existence of the private signal, any equilibrium would then involve mixed strategies. But here, the noisy signal is employed to purify the mixed strategy equilibrium in a natural way. While in the pure-noise limit, every player is held indifferent among forecasts $m = -1, 1$, for any $t > 0$ there was no such global indifference in the equilibria.

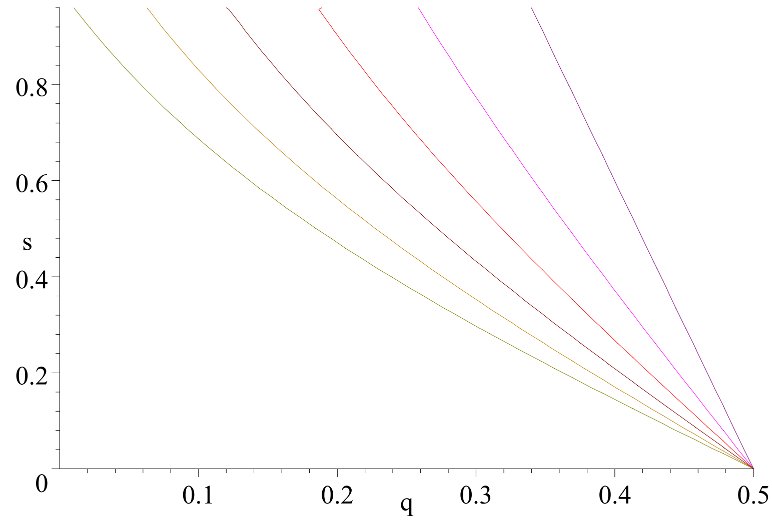

Figure 4.2: The equilibrium cutoff signal s^* is plotted against the prior belief $q \in [0, 1/2]$ for $t = 1, .8, .6, .4, .2, 0$, progressively further to the right. As the signal precision increases, the cutoff signal decreases closer to zero.

5. Continuous State: Characterizations of Winner-Takes-All Contest

Although we have not yet extended the equilibrium existence result to this setting, we now offer some characterizations and simulations of the equilibria when the joint distribution of $(x, s_1, \ldots, s_n)$ is continuous, with each coordinate living on the real line $\mathbb{R}$.

We assume that the distribution is such that there exists an equilibrium in which all forecasters employ an increasing, pure strategy. Given that i has sent message m_i and the state is x, forecaster i wins if all the other m_j are outside the interval with length $2m_i$ centered around x. Let the strategy of all forecasters be a strictly increasing function $m_j(s)$. Consider the case $m_i \leq x$, then that interval is $[m_i, 2x - m_i]$ so that the signal interval for s_j is given by

$$m_i \leq m_j(s_j) \leq 2x - m_i$$

or

$$m_j^{-1}(m_i) \leq s_j \leq m_j^{-1}(2x - m_i).$$

Since $s_j|x, s_i \sim f_j(s_j|x, s_i)$, allowing for dependence across i conditional on the state x, the chance that i wins over j when $m_i \leq x$ is

$$\Pr(i \text{ wins over } j|m_i, x, s_i) = 1 - \int_{m_j^{-1}(m_i)}^{m_j^{-1}(2x-m_i)} f_j(s_j|x, s_i)\, ds_j \tag{5.1}$$

Similarly, for the case $m_i \geq x$ we get

$$\Pr(i \text{ wins over } j|m_i, x, s_i) = 1 - \int_{m_j^{-1}(2x-m_i)}^{m_j^{-1}(m_i)} f_j\left(s_j|x, s_i\right) ds_j. \tag{5.2}$$

The expected value of sending message m_i after receiving signal s_i is

$$V_i(m_i|s_i) = \int_X \prod_{j \neq i} \left[\Pr(i \text{ wins over } j|m_i, x, s_i)\right] q_i(x|s_i) dx.$$

Forecaster i wins the overall contest if and only if it wins against all other forecasters. Using (5.1) and (5.2), we then have

$$\begin{aligned} V_i(m_i|s_i) &= \int_{-\infty}^{m_i} \prod_{j \neq i} \left(1 - \int_{m_j^{-1}(2x-m_i)}^{m_j^{-1}(m_i)} f_j\left(s_j|x, s_i\right) ds_j\right) q_i(x|s_i) dx \qquad (5.3) \\ &+ \int_{m_i}^{+\infty} \prod_{j \neq i} \left(1 - \int_{m_j^{-1}(m_i)}^{m_j^{-1}(2x-m_i)} f_j\left(s_j|x, s_i\right) ds_j\right) q_i(x|s_i) dx \end{aligned}$$

Taking the derivative of each $V_i(m_i|s_i)$ with respect to m_i, one derives a system of first order conditions jointly necessary for a pure-strategy monotonic equilibrium profile. This system of differential equations reduced to a single equation under the assumption that signals are symmetrically distributed, $f_i\left(s_i|x, s_j\right) = f_j\left(s_j|x, s_i\right)$. As shown in the appendix, if in addition signals are conditionally independent, $f\left(s_i|x, s_j\right) = f\left(s_i|x\right)$, we have:

Proposition 4 (First Order Condition). *Under signal symmetry and conditional independence, the first order condition describing the symmetric monotonic equilibrium is given by the differential equation*

$$\begin{aligned} &\int_m^{+\infty} \left(1 - \int_{m^{-1}(2x-m)}^{s} f\left(z|x\right) dz\right)^{n-2} \left[\frac{f(s|x)}{m'(s)} + \frac{f\left(m^{-1}(2x-m)|x\right)}{m'(m^{-1}(2x-m))}\right] f\left(s|x\right) q(x) dx \\ = &\int_{-\infty}^{m} \left(1 - \int_{s}^{m^{-1}(2x-m)} f\left(z|x\right) dz\right)^{n-2} \left[\frac{f(s|x)}{m'(s)} + \frac{f\left(m^{-1}(2x-m)|x\right)}{m'(m^{-1}(2x-m))}\right] f\left(s|x\right) q(x) dx, \qquad (5.4) \end{aligned}$$

to be solved for a strictly increasing $m(s)$ with initial condition $m\left(\mu\right) = \mu$.

As shown in the appendix, (5.4) can be rewritten as

$$\begin{aligned}
&\int_s^{+\infty}\left(1-\int_y^s f\left(z|\tfrac{m(y)+m(s)}{2}\right)dz\right)^{n-2}\tfrac{m'(y)}{m'(s)}\left(f\left(s|\tfrac{m(y)+m(s)}{2}\right)\right)^2 q(\tfrac{m(y)+m(s)}{2})dy \qquad (5.5)\\
&+\int_s^{+\infty}\left(1-\int_y^s f\left(z|\tfrac{m(y)+m(s)}{2}\right)dz\right)^{n-2} f\left(y|\tfrac{m(y)+m(s)}{2}\right)f\left(s|\tfrac{m(y)+m(s)}{2}\right)q\left(\tfrac{m(y)+m(s)}{2}\right)dy\\
=&\int_{-\infty}^s\left(1-\int_s^y f\left(z|\tfrac{m(y)+m(s)}{2}\right)dz\right)^{n-2}\tfrac{m'(y)}{m'(s)}\left(f\left(s|\tfrac{m(y)+m(s)}{2}\right)\right)^2 q(\tfrac{m(y)+m(s)}{2})dy\\
&+\int_{-\infty}^s\left(1-\int_s^y f\left(z|\tfrac{m(y)+m(s)}{2}\right)dz\right)^{n-2} f\left(y|\tfrac{m(y)+m(s)}{2}\right)f\left(s|\tfrac{m(y)+m(s)}{2}\right)q\left(\tfrac{m(y)+m(s)}{2}\right)dy
\end{aligned}$$

which is nicely free from the inverse strategy. With given functional forms for q and f, this equation can be solved numerically.

