LongRun Technical Document

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The development of LongRun

LongRun was created by the RiskMetrics Group in conjunction with a variety of groups within J.P. Morgan, including risk management services, corporate finance, advisory, foreign exchange, and capital markets.

All data used in the production of the RiskMetrics Group’s LongRun forecasts are provided by Reuters.
# Table of contents

<table>
<thead>
<tr>
<th>Chapter</th>
<th>Title</th>
<th>Pages</th>
</tr>
</thead>
<tbody>
<tr>
<td>Preface</td>
<td></td>
<td>v</td>
</tr>
<tr>
<td><strong>Chapter 1.</strong> Introduction to <em>LongRun</em></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>1.1</td>
<td>What is <em>LongRun</em>?</td>
<td>2</td>
</tr>
<tr>
<td>1.2</td>
<td>Scenario generation</td>
<td>5</td>
</tr>
<tr>
<td>1.2.1</td>
<td>Forecasting</td>
<td>5</td>
</tr>
<tr>
<td>1.2.2</td>
<td>Scenario simulation</td>
<td>10</td>
</tr>
<tr>
<td>1.3</td>
<td>Summary</td>
<td>12</td>
</tr>
<tr>
<td><strong>Chapter 2.</strong> Forecasts based on current market prices</td>
<td></td>
<td>15</td>
</tr>
<tr>
<td>2.1</td>
<td>Forecasts using futures, forwards, and options</td>
<td>15</td>
</tr>
<tr>
<td>2.1.1</td>
<td>Cash and derivatives contracts and markets</td>
<td>16</td>
</tr>
<tr>
<td>2.1.2</td>
<td>The performance of derivatives prices in forecasting cash prices</td>
<td>26</td>
</tr>
<tr>
<td>2.1.3</td>
<td>Efficient markets theory</td>
<td>31</td>
</tr>
<tr>
<td>2.1.4</td>
<td>Risk neutral forecasts from derivative asset prices</td>
<td>38</td>
</tr>
<tr>
<td>2.2</td>
<td>Forecasts of extreme moves in asset prices</td>
<td>51</td>
</tr>
<tr>
<td>2.2.1</td>
<td>Statistical behavior of asset prices</td>
<td>51</td>
</tr>
<tr>
<td>2.2.2</td>
<td>The volatility smile in option markets</td>
<td>54</td>
</tr>
<tr>
<td>2.2.3</td>
<td>Interpreting the volatility smile</td>
<td>57</td>
</tr>
<tr>
<td>2.2.4</td>
<td>Implied probability distributions using the volatility smile</td>
<td>61</td>
</tr>
<tr>
<td>Appendix 2.A</td>
<td>Covered parity in the currency, interest rate, commodity and equity index markets</td>
<td>69</td>
</tr>
<tr>
<td>2.A.1</td>
<td>Cost-of-carry and the mechanics of forward prices</td>
<td>69</td>
</tr>
<tr>
<td>2.A.2</td>
<td>Foreign exchange</td>
<td>70</td>
</tr>
<tr>
<td>2.A.3</td>
<td>Equities and commodities</td>
<td>71</td>
</tr>
<tr>
<td>2.A.4</td>
<td>Interest rates</td>
<td>72</td>
</tr>
<tr>
<td>Appendix 2.B</td>
<td>Why do option prices contain so much information?</td>
<td>76</td>
</tr>
<tr>
<td><strong>Chapter 3.</strong> Forecasts based on economic structure</td>
<td></td>
<td>81</td>
</tr>
<tr>
<td>3.1</td>
<td>Introduction</td>
<td>81</td>
</tr>
<tr>
<td>3.2</td>
<td>Constructing an econometric ‘forecasting system’</td>
<td>82</td>
</tr>
<tr>
<td>3.3</td>
<td>Econometric framework</td>
<td>84</td>
</tr>
<tr>
<td>3.3.1</td>
<td>Advantages of the vector autoregressive model</td>
<td>84</td>
</tr>
<tr>
<td>3.3.2</td>
<td>Introduction to the error correction model</td>
<td>85</td>
</tr>
<tr>
<td>3.4</td>
<td>VECM: The functional form of our chosen model</td>
<td>94</td>
</tr>
<tr>
<td>3.4.1</td>
<td>Three types of VARM</td>
<td>95</td>
</tr>
<tr>
<td>3.4.2</td>
<td>Two types of ECM</td>
<td>96</td>
</tr>
<tr>
<td>3.4.3</td>
<td>Backtesting</td>
<td>100</td>
</tr>
<tr>
<td>3.4.4</td>
<td>Summary</td>
<td>102</td>
</tr>
<tr>
<td>3.5</td>
<td>Specification of the VECM</td>
<td>103</td>
</tr>
<tr>
<td>3.5.1</td>
<td>Foreign exchange</td>
<td>104</td>
</tr>
<tr>
<td>3.5.2</td>
<td>Interest rate</td>
<td>105</td>
</tr>
<tr>
<td>3.5.3</td>
<td>Equity index</td>
<td>107</td>
</tr>
<tr>
<td>3.5.4</td>
<td>Commodity price</td>
<td>107</td>
</tr>
<tr>
<td>3.6</td>
<td>Economic regimes in forecasts</td>
<td>108</td>
</tr>
<tr>
<td>3.6.1</td>
<td>Accounting for economic regimes in forecasts</td>
<td>109</td>
</tr>
<tr>
<td>3.6.2</td>
<td>Applying structural break tests</td>
<td>111</td>
</tr>
<tr>
<td>3.6.3</td>
<td>Incorporating structural breaks</td>
<td>112</td>
</tr>
<tr>
<td>3.7</td>
<td>Estimating the VECM</td>
<td>114</td>
</tr>
<tr>
<td>3.7.1</td>
<td>Selecting order of lags and cointegration vectors</td>
<td>114</td>
</tr>
<tr>
<td>3.7.2</td>
<td>USD per ZAR exchange rate example</td>
<td>115</td>
</tr>
</tbody>
</table>
Preface