To further understand the game when the number of players is 2, let us re-derive the first order condition from a different point of view. Above, we have always let the forecaster first consider the possible location of x, and second consider the conditional distribution of the opponent's signal. Equally naturally, the forecaster could first consider the opponent's signal, and second consider the conditional distribution of x. Recall that player i wins with m_i if it is closer than $m_j(s_j)$ to x. When $m_i \leq m_j(s_j)$, i.e. $m_j^{-1}(m_i) \leq s_j$, then i wins when $x \in (-\infty, (m_i + m_j(s_j))/2]$. Likewise, when $m_i \geq m_j$, i.e. $m_j^{-1}(m_i) \geq s_j$, then i wins when $x \in [(m_i + m_j(s_j))/2, \infty)$. The expected value of sending message m_i after receiving signal s_i is then

$$\begin{aligned}
V(m_i|s_i) &= \int_{-\infty}^{m_j^{-1}(m_i)}\int_{\frac{m_i+m_j(s_j)}{2}}^{+\infty} q(x|s_i,s_j)\,dx f(s_j|s_i)\,ds_j\\
&\quad+\int_{m_j^{-1}(m_i)}^{+\infty}\int_{-\infty}^{\frac{m_i+m_j(s_j)}{2}} q(x|s_i,s_j)\,dx f(s_j|s_i)\,ds_j
\end{aligned}$$

where $f(s_j|s_i)$ denotes the conditional p.d.f. for s_j given s_i. The first order condition is derived using Leibnitz's rule, as before. We find that $V_m(m_i|s_i)$ is

$$\begin{aligned}
&\frac{f\left(m_j^{-1}(m_i)|s_i\right)}{m_j'\left(m_j^{-1}(m_i)\right)}\left[\int_{m_i}^{+\infty} q\left(x|s_i,m_j^{-1}(m_i)\right)dx-\int_{-\infty}^{m_i} q\left(x|s_i,m_j^{-1}(m_i)\right)dx\right]\\
&-\frac{1}{2}\left[\int_{-\infty}^{m_j^{-1}(m_i)} q\left(\tfrac{m_i+m_j(s_j)}{2}|s_i,s_j\right)f(s_j|s_i)\,ds_j-\int_{m_j^{-1}(m_i)}^{+\infty} q\left(\tfrac{m_i+m_j(s_j)}{2}|s_i,s_j\right)f(s_j|s_i)\,ds_j\right].
\end{aligned}$$

Consider the first term. A marginal increase in m_i will push the opponent's signal values s_j at $m_j^{-1}(m_i)$ over from giving a message m_j above m_i to giving a message below m_i. The mass of those signals pushed is given by the ratio $f\left(m_j^{-1}(m_i)|s_i\right)/m_j'\left(m_j^{-1}(m_i)\right)$. The marginal gain in the chance of outcomes x where m_i beats m_j is the difference

$\int_{m_i}^{+\infty} q\left(x|s_i, m_j^{-1}(m_i)\right) dx - \int_{-\infty}^{m_i} q\left(x|s_i, m_j^{-1}(m_i)\right) dx$. This term alone will vanish precisely when there is equal chance of winning on both sides, i.e. when m_i is the median of the distribution over x given signal realizations s_i and $s_j = m_j^{-1}(m_i)$. This finding is related to a result of Bernhardt, Duggan and Squintani (2002). Having assumed that the signal space is discrete, they show that a pure strategy equilibrium must employ this posterior median.

However, here we also have a second term arising from our continuous signal space. When m_i is increased marginally, for every s_j such that $m_j(s_j) \leq m_i$ the opponent wins with greater frequency (of x), as expressed by the factor $q\left(\left(m_i + m_j(s_j)\right)/2|s_i, s_j\right)/2$. Averaging over these values of s_j gives the factor $-\int_{-\infty}^{m_j^{-1}(m_i)} q\left(\left(m_i + m_j(s_j)\right)/2|s_i, s_j\right) f(s_j|s_i)\, ds_j/2$. This loss to the opponent is balanced by the similar greater frequency with which one beats an opponent with $m_j(s_j) \geq m_i$.

5.1. Symmetric Location Models

Under a symmetric location assumption that $q(x)$ is symmetric around μ and $f(s|x) = g(|s-x|)$, the increasing strategy $m(\cdot)$ naturally satisfies anti-symmetry around μ, meaning that $m(\mu+s) - \mu = \mu - m(\mu - s)$. Observe that anti-symmetry of $m(\cdot)$ through μ implies anti-symmetry of $m^{-1}(\cdot)$ through μ, that $m^{-1}(\mu+m) - \mu = \mu - m^{-1}(\mu - m)$. The appendix provides the straightforward verification, that if all opponents employ anti-symmetric strategies, then $V_i(\mu + m_i|\mu + s_i) = V_i(\mu - m_i|\mu - s_i)$. This implies that an anti-symmetric strategy is a best response, so anti-symmetry is preserved by the best-reply correspondence.

Proposition 5 (Anti-Symmetry). *Assume that q and f are a symmetric location model. Anti-symmetry of the opponents' strategy implies an anti-symmetric best response.*

In the particular case with $n = 2$ players, we can obtain a further necessary condition from the first order equation. Differentiating the identity (5.5) with respect to s, and evaluating at μ, we obtain (calculations available on request) the criterion

$$\begin{aligned} 0 \;=\;& -3g^2(0)\, q(\mu) + 2\int_{\mu}^{+\infty} \frac{2}{m'(\mu)} g'(\mu - x)\, g(\mu - x)\, q(x) dx \qquad (5.6) \\ & - \int_{\mu}^{+\infty} \frac{m'(\mu)}{2} g'\left(\frac{2y - m(y) - \mu}{2}\right) g\left(\frac{\mu - m(y)}{2}\right) q\left(\frac{m(y) + \mu}{2}\right) dy \\ & + \int_{\mu}^{+\infty} \frac{2 - m'(\mu)}{2} g\left(\frac{2y - m(y) - \mu}{2}\right) g'\left(\frac{\mu - m(y)}{2}\right) q\left(\frac{m(y) + \mu}{2}\right) dy \\ & + \int_{\mu}^{+\infty} \frac{m'(\mu)}{2} g\left(\frac{2y - m(y) - \mu}{2}\right) g\left(\frac{\mu - m(y)}{2}\right) q'\left(\frac{m(y) + \mu}{2}\right) dy. \end{aligned}$$

This criterion can be checked. Assume, for the sake of argument, that the right hand side turns out to be positive. This means that we have found $V_m(m(s)|s)$ to be an increasing function of s (at μ). Then for s just above μ, we have $V_m(m(s)|s) > 0$. This means that the forecaster can improve his reply from $m(s)$ by further increasing his message. So, such a finding implies an incentive to exaggerate, locally around μ. Conversely, if the derivative were found to be negative, the incentive leads to conservative deviations.

5.2. Normal Model

Consider the normal location model. The state has a normal prior distribution, $x \sim N(\mu, 1/\nu)$ with p.d.f.

$$q(x) = \sqrt{\frac{\nu}{2\pi}} e^{-\frac{\nu}{2}(x-\mu)^2}. \tag{5.7}$$

Signals $s_i|x \sim N(x, 1/\tau_i)$ are conditionally independent, with conditional p.d.f.

$$f(s_i|x) = \sqrt{\frac{\tau_i}{2\pi}} e^{-\frac{\tau_i}{2}(s_i-x)^2}. \tag{5.8}$$

The posterior on the state conditional on observation of signal s_i is $x|s_i \sim N\left(\frac{\nu\mu+\tau_i s_i}{\nu+\tau_i}, \frac{1}{\nu+\tau_i}\right)$, so that

$$q(x|s_i) = \sqrt{\frac{\nu+\tau_i}{2\pi}} e^{-\frac{(\nu+\tau_i)}{2}\left(x-\frac{\nu\mu+\tau_i s_i}{\nu+\tau_i}\right)^2}. \tag{5.9}$$

In this setting, the classic Hotelling "median voter theorem" does not generalize with two players. It suffices to make this point when the model is symmetric, i.e., $\tau_i = \tau_j$. There is no equilibrium in which each forecaster i reports the median (equal to the mean) of the posterior distribution on the state conditional on the signal received s_i.

To prove this point, we first note that with symmetric linear strategies $m(s) = ks + (1-k)\mu$, the necessary criterion (5.6) reduces to (calculations in appendix)

$$3 = \frac{4\tau}{(2\tau+\nu)k} + \frac{k(2\tau(2-k)+\nu k)}{\tau(2-k)^2 + (\tau+\nu)k^2}.$$

In case of the truthtelling strategy, $k = \tau/(\tau+\nu)$ the criterion further reduces to

$$3 = \frac{4(\tau+\nu)}{2\tau+\nu} + \frac{\tau(5\nu+2\tau)}{4\nu^2+5\nu\tau+2\tau^2}.$$

Equivalently, this can be written as a condition for the parameter $t = \tau/\nu$, the ratio of quality of private information to quality of prior. The condition is then equivalent to $0 = 4t^2 + 2t + 4$ which fails. The right hand side always exceeds the left hand side, so we are in the case where the derivative of $V_m(m(s)|s)$ is found to be positive at $s = \mu$. In fact, this implies a local (near $s = \mu$) best deviation to exaggeration.

We can now return to Bernhardt, Duggan and Squintani's (2002) generalization of Hotelling's rule, that the equilibrium strategy should equal the posterior median given one's own signal, and given that the opponent received the same signal. In our normal model with conditional independence of signals, this forecast is linear with weight $k = 2\tau/(\nu + 2\tau)$. Inserting this in our criterion, we find

$$3 = 2 + \frac{\tau(3\nu + 2\tau)}{(\tau + \nu)(2\tau + \nu)}.$$

This condition fails since it is equivalent to $0 = \nu^2$. The left hand side always exceeds the right hand side, so $V_m(m(s)|s) < 0$ and it pays locally (at $s = \mu$) to deviate to a more conservative response.[15]

5.3. Infinitely Many Players

This section considers the winner-take-all contest with infinitely many players, introduced in our companion paper Ottaviani and Sørensen (2002). Consider a pure strategy symmetric equilibrium, whereby all players use the same pure $m(s)$ map. For now assume that m is twice differentiable and increasing. Conditionally on x, the signals of others are distributed as $f(s|x)$ and thus their messages are distributed

$$g(m|x) = \frac{f(s|x)}{m'(s)} = f(s(m)|x)s'(m).$$

In order to define the individual payoff, we take the limit as the number of players goes to infinity.

Intuitively, with many players, you only win if $m = x$ and the tendency to win is inversely proportional to the mass of opponents who then also guessed in the range m. Alas,

$$V(m|s) = \frac{q(m|s)}{g(m|m)}.$$

Formally, consider the forecasting contest with n homogenous players competing for a prize of size n. Without loss of generality, focus on player 1, and let this player conjecture that all the other forecasters apply the same strictly increasing, continuously differentiable strategy $\hat{m}$. Once 1 has sent message m and the state is x, any opponent beats i with a message in the interval closer to x. In case $m \leq x$, then that message interval is $[m, 2x - m]$. Similarly, for the case $m \geq x$ the opponent wins when the message is in the

[15] Our characterizations are based on the assumption of the existence of a symmetric pure-strategy equilibrium. Logically, it is also possible that no such pure-strategy equilibrium exists. Bernhardt, Duggan and Squintani notice that there may fail to exist a pure strategy equilibrium in their discrete signal setting. However, the simulations reported below have run smoothly. So, we believe the simulations lead (approximately) to a pure strategy equilibrium.

interval $[2x-m, m]$. Player 1 is the winner only when he beats all of the opponents. The expected value of sending message m after receiving signal s_1 is then

$$\begin{aligned} U^n(m|s_1) &= \int_{-\infty}^{m} n\left[1-\int_{2x-m}^{m} g\left(m_j|x\right) dm_j\right]^{n-1} q(x|s_1)dx \\ &+ \int_{m}^{+\infty} n\left[1-\int_{m}^{2x-m} g\left(m_j|x\right) dm_j\right]^{n-1} q(x|s_1)dx. \end{aligned}$$

Lemma 1. *Assume that the opponents' common strategy is linear with a slope no greater than one. In the limit as n tends to infinity, player 1's expected payoff $U^n(m_1|s_1)$ from forecast m_i given private signal s_i tends to*

$$U(m_1|s_1) = \frac{q(m_1|s_1)}{g(m_1|m)}.$$

Proof. See the Appendix. □

This result provides a foundation for the payoff function assumed in the limit winner-take-all forecasting contest analyzed by Ottaviani and Sorensen (2002). Heuristically, focus on the case in which the opponents use a linear strategy, with a slope no greater than one.[16] The strategies of the opponents can be used to derive the conditional density $\gamma(m|x)$ of their forecasts, with full support on the real line. Issuing forecast m_i, forecaster i wins only if $x = m_i$, which occurs with chance $q(m_i|s_i)$. Conditional on being on the mark, the prize pool is divided among all the winning forecasters, and their density computed at $x = m_i$ is equal to $\gamma(m_i|m_i)$.

To calculate

$$V_m(m|s) = \frac{q_x(m|s)g(m|m) - q\left(m|s\right)\left[g_m(m|m) + g_x(m|m)\right]}{\left(g(m|m)\right)^2}$$

we make use of some facts. First,

$$g_m(m|x) = f_s(s(m)|x)\left(s'(m)\right)^2 + f(s(m)|x)s''(m)$$

and since $s'(m) = 1/m'(s(m))$ we have

$$s''(m) = -\frac{1}{\left(m'(s)\right)^2} m''(s)s'(m).$$

Second, $g_x(m|x) = f_x\left(s(m)|x\right)s'(m)$. Third, $q\left(x|s\right) = f\left(s|x\right)q(x)/f(s)$ so

$$q_x\left(x|s\right) = \frac{f_x\left(s|x\right)q(x) + f\left(s|x\right)q_x(x)}{f(s)}.$$

[16] Note that if the opponents are honest, they are using a linear strategy with slope $\tau/\left(\tau+\nu\right) < 1$. As shown below, there exists a symmetric equilibrium where forecasters also adopt linear strategies with slope less than one.