This technical document details the long-term forecasting and scenario generation methodologies in LongRun. It contains two sets of techniques for computing forecast values and confidence intervals for asset prices and a procedure for generating scenarios for use in Monte Carlo.

In some circles of the economics and finance professions, forecasting is not a highly regarded activity. For some, it evokes images of speculators, chart analysts and questionable investor newsletters; for others, memories of the grandiose econometric forecasting failures of the 1970’s. There is nonetheless a need for forecasting in risk management. A prudent corporate treasurer or fund manager must have some way of measuring the risk to earnings, cash flows or returns. Any measure of risk must incorporate some estimate of the probability distribution of the future asset prices on which financial performance depends. Forecasting is an indispensable element of prudent financial management.

How should corporate treasurers and fund managers approach forecasting? Forecasting accuracy per se is not the object of the exercise: every currently known forecasting tool often falls wide of the mark. In a risk management context, the forecasts should rather be practical, based on objective techniques, it must be possible to examine how the methodologies would have performed had they been applied in the past, and it should be possible to articulate the techniques to shareholders, investors, and regulators. It is also desirable to have available different, methodologically independent forecasting techniques. The risk manager can then compare the results with one another and with his own judgements about future asset prices. We believe that LongRun meets these criteria for forecasting techniques.

The RiskMetrics Group’s policy is to make public its risk management methodologies. In doing so we aim to foster public discussion of our approach, to help our clients grasp the methodologies which underline our products, and more generally to promote public understanding of risk management issues. We hope that interested practitioners and scholars will examine the LongRun methodology and look forward to studying their criticisms, alternative approaches, and suggested applications.

The authors have enjoyed support and constructive comments from a number of colleagues. We would particularly like to thank Ethan Berman, Alvin Lee and Jim Ye of the RiskMetrics Group, Mark Everson of Ford Motor Company, and John Byma of the Procter & Gamble Company for their detailed comments on several drafts of this document. Christopher Finger of the RiskMetrics Group helped formulate LongRun’s simulation procedures and had many useful suggestions throughout. Peter Zangari was instrumental in the early stages of this project. We would also like to thank Tatiana Kolubayev for editing and producing this document.