Now we find that the first order condition $V_m(m(s)|s) = 0$, equivalent to

$$\frac{q_x(m|s)}{q(m|s)} = \frac{g_m(m|m) + g_x(m|m)}{g(m|m)},$$

can be rewritten as

$$\frac{f_x(s|m)\, q(m) + f(s|m)\, q_x(m)}{f(s|m)\, q(m)} = \frac{f_s(s|m)\,(s'(m))^2 + f(s|m)s''(m) + f_x(s|m)\, s'(m)}{f(s|m)s'(m)}$$

i.e.

$$f(s|m)\,\frac{q_x(m)}{q(m)} = f_s(s|m)\,(s'(m)) + f(s|m)\frac{s''(m)}{s'(m)}$$

or

$$m'(s)\frac{q_x(m)}{q(m)} = \frac{f_s(s|m)}{f(s|m)} - \frac{m''(s)}{m'(s)}.$$

In the normal model, we have $\frac{f_s(s|x)}{f(s|x)} = -\tau(s-x)$ and $\frac{q_x(x)}{q(x)} = -\nu(x-\mu)$, so that the necessary first-order condition is

$$m'\nu(m-\mu) = \tau(s-m) - \frac{m''}{m'}.$$

It is seen that there is a solution in linear strategies, $m(s) = As + C\mu$. The equation is then

$$A\nu(As + C\mu - \mu) = \tau(s - As - C\mu),$$

so that the parameters A and C must solve $\nu A^2 = \tau(1-A)$ and $A\nu(C-1) = -\tau C$. Next, notice that $m(\mu) = \mu$ must hold, since otherwise the necessary first-order condition computed at $s = \mu$ would give $m'(\mu) = -\tau/\nu < 0$, in contradiction with the conjecture that $m(.)$ is increasing. Clearly, $m(\mu) = \mu$ implies $A + C = 1$. There is a unique solution

$$A = \frac{\sqrt{(\tau^2 + 4\nu\tau)} - \tau}{2\nu} = 1 - C = 1 - \frac{\sqrt{(\tau^2 + 4\nu\tau)} - \tau}{\sqrt{(\tau^2 + 4\nu\tau)} + \tau} \in [\frac{\tau}{(\nu+\tau)}, 1].$$

To gain intuition on the forces driving the equilibrium, it is useful to consider the best reply of a player against opponents who all use a linear strategy of the form

$$m(s) = As + (1-A)\mu.$$

The mean location of an opponent conditional on the state is $E[m|x] = Ax + (1-A)\mu$ and the conditional variance is $V[m|x] = \frac{A^2}{\tau}$. The honest strategy h is the special case with $A = \tau/(\tau+\nu)$, for which $V[h|x] = \tau/(\tau+\nu)^2$. The best reply solves

$$\max_m \frac{q(m|s)}{g(m|m)}$$

Taking the logarithm, using the signal distributions and simplifying, the maximand becomes

$$-\frac{\tau+\nu}{2}\left(m-E\left[x|s\right]\right)^{2}+\frac{\tau\left(1-A\right)^{2}}{2A^{2}}\left(m-\mu\right)^{2}$$

with first order condition $0=-\left(\tau+\nu\right)\left(m-E\left[x|s\right]\right)+\tau\left(1-A\right)^{2}\left(m-\mu\right)/A^{2}$. Solving the first order condition for m, we get a linear best reply $m=Bs+\left(1-B\right)\mu$ with

$$B\left(A\right)=\frac{\tau}{\tau+\nu-\frac{\tau(1-A)^{2}}{A^{2}}}.$$

The second order condition is

$$-\left(\tau+\nu\right)+\frac{\tau\left(1-A\right)^{2}}{A^{2}}<0\Leftrightarrow A>\left(\sqrt{\tau\nu+\tau^{2}}-\tau\right)/\nu.$$

The denominator of B is positive under this same condition.

Locating at the posterior expectation ($B=\frac{\tau}{\tau+\nu}$) is the best reply to $A=1$, i.e. when all the other players locate at their signal with no weight on the prior. If instead the opponents have a positive weight on the prior, the best reply for an individual player is to put more weight on the signal than according to the honest forecast. Note that the best reply weight on own signal $B\left(A\right)$ is decreasing in A for $A\leq 1$. When $A>1$, $B\left(A\right)$ is increasing in A.

A player's best location maximizes the ratio between the probability of winning the first prize and the number of opponents with whom this prize is shared. First, the probability of winning conditional on signal s is equal to the posterior belief on the state $x|s$, a normal distribution centered at $E[x|s]$. The denominator of the objective function is $g(x|x)$, equal to the mass of opponents who also locate at x when the state is x.

The shape of $g(x|x)$ depends on the weight assigned by the other forecasters to their signal.[17] We now argue that since the other forecasters put some weight on the prior mean, $g(x|x)$ is bell shaped around μ. To understand why this is the case consider first the case in which the weight on the prior mean is positive so that $A<1$. Take two states x and x', with $x'>x>\mu$ so that x is closer to μ. The value taken by the p.d.f. $g(m|x)$ at $m=x$ is higher than the the value taken by the p.d.f. $g(m|x')$ at $m=x'$. Note that $g(x|x)$ is *not* a probability density function. For example, when the other forecasters have perfectly informative signals $g(x|x)$ is constantly equal to 1. Note that $g(x|x)$ is also bell shaped around μ when the opponents put a negative weight on the prior mean, $A>1$.

The probability of winning is flat at the honest $E[x|s]$ with a unimodal and symmetric posterior distribution, while the fraction of correct forecasts is decreasing in the distance

[17] The shape of $g(x|x)$ does not depend on the forecaster's signal s_i because the state x is sufficient for s_i in the inference of the other forecasters' signals s_{-i} due to the assumption that the signals of different forecasters are independent conditional on the state x.

of the state x from the prior mean μ. The forecaster is unsure about where to find x, but understands that the opponents are concentrated around μ. It is then optimal to move away from the prior mean.

The equilibrium solves $B(A) = A$, resulting in the weight on the signal

$$A = \frac{\sqrt{\tau^2 + 4\nu\tau} - \tau}{2\nu}$$

Notice that this is increasing in τ and decreasing in ν, like the honest weight.

6. Normal Model: Simulations

We now return to the general prize structure, but continue with the normal model. We report results of simulations aimed at characterizing the effect of the prize structure and the number of forecasters on the equilibrium.[18]

We initialize with the symmetric strategy $m_0(s)$ that each forecaster uses the best predictor in the normal model, given own signal. The simulation contains a main loop of symmetric strategy improvement, repeated T times. For each signal s in a grid with S points, we use a Monte Carlo method (with I random draws) to approximate the expected return to any of M small deviations in a grid around $m_t(s)$. The strategy that achieves the highest return becomes the new value of $m_{t+1}(s)$. After the T-th loop, we have a final simulated equilibrium strategy $m_T(s)$. Due to the randomness of the Monte Carlo approach, this strategy is not smooth, so the pictures below report a best fit anti-symmetric polynomial $p_5(s-\mu)^5 + p_3(s-\mu)^3 + p_1(s-\mu) + \mu$ to $m_T(s)$. In all the simulations reported below, we have set the location model's parameters at $\mu = 0$, $\nu = 1$, and $\tau = .2$, and the simulation parameters at $T = 50, I = 15, M = S = 31$.

6.1. Winner-Takes-All Contest

Our first comparative statics exercise investigates how the pressure to deviate from honesty is affected by the number of opponents in the winner-takes all contest. Intuitively, the greater is the number of forecasters, the denser are their forecasts, and the more one needs to exaggerate in order to be the unique winner of the contest.

For $n = 2$, at extreme signals the equilibrium strategy puts a lower weight on the signal compared to the conditional expectation, as illustrated in Figure 6.1. In this sense, the equilibrium is "conservative". This result again contrasts with Bernhardt, Duggan and Squintani's (2002) prediction of equilibrium exaggeration obtained with discrete signals. The Figure suggest that the weight assigned to the signal by the (symmetric) equilibrium

[18] Our Matlab code is available upon request.

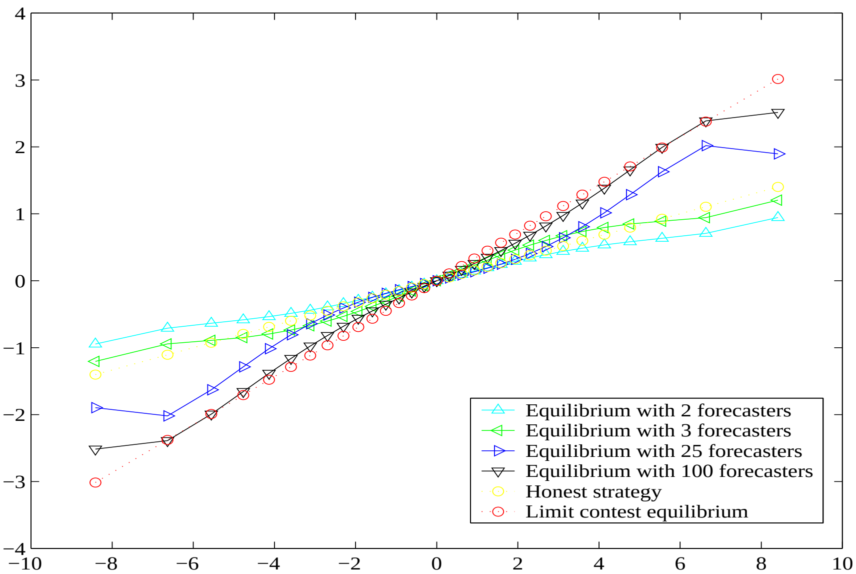

Figure 6.1: This shows the equilibrium strategies predicted in a simulation of the winner-takes-all contest, where the number of forecasters is $n = 2, 3, 25, 100$. We also show the honest strategy, and the limit strategy that is a theoretical prediction of the model as n tends to infinity.

strategy increases with the number of players, eventually approaching the theoretical prediction of Ottaviani and Sørensen (2002) for the limit contest with an infinite number of players. Since the state realizations at the ends of the interval are very unlikely, the strategy for those states can easily be far from optimal. This explains the non-monotonicities of the strategy for those extreme states.

6.2. Loser-Loses-All Contest

Our second comparative statics exercise is similar to the first one. In a loser-loses-all contest, the worst forecaster is singled out to take a discrete loss. We investigate again how the pressure to deviate from honesty is affected by the number of opponents. Intuitively, the greater is the number of forecasters, the more one is able to secure no loss by hiding among them, and thus the greater the pressure for conservatism.

Figure 6.2 shows the resulting picture. Already at $n = 2$ there is some conservatism and this phenomenon quickly becomes very pronounced as the number of forecasters increases. The simulations suggest that the strategy moves ever closer to $m(s) = \mu$, as n tends to infinity.

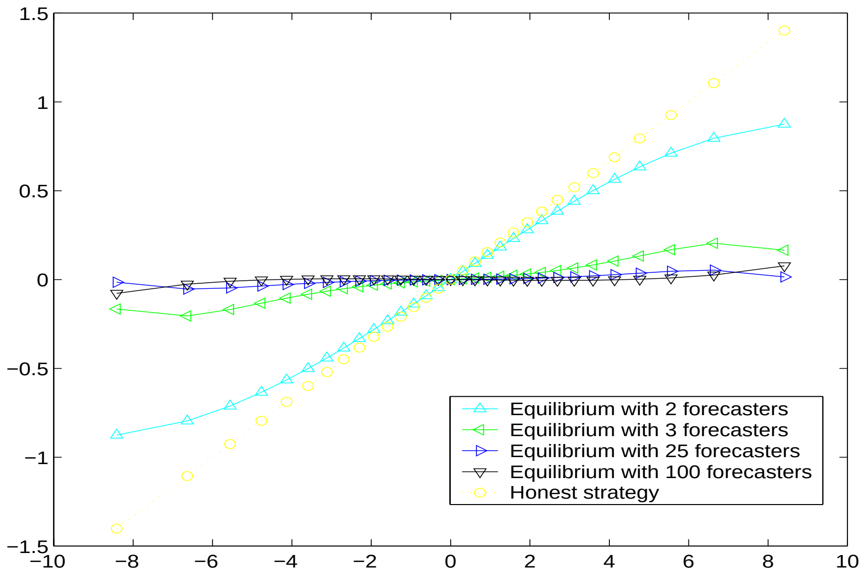

Figure 6.2: This shows the equilibrium strategies predicted in a simulation of the loser-loses-all contest, where the number of forecasters is $n = 2, 3, 25, 100$. We also show the honest strategy.

6.3. Effect of Incentives

The winner-takes-all prize structure offers a reward to the best forecaster, and nothing to the rest. It is in some sense a very convex prize structure, inducing a risk-taking behavior that leads to excessive differentiation. On the opposite end of the spectrum, the loser-punished prize structure is flat for all but the lower, and it is in this sense very concave. We now illustrate in one picture the effect on the equilibrium as we move between these two extremes. We offer here a convex prize structure defined by a quadratic function $Z_k = (n-k)^2$ of the realized forecast rank k, a linear function of the rank, and a concave function $Z_k = n^2 - k^2$.

In Figure 6.3 we used $n = 25$ forecasters. The winner-takes-all and loser-loses-all equilibria are the most extreme, and equivalent to the outcomes of the previous two simulations for $n = 25$. For the three intermediate prize structures we see the expected result, that the greater is the convexity, the greater is the pressure to differentiate. However, locally around μ, where the simulation is actually to be expected to yield the most precise results, the winner-takes-all strategy is rather conservative. Notice, that this conservativeness is greatly reduced in the winner-takes-all contest as the number of the forecasters increases further, as seen in Figure 6.1.

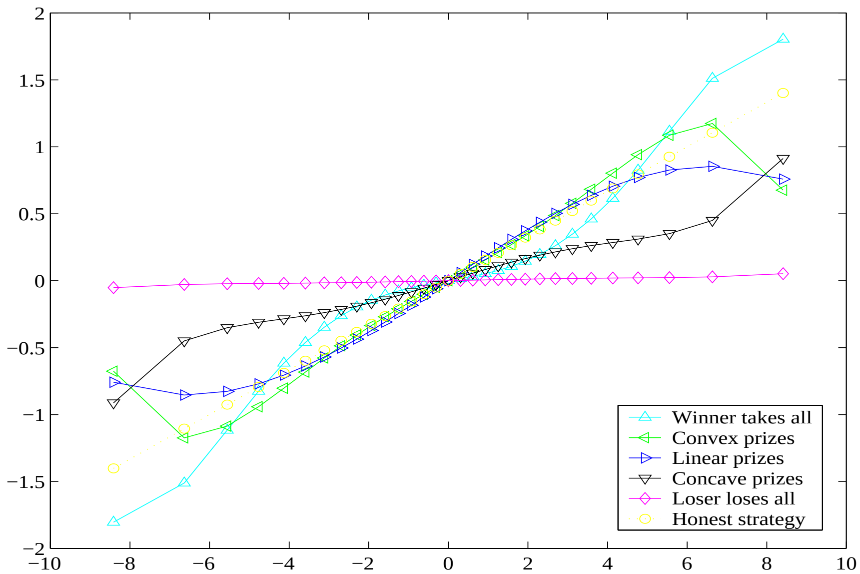

Figure 6.3: This shows the equilibrium strategies predicted in a simulation of the contest with $n = 25$ forecasters, as the prize struture displays varying degrees of convexity.