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Chapter 1. Introduction to LongRun

The implementation of a comprehensive risk management system has become an essential step for conducting business in a global and increasingly liberalized trade environment. Given the numerous risks faced by corporations and financial institutions, the growing number of hedging alternatives and financing strategies, and the risk information disclosure required by regulators and sophisticated investors, the need for a framework to implement an adequate risk management process is imminent. In the case of financial institutions, an additional incentive for developing a risk management system can be found in the use of internal risk models to estimate the regulatory capital charges to cover market risks.\(^1\)

The Risk\textit{Metrics}® methodology, first introduced in 1994, provides an answer to the market risk measurement problem when the relevant horizon is short. Risk\textit{Metrics} has been used and tested by professionals in financial institutions and regulatory bodies throughout the world, and has proven to be a consistent and reliable framework with which one can measure market risk over horizons up to 3 months. However, as the demand for market risk measurement over long horizons increases, we are faced with the challenge of developing a dedicated and robust long-term risk measurement framework.

For example, corporations have expressed a concern regarding the application of risk management principles in their own environment, especially since shareholders, potential investors and other market participants have taken a strong interest in companies’ risk profiles. Regulatory bodies have also recently presented their views on how corporations should disclose their risk information.\(^2\)

From a risk management perspective, corporations are less sensitive to daily market moves and focus more on longer term fluctuations when gauging performance. Moreover, many companies have budgeting and planning horizons up to 1 year or longer. Therefore, risk measurement in a corporate environment requires estimates of market risk for longer time horizons than is addressed by Risk-Metrics.\(^3\)

In the financial industry, mutual funds and pension plan managers want to control their long-term risk/return profile; that is, they want to obtain acceptable returns while keeping only the risks to which they want to be exposed. The main source of risk for mutual funds and pension plans arises from long-term fluctuations in market prices and rates. In addition, pension plans and insurance companies have long-term liabilities in the form of annuities or insurance claim payments. As these institutions seek to match those liabilities with their assets, they become exposed to long-term market risks, making long-term risk management imperative for running their operations.

As these examples suggest, the relevant horizon for risk management is not uniquely defined across all businesses. Instead, the relevant horizon commonly varies by asset class (e.g., bonds versus commodities), position in the firm (e.g., CFO versus trading desk), and industry sector (e.g., pension funds versus banking), among other factors, and the appropriate relevant horizon must be considered on an application-by-application basis.

Currently, various models are being used that rely on long-horizon forecasts to measure risk. Long-horizon forecasts are already routine in corporate finance, as well as in other risk management situations:

\(^{1}\) The use of internal models for this purpose was approved by the Bank for International Settlements in the January 1996 Amendment to the 1988 Capital Accord.

\(^{2}\) The Securities and Exchange Commission (SEC) and the Financial Accounting Standards Board (FASB) have recently addressed the issue of market derivative activities disclosure in the case of companies.

\(^{3}\) See the \textit{CorporateMetrics Technical Document} for a complete discussion of corporate risk management.
• In corporate finance, project evaluation frequently requires long-horizon forecasts of key economic and financial variables.

• Credit risk models make long-term projections about the credit standing of corporate entities. Some models rely on transition matrices that use historical information to specify how debt instruments’ ratings will change over an extended period of time.

• In asset and liability management, risk calculations are often based on simulated term structures of interest rates over long-horizons across a variety of currencies.

• Asset allocation models, which determine the “optimal” investment mix across a broad class of assets, typically require forecasts of the expected return on assets over long investment horizons.

Given the necessity for systematic and comprehensive risk management solutions over long-term horizons, and the difficulty of the problem at hand, our objective in this document is to provide a tool for measuring long-term financial risk. This document is intended to provide the industry professional with a rigorous yet accessible treatment on the “nuts and bolts” of long-term scenario generation. Readers will learn not only how to forecast financial variables and generate scenarios, but also the rationale behind the proposed methodologies. We also provide and explain the necessary data to enable the reader to implement the methodology set forth in this document.

1.1 What is LongRun?

We provide an integrated and self contained framework, including data and methodology, for generating market rate scenarios that allow us to measure market risk over long horizons. We call this framework LongRun. In this section we present a general introduction to LongRun and explain the rationale behind it.

Industry professionals responsible for hedging financial exposures, managing credit exposures, or determining the interest expense of different funding strategies, share a common requirement: they need a consistent and reliable estimate of their positions’ distribution of potential changes in value.\(^4\) How forecasts are made, and how value is defined, depends on the definitions of risk and value.\(^5\) LongRun is flexible enough to conform to a variety of definitions. For example, in the standard Value-at-Risk calculation, a portfolio’s value refers to its “marked-to-market” value and forecasts of value changes are based (typically) on historical time series of relevant prices and rates. In Cash-Flow-at-Risk, value is measured in terms of the nominal amount of a set of cash flows and forecasts are based on either historical time series or subjective, user-defined scenarios. Earnings-at-Risk uses the same forecasting methods as Cash-Flow-at-Risk but defines value in terms of earnings.