7. Discussion

While we have focused on a positive theory of forecasting contests, it is natural to ask normative questions. What purpose do forecasting contests serve? The statistical literature on scoring rules and probability forecasting is normative in nature, but has focused on the case of single forecasters, disregarding the possible strategic elements arising with multiple forecasters.[19] In analogy with Holmström's (1979 and 1982) informativeness principle, we conjecture that in our setting whenever the realization of the state is sufficient for the message reported by the other forecasters, there is no gain from conditioning the state-contingent reward also on such messages.[20] When instead the state is observed with noise, or equivalently with conditionally dependent signals, conditioning the reward also on the forecasts of the competitors might instead be beneficial.

Finally, we intend to compare the incentives for information acquisition in different forecasting contests. See also Osband (1989) for a normative approach to incentives for information acquisition for individual forecasters.

[19] See Dawid (1986) for an excellent review.

[20] The statement is in the weak form because of the assumption of risk neutrality

8. Appendix

Proof of Proposition 1. Let $W_i(m_i|x)$ denote the expected prize to forecaster i, conditionally on state x being realized and message m_i being sent. We can express $W_i(m_i|x)$ from the prizes $Z_n \leq \ldots \leq Z_1$ as $W_i(m_i|x) = \sum_{k=1}^{n} \Pr(i \text{ places } k-\text{th}|m_i, x) Z_k$. We will argue that $W_i(m_i|x)$ is a weakly decreasing function of $|m_i - x| \in [0, \pi]$. By assumption, for any opponent $j \neq i$, $\Pr(i \text{ beats } j|m_i, x)$ is a function of $|m_i - x| \in [0, \pi]$. This function is weakly decreasing, for the smaller the gap from m_i to x, the smaller the probability that opponent j sent a message closer to x. By conditional independence of opponents' signals, beating opponent j and beating opponent j' are independent events (with opponent-dependent probabilities). It now follows from primitive probability analysis, that $\Pr(i \text{ places } k-\text{th or better}|m_i, x)$ is a weakly decreasing function of $|m_i - x| \in [0, \pi]$. This result states that an increase in $|m_i - x|$ implies a first-order stochastic decrease in the distribution of player i's rank: the player is more likely to take low ranks. By the weak monotonicity of the prize structure, we conclude that $W_i(m_i|x)$ is a weakly decreasing function of $|m_i - x| \in [0, \pi]$.

Since $q(x)$ is the uniform distribution, the posterior belief on x is described by the p.d.f. $q_i(x|s_i) = f_i(s_i|x)/f_i(s_i) = f_i(s_i|x)$. By symmetry and unimodality of $f_i(s_i|x)$, this distribution of x is symmetric and unimodal around s_i. Thus, $q_i(x|s_i)$ depends only on $|x - s_i|$ and is decreasing in $|x - s_i| \in [0, \pi]$.

These properties imply that $m_i = s_i$ maximizes the expected prize to i, so that truthtelling is optimal. By symmetry, it suffices to consider any possible $m_i \in [s_i, s_i + \pi]$. Let $V_i(m_i|s_i) = \int_X W_i(m_i|x) q(x|s_i)\, dx$ denote the expected prize to forecaster i from sending message m_i when the signal is s_i. We must prove that $V_i(s_i|s_i) \geq V_i(m_i|s_i)$. Note that half of the space X is closer to s_i than to m_i, namely the values of x in the interval $[(s_i + m_i - 2\pi)/2, (s_i + m_i)/2]$. We have

$$
\begin{aligned}
& V_i(s_i|s_i) - V_i(m_i|s_i) \\
= & \int_{(s_i+m_i-2\pi)/2}^{(s_i+m_i)/2} [W_i(s_i|x) - W_i(m_i|x)]\, q_i(x|s_i)\, dx \\
& + \int_{(s_i+m_i)/2}^{(s_i+m_i+2\pi)/2} [W_i(s_i|x) - W_i(m_i|x)]\, q_i(x|s_i)\, dx \\
= & \int_{(s_i+m_i-2\pi)/2}^{(s_i+m_i)/2} [W_i(s_i|x) - W_i(m_i|x)]\, q_i(x|s_i)\, dx \\
& + \int_{(s_i+m_i-2\pi)/2}^{(s_i+m_i)/2} [W_i(s_i|s_i+m_i-x) - W_i(m_i|s_i+m_i-x)]\, q_i(s_i+m_i-x|s_i)\, dx \\
= & \int_{(s_i+m_i-2\pi)/2}^{(s_i+m_i)/2} [W_i(s_i|x) - W_i(m_i|x)]\, [q(x|s) - q(x|m)]\, dx
\end{aligned}
$$

where the first equality is by definition, the second uses the change of variable $y = m_i + s_i - x$ in the second integral, and the last follows from $W_i(s_i|x)$ and $q_i(x|s_i)$ depending on their arguments only through their distance. Since $[(s_i + m_i - 2\pi)/2, (s_i + m_i)/2]$ is the interval of x values closer to s_i than m_i, we have $W_i(s_i|x) \geq W_i(m_i|x)$ and $q_i(x|s_i) \geq q_i(x|m_i)$. Then the integrand is always non-negative and so the integral is non-negative, proving $V_i(s_i|s_i) \geq V_i(m_i|s_i)$. □

Proof of Proposition 4. The derivative of this with respect to m_i, $V_i'(m_i|s_i)$, is found by iterative application of Leibnitz's rule. To compute the last term, note that

$$\frac{\partial\left(\prod_{j\neq i}\left(1-\int_{m_j^{-1}(2x-m_i)}^{m_j^{-1}(m_i)} f_j(s_j|x)\,ds_j\right)\right)}{\partial m_i}$$
$$= -\sum_{k\neq i}\left\{\left(\frac{\partial}{\partial m_i}\int_{m_k^{-1}(2x-m_i)}^{m_k^{-1}(m_i)} f_k(s_k|x)\,ds_k\right)\prod_{j\neq i,k}\left(1-\int_{m_j^{-1}(2x-m_i)}^{m_j^{-1}(m_i)} f_j(s_j|x)\,ds_j\right)\right\}$$

where

$$\frac{\partial}{\partial m_i}\int_{m_k^{-1}(2x-m_i)}^{m_k^{-1}(m_i)} f_k(s_k|x)\,ds_k = \frac{dm_k^{-1}(m_i)}{dm_i} f_k\left(m_k^{-1}(m_i)|x\right) + \frac{dm_k^{-1}(2x-m_i)}{dm_i} f_k\left(m_k^{-1}(2x-m_i)|x\right).$$