Once we settle on a definition of value, in order to measure, and ultimately manage financial risk, we need to forecast its potential changes. The complete process for measuring financial risk can be summarized in five steps.

**STEP 1: Defining risk.** Define the type of financial risk that we plan to measure. In other words, do we want to measure future changes in “marked-to-market” value, cashflow amount, reported earnings, or other?

\(^4\) Financial exposures would include exposures to foreign exchange, interest rate, commodity and equity markets.

\(^5\) See the CorporateMetrics Technical Document for a detailed discussion on risk and value definitions.
STEP 2: Cashflow identification and mapping. Given the definition of value in Step 1, identify all cash flows whose values are subject to change and allocate those cash flows to a price series, where the price series will determine the risk of the cash flows.

STEP 3: Forecasting. Specify a set of future (forecast) dates and obtain the joint distribution of these prices at each of these dates.

STEP 4: Scenario simulation. Obtain the position’s distribution of potential changes in value. We generate market prices and rates scenarios and then value the positions at each of this scenarios.6

STEP 5: Risk estimation. Calculate a risk measure (e.g., VaR) given the positions’ distribution of potential changes in value.

Except for Steps 3 and 4, the setup for measuring financial risk is identical to the methodology described in the RiskMetrics Technical Document. Whereas the RiskMetrics methodology is geared toward measuring market risks for short-term horizons, up to approximately 3 months, LongRun handles longer-term market risk up to 2 years. The purpose of this document is to describe a framework for carrying out the third and fourth steps of measuring financial risk—forecasting and scenario simulation—over long-term horizons.7

It is important to realize that long-term and short-term forecasting—and hence risk measurement—are two different problems. Most of the simplifying assumptions used in short-term forecasting are not applicable to longer horizons. Three of these common assumptions are:

- The mean of the returns can be safely assumed to be zero. With this assumption we are basically ignoring any upward and downward trend in the asset price, as well as growth due to dividends and the time value of money.

- Successive 1-day returns have the same volatility and are independent of one another. This implies that the variance of returns is a linear function of time.8

- Returns follow a random walk. Broadly speaking, this means that returns wander away from any starting point with no particular direction and have a normal distribution at each point in time.

As discussed in the RiskMetrics Technical Document, these assumptions do not have a great impact on risk calculations for a short horizon. Any upward or downward drift in prices is not likely to affect the mean returns in a short period of time. Similarly, scaling up daily volatility to obtain a 10-day volatility does not introduce a large bias in the calculations. Such approximations, however, cannot be applied to long-term horizon forecasting. The mean of the returns will not be zero in the long term, and scaling up daily volatilities to a long-term horizon would introduce a considerable error in the estimate. Moreover, as we will explore in this document, the random walk model does not always provide the best explanation for the dynamics of financial returns over long horizons.

For example, Chart 1.1 shows a 90% confidence interval for the MXP per USD exchange rate based on a random walk model with zero drift. As we can see, the confidence interval is wide and the realized price of the dollar is always in the upper half of the interval. This example suggests that the random walk model with zero drift is not properly capturing either the trend of the MXP per USD

6 In simple cases it is possible to obtain an analytic solution for the position’s distribution of changes in value. In this situation we can skip the scenario simulation step.

7 By long-term horizon we mean a time frame of up to 2 years.

8 In other words, the daily standard deviation of returns can be scaled up by using the “square root of time” rule.
exchange rate, or its volatility over long time horizons. In the next two chapters, we discuss these and other problems related with long-term forecasting, and propose various alternative models to solve them.

Chart 1.1
90% confidence interval for the MXP per USD exchange rate

It is essential to understand that long-term prices and rate forecasts are almost never equal (and often not very close) to the realized prices and rates. However, if we want to estimate risk due to fluctuations in future market prices and rates, we need an approximation to the realized future values. Therefore, forecasting asset prices and rates is an indispensable component of the risk measurement process. Given the necessity to calculate forecasts, we chose to take a quantitative (rather than a subjective or “gut feeling”) approach to obtain educated long-term forecasts of market prices and rates. In addition, all the forecasting methodologies presented in LongRun are backtested against historical profits and losses to ensure their credibility.

We develop and explain the data, forecasting models, and simulation algorithms required to generate paths of prices and rates over long horizons for a broad set of prices and rates. We refer to the entire framework—the data, forecasting models and simulation algorithms—as LongRun. Its primary purpose is to provide industry professionals with a flexible, consistent, and transparent mechanism to create scenarios that serve as inputs to any risk management system. Its distinguishing features can be summarized as follows:

- **Integrated structure:** LongRun’s forecasting models and simulation algorithms provide a tool to produce scenarios over long horizons. LongRun’s methodology is broad enough to apply to all asset classes including foreign exchange, commodities, interest rates and equities. LongRun’s general framework includes a suite of data sets, forecasting models and simulation methodologies rather than a specific type of data set and model.

- **Transparent methodology:** Helping to understand “what is going on” is one of the most important properties of any risk measurement system. In order to quantify the long-term risk of a particular position, we must make predictions about potential changes in asset prices.

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9 It is important to note that this document is not about telling users where, say, the S&P500 or the USD per GBP exchange rate will trade at one year from now. Instead, this document is about providing risk managers with a comprehensive framework for scenario generation, which is a necessary input to risk management.

10 In this document we will obtain mean and volatility forecasts which will then be used to obtain forecasts of price distributions.
often over long horizons (e.g., 1 year). Although asset price forecasts over long-horizons are fraught with error, they are nonetheless a prerequisite to measuring risk. We can go a long way towards mitigating the dangers associated with such forecasts through open discussion of the key assumptions that underpin the models. To this end, LongRun’s entire methodology, consisting of all relevant assumptions, formulas and equations, are publicly available.

- **Comprehensive data set**: LongRun’s scenarios are derived from two different types of data (inputs): (1) historical information on asset prices and macroeconomic fundamentals, and (2) current market prices such as forward rates and option prices.

- **Long-horizon forecasts**: LongRun was built to forecast and simulate any number of price and rate distributions over horizons ranging from 1 day to 2 years. For example, LongRun is capable of providing forecasts of the distributions of the U.S. dollar price of pound sterling, Japanese yen and Canadian dollar every day, for the next 2 years, while accounting for the daily correlation structure among these rates. LongRun uses two different models to produce forecasts. The first is based on current market prices and rates, whereas the second is based on historical financial and economic data.

An important objective of LongRun is to provide readers with a framework to make more educated decisions on scenario generation. The scenario generation in LongRun is done in two separate steps:

1. We forecast the mean, standard deviation, and distribution of the market rates, as described in Step 3 on page 3.

2. We simulate scenarios based on the forecasted variables, as described in Step 4 on page 3.

An introduction to each of the components of the scenario generation procedure—forecasting and simulation—is presented in the next section.

### 1.2 Scenario generation

#### 1.2.1 Forecasting

While it is easy to agree that market risk measurement requires information about market price and rate probability distributions, it is far more difficult to agree on a method for determining these distributions. A wide variety of forecasting procedures are available, and it is very important to note that no single model is universally applicable. For long-term forecasting, it is desirable to produce different forecasts under different sets of assumptions so that results can be compared under alternative scenarios. Since there is no ‘best’ forecasting procedure, the choice of method should be based on the properties of the given data sets and the particular objective in producing forecasts. In general, risk managers may consider using more than one method in order to meet a diverse set of objectives. The choice of methods may be based on the following concerns:

- Perceived accuracy in specifying the distribution of future market rates
- Ease of implementation
- Closeness to market views or consensus
- Ability to
  - incorporate proprietary information or views on market relationships
  - test extreme events
  - incorporate current market information (e.g., forward rates, option implied volatility)
  - account for macroeconomic conditions
Models that are used to make forecasts typically specify (1) how the variables evolve over time, (2) how they relate to other variables, and (3) how they are distributed at any point in time. *LongRun’s* forecasting procedure involves specifying a set of forecast dates and producing forecasts for the means, volatilities, and distribution of prices at each of those dates. Chart 1.2 shows a distribution of prices at a set of forecast dates of 3, 6, and 9 months.