Then

$$\begin{aligned}
& V_i'(m_i|s_i) \\
= & \prod_{j\neq i}\left(1-\int_{m_j^{-1}(2m_i-m_i)}^{m_j^{-1}(m_i)} f_j(s_j|m_i)\,ds_j\right) q_i(m_i|s_i) \\
& -\int_{-\infty}^{m_i}\sum_{k\neq i}\left\{\left(\frac{dm_k^{-1}(m_i)}{dm_i} f_k\left(m_k^{-1}(m_i)|x\right) + \frac{dm_k^{-1}(2x-m_i)}{dm_i} f_k\left(m_k^{-1}(2x-m_i)|x\right)\right)\right. \\
& \left.\prod_{j\neq i,k}\left(1-\int_{m_j^{-1}(2x-m_i)}^{m_j^{-1}(m_i)} f_j(s_j|x)\,ds_j\right)\right\} q_i(x|s_i)dx \\
& -\prod_{j\neq i}\left(1-\int_{m_j^{-1}(m_i)}^{m_j^{-1}(2m_i-m_i)} f_j(s_j|m_i)\,ds_j\right) q_i(m_i|s_i) \\
& +\int_{m_i}^{+\infty}\sum_{k\neq i}\left\{\left(\frac{dm_k^{-1}(m_i)}{dm_i} f_k\left(m_k^{-1}(m_i)|x\right) + \frac{dm_k^{-1}(2x-m_i)}{dm_i} f_k\left(m_k^{-1}(2x-m_i)|x\right)\right)\right. \\
& \left.\prod_{j\neq i,k}\left(1-\int_{m_j^{-1}(m_i)}^{m_j^{-1}(2x-m_i)} f_j(s_j|x)\,ds_j\right)\right\} q_i(x|s_i)dx \\
= & \int_{m_i}^{+\infty} U_i^1(x, m_i, m_{-i}(\cdot))\, q_i(x|s_i)dx - \int_{-\infty}^{m_i} U_i^2(x, m_i, m_{-i}(\cdot))\, q_i(x|s_i)dx
\end{aligned}$$

where we define

$$
\begin{aligned}
& U_i^1\left(x, m_i, m_{-i}(\cdot)\right) \\
= & \sum_{k \neq i}\left\{\left(\frac{d m_k^{-1}\left(m_i\right)}{d m_i} f_k\left(m_k^{-1}\left(m_i\right) | x\right)+\frac{d m_k^{-1}\left(2 x-m_i\right)}{d m_i} f_k\left(m_k^{-1}\left(2 x-m_i\right) | x\right)\right)\right. \\
& \left.\prod_{j \neq i, k}\left(1-\int_{m_j^{-1}\left(2 x-m_i\right)}^{m_j^{-1}\left(m_i\right)} f_j\left(s_j | x\right) d s_j\right)\right\}
\end{aligned}
$$

and

$$
\begin{aligned}
& U_i^2\left(x, m_i, m_{-i}(\cdot)\right) \\
= & \sum_{k \neq i}\left\{\left(\frac{d m_k^{-1}\left(m_i\right)}{d m_i} f_k\left(m_k^{-1}\left(m_i\right) | x\right)+\frac{d m_k^{-1}\left(2 x-m_i\right)}{d m_i} f_k\left(m_k^{-1}\left(2 x-m_i\right) | x\right)\right)\right. \\
& \left.\prod_{j \neq i, k}\left(1-\int_{m_j^{-1}\left(m_i\right)}^{m_j^{-1}\left(2 x-m_i\right)} f_j\left(s_j | x\right) d s_j\right)\right\} .
\end{aligned}
$$

Consider now the special case in which the signals are independent conditional on x and are identically distributed, so that $f_i\left(s_i | x, s_j\right)=f\left(s_i | x\right)=f\left(s_j | x\right)=f_j\left(s_j | x, s_i\right)$. We look for a symmetric equilibrium strategy $m_i(s)=m_j(s)=m(s)$. The first order condition requires that

$$
\begin{aligned}
& \int_m^{+\infty}\left(\frac{d m^{-1}(m)}{d m} f\left(m^{-1}(m) | x\right)+\frac{d m^{-1}(2 x-m)}{d m} f\left(m^{-1}(2 x-m) | x\right)\right) \\
& \left(1-\int_{m^{-1}(2 x-m)}^{m^{-1}(m)} f(z | x) d z\right)^{n-2} f(s | x) q(x) d x \\
= & \int_{-\infty}^m\left(\frac{d m^{-1}(m)}{d m} f\left(m^{-1}(m) | x\right)+\frac{d m^{-1}(2 x-m)}{d m} f\left(m^{-1}(2 x-m) | x\right)\right) \\
& \left(1-\int_{m^{-1}(m)}^{m^{-1}(2 x-m)} f(z | x) d z\right)^{n-2} f(s | x) q(x) d x
\end{aligned}
$$

holds at $m=m(s)$. After substitution of $dm^{-1}(m)/dm=1/m'(m^{-1}(m))$ and $m^{-1}(m)=s$ we obtain the differential equation (5.4) describing the symmetric equilibrium, to be solved for a strictly increasing $m(s)$ with initial condition $m(\mu)=\mu$. □

Elimination of Inverse from First Order Condition. We now apply a change of variable to (5.4) so as to eliminate the presence of the inverse. We let $y=m^{-1}(2x-m(s))$, with $m'(y)\,dy=2dx$, so that $m(y)=2x-m(s)$ and $x=(m(y)+m(s))/2$. The lower bound of integration $x = m$ becomes $y = m^{-1}(2m(s) - m(s)) = s$ and the upper bound $x = \infty$ becomes $y = m^{-1}(2\infty - m(s)) = m^{-1}(\infty) = \infty$, so that

$$\begin{aligned} & \int_m^{+\infty} \left(1 - \int_{m^{-1}(2x-m)}^{s} f(z|x)\, dz\right)^{n-2} \tfrac{1}{m'(s)} f(s|x)\, f(s|x)\, q(x) dx \\ = \; & \tfrac{1}{2} \int_s^{+\infty} \left(1 - \int_y^s f\left(z|\tfrac{m(y)+m(s)}{2}\right) dz\right)^{n-2} \tfrac{m'(y)}{m'(s)} \left(f\left(s|\tfrac{m(y)+m(s)}{2}\right)\right)^2 q(\tfrac{m(y)+m(s)}{2}) dy \end{aligned}$$

and

$$\begin{aligned} & \int_m^{+\infty} \left(1 - \int_{m^{-1}(2x-m)}^{s} f(z|x)\, dz\right)^{n-2} \tfrac{f\left(m^{-1}(2x-m(s))|x\right)}{m'(m^{-1}(2x-m(s)))} f(s|x)\, q(x) dx \\ = \; & \tfrac{1}{2} \int_s^{+\infty} \left(1 - \int_y^s f\left(z|\tfrac{m(y)+m(s)}{2}\right) dz\right)^{n-2} f\left(y|\tfrac{m(y)+m(s)}{2}\right) f\left(s|\tfrac{m(y)+m(s)}{2}\right) q\left(\tfrac{m(y)+m(s)}{2}\right) dy. \end{aligned}$$

We have thus obtained (5.5). □

Proof of Proposition 5. We verify that $V(\mu + m_i|\mu + s_i) = V(\mu - m_i|\mu - s_i)$. We calculate

$$\begin{aligned} V(\mu + m_i|\mu + s_i) \; = \; & \int_{-\infty}^{\mu+m_i} \left(1 - \int_{m^{-1}(2x-\mu-m_i)}^{m^{-1}(\mu+m_i)} f(s|x)\, ds\right)^{n-1} q(x|\mu + s_i) dx \\ & + \int_{\mu+m_i}^{+\infty} \left(1 - \int_{m^{-1}(\mu+m_i)}^{m^{-1}(2x-\mu-m_i)} f(s|x)\, ds\right)^{n-1} q(x|\mu + s_i) dx. \end{aligned}$$

Changing variables in the outer integrals to $y = 2\mu - x$, we find $V(\mu + m_i|\mu + s_i)$ equal to

$$\begin{aligned} & \int_{\mu-m_i}^{+\infty} \left(1 - \int_{m^{-1}(3\mu-m_i-2y)}^{m^{-1}(\mu+m_i)} f(s|2\mu - y)\, ds\right)^{n-1} q(2\mu - y|\mu + s_i) dy \\ & + \int_{-\infty}^{\mu-m_i} \left(1 - \int_{m^{-1}(\mu+m_i)}^{m^{-1}(3\mu-m_i-2y)} f(s|2\mu - y)\, ds\right)^{n-1} q(2\mu - y|\mu + s_i) dy. \end{aligned}$$

Changing variables in the inner integrals to $\sigma = 2\mu - s$, we find $V(\mu + m_i|\mu + s_i)$ equal to

$$\begin{aligned} & \int_{\mu-m_i}^{+\infty} \left(1 - \int_{2\mu-m^{-1}(\mu+m_i)}^{2\mu-m^{-1}(3\mu-m_i-2y)} f(2\mu - \sigma|2\mu - y)\, d\sigma\right)^{n-1} q(2\mu - y|\mu + s_i) dy \\ & + \int_{-\infty}^{\mu-m_i} \left(1 - \int_{2\mu-m^{-1}(3\mu-m_i-2y)}^{2\mu-m^{-1}(\mu+m_i)} f(2\mu - \sigma|2\mu - y)\, d\sigma\right)^{n-1} q(2\mu - y|\mu + s_i) dy. \end{aligned}$$

Notice now that anti-symmetry of the inverse strategy gives $2\mu - m^{-1}(3\mu - m_i - 2y) = m^{-1}(-\mu + m_i + 2y)$ and $2\mu - m^{-1}(\mu + m_i) = m^{-1}(\mu - m_i)$, that the location assumption gives $f(2\mu-\sigma|2\mu-y)=f(\sigma|y)$, and that the symmetry of q implies $q(2\mu-y|\mu+s_i)=q(y|\mu-s_i)$. Thus,

$$\begin{aligned} V(\mu+m_i|\mu+s_i) &= \int_{\mu-m_i}^{+\infty}\left(1-\int_{m^{-1}(\mu-m_i)}^{m^{-1}(2y-\mu+m_i)} f(\sigma|y)\,d\sigma\right)^{n-1} q(y|\mu-s_i)dy \\ &\quad + \int_{-\infty}^{\mu-m_i}\left(1-\int_{m^{-1}(2y-\mu+m_i)}^{m^{-1}(\mu-m_i)} f(\sigma|y)\,d\sigma\right)^{n-1} q(y|\mu-s_i)dy \\ &= V(\mu-m_i|\mu-s_i) \end{aligned}$$

as desired. □

Proof of Lemma 1. Forecaster i wins with m_i when all opponents locate outside the interval $[x-|x-m_i|, x+|x-m_i|]$. Given signal s_i, the expected payoff is then

$$\int_{-\infty}^{m_i} n\left[1-\int_{2x-m_i}^{m_i} g(m_j|x)\,dm_j\right]^{n-1} q(x|s_i)dx+\int_{m_i}^{+\infty} n\left[1-\int_{m_i}^{2x-m_i} g(m_j|x)\,dm_j\right]^{n-1} q(x|s_i)dx. \tag{8.1}$$

We first study the limit as $n\to\infty$ of the first term $\int_{-\infty}^{m_i} n\left[1-\int_{2x-m_i}^{m_i} g(m_j|x)\,dm_j\right]^{n-1} q(x|s_i)dx$. Define the function $H(x)=1-\int_{2x-m_i}^{m_i} g(m_j|x)\,dm_j$. Observe that in general,

$$\frac{d}{dx}\left[\frac{(H(x))^n}{H'(x)}\right]=(n-1)(H(x))^{n-1}-\frac{H''(x)(H(x))^n}{(H'(x))^2}.$$

Integrating by parts, we obtain

$$\begin{aligned} &\int_{-\infty}^{m_i}(n-1)[H(x)]^{n-1} q(x|s_i)dx \\ =\ &\left[\tfrac{(H(x))^n}{H'(x)}q(x|s_i)\right]_{-\infty}^{m_i}-\int_{-\infty}^{m_i}\tfrac{(H(x))^n}{H'(x)}\tfrac{dq(x|s_i)}{dx}dx+\int_{-\infty}^{m_i}\tfrac{H''(x)(H(x))^n}{(H'(x))^2}q(x|s_i)dx. \end{aligned} \tag{8.2}$$

We now analyze the limit of this expression as $n\to\infty$.

First, consider $H'(x)=2g(2x-m_i|x)-\int_{2x-m_i}^{m_i}\frac{dg(m_j|x)}{dx}dm_j$. Under a linear strategy $\hat{c}(s_j)=As_j+(1-A)\mu$, conditionally on x the message m_j is normally distributed with mean $Ax+(1-A)\mu$ and variance A^2/τ. Thus

$$g(m_j|x)=\sqrt{\frac{\tau/A^2}{2\pi}}e^{-\frac{\tau}{2A^2}(m_j-Ax+(1-A)\mu)^2}.$$

Since $dg(m_j|x)/dx=-Adg(m_j|x)/dm_j$, we can rewrite $H'(x)=(2-A)g(2x-m_i|x)+Ag(m_i|x)$. The expression for g shows that for large $|x|$, $g(m_i|x)$ is of order $\exp(-\tau x^2)$ while $g(2x-m_i|x)$ is of order $\exp\left(-\tau(2-A)^2x^2/A^2\right)$. By the assumption that $A\in(0,1]$, we have $(2-A)^2/A^2\geq 1$, so that $H'(x)$ is of order $\exp(-\tau x^2)$ for large $|x|$.

Second, since $q(x|s_i)$ is of order $\exp\left(-(\tau+\nu)x^2\right)$, we obtain that $q(x|s_i)/H'(x)$ is of order $\exp\left(-\nu x^2\right)$ for large $|x|$, and therefore tends to zero as $x \to -\infty$. Furthermore, $H''(x)$ is of order $x\exp\left(-\tau x^2\right)$ and $dq(x|s_i)/dx$ is of order $x\exp\left(-(\tau+\nu)x^2\right)$. Hence, $\left(dq(x|s_i)/dx\right)/H'(x)$ is of order $x\exp\left(-\nu x^2\right)$ and $H''(x)\,q(x|s_i)/\left(H'(x)\right)^2$ is of the order $x\exp\left(-\nu x^2\right)$ so both of the integrals in (8.2) are finite.

Finally, consider the limit as n tends to infinity. For any $x < m_i$, so that $\lim_{n\to\infty}(H(x))^n$ for $H(x) < 1$. However, $H(m_i) = 1$ so $(H(m_i))^n = 1$. Inserting in (8.2), first addend tends to $q(m_i|s_i)/H'(m_i) = q(m_i|s_i)/\left(2g(m_i|m_i)\right)$, while the second and third addends tend to zero as $n \to \infty$. It follows that $\int_{-\infty}^{m_i} n\,[H(x)]^{n-1}\, q(x|s_i)dx$ tends to the same limit. A symmetric argument proves that also $\int_{m_i}^{+\infty} n\left[1-\int_{m_i}^{2x-m_i} g(m_j|x)\,dm_j\right]^{n-1} q(x|s_i)dx$ tends to $q(m_i|s_i)/\left(2g(m_i|m_i)\right)$. Overall, we conclude that the sum (8.1) tends to $q(m_i|s_i)/g(m_i|m_i)$.□

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