**Chart 1.2**

**Distribution of prices at forecast dates**

In this document we present two forecasting methodologies: one based on current market information, the other based on econometric models. These methodologies differ in their underlying assumptions and in the data sets used to obtain the forecasts. Forecasts based on current market prices make intensive use of spot, futures, forwards and options price data and apply some basic facts about derivatives theory, while the forecasts based on economic fundamentals rely on historical time series of financial and/or economic data and the econometric modeling of time series.

A. **FORECASTS BASED ON CURRENT MARKET INFORMATION**

Forecasts based on current market information are founded in the notion that market expectations are embedded in an up-to-date way in current spot and derivatives prices. Therefore, it is possible to study what the market “thinks” by analyzing current spot, forward, futures and option prices.

Let us ask ourselves the question: Does the market believe U.S. Treasury rates will rise or fall over the next year? Market professionals frequently answer such questions by observing the term structure of interest rates. An upward-sloping term structure indicates that rates are expected to rise, while a downward-sloping term structure indicates falling rates. In this view, information about market expectations of future interest rates is embedded in current bond prices.

The common approach of using market-implied forecasts applies, in principle, to any traded asset. Modern financial theory views asset prices as forward-looking; the diverse beliefs of market participants concerning the future are filtered through the market process (i.e., the buying and selling of assets).

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11 Some models just specify the distribution of prices at the forecast date and say nothing about how prices got there.
of financial assets by market participants) and therefore embedded in market prices. A decent approximation of market dynamics is the view that markets are efficient, so that asset prices accurately reflect the consensus market view. Formally, this assumption is known as the efficient markets hypothesis.

The efficient markets hypothesis states that everything currently known about the future prospects of an asset is embedded in market prices. The hypothesis is often treated as a tautology, which merely validates whatever market prices prevail. To have real empirical content, the hypothesis must exclude certain types of market behavior, for example:

- The possibility of long lags between perturbations and the onset of a new equilibrium level. The efficient markets hypothesis implies that markets react instantly to new information concerning the asset’s future prospects. More realistically, markets adjust quickly to perturbations, which excludes the possibility of long lags between the arrival of news and the attainment of new equilibrium prices.

- Responses to fads and fashions. Prices are determined, in part, by market participants’ tastes, that is, their willingness to bear risk over various time horizons. There is a presumption that tastes are stable, so fads and fashions do not rule the market. Current research implements this part of the hypothesis empirically by using the assumption of rational expectations.

The efficient markets approach seeks to resolve one of the paradoxes of financial markets: On the one hand, all currently available information relevant to future returns is already impounded in current returns. On the other hand, returns are random and unpredictable. The paradox’s solution is that, if current prices contain all available information, only new information can have an impact on next period’s prices. New information is, by nature, random and unpredictable—until it arrives: we can as easily imagine news provoking a price decline as a rally.

In an efficient market, current market information incorporates not only the market’s views, but also the risk preferences of market participants. For example, different market participants will have different appetites or aversions to the future returns implied by current market information. Forecasts based on current market prices reflect aggregate risk preferences as well as aggregate views about the future in the form of risk neutral probabilities.

We can extract the mean, volatility, and even the entire probability distribution of future spot prices from forward, futures, and option prices. The probabilities underlying these forecasts are referred to as risk neutral probabilities and usually differ from the “market consensus” means, volatilities and distributions. The difference resides in that the risk neutral parameters contain information about the risk preferences of the agents in the economy and thus we can’t disregard risk aversion when interpreting risk neutral probabilities and the forecasts obtained using them. In other words, in addition to “market’s consensus” probabilities, the risk neutral probabilities contain information about aggregate risk preferences, supply, and demand.

To understand the concept of risk neutrality and the link with the efficient markets hypothesis, we make use of a simple example:

Suppose that you are a bookmaker accepting bets on a football game (team A versus team B), and you believe that both teams have the same probability of winning the game (we can even assume that you know with certainty that the probabilities are equal, i.e., 1/2). It seems fair to receive one dollar now for a bet on team A, pay two dollars in the event that team A wins the game, and nothing otherwise. Let us assume that there are 80 people betting on team A and 20 people betting on team B. If you were willing to accept bets based on the true probabilities (i.e., 1/2) of victory, you will lose 60 dollars if team A wins (80 people × $2 − $100 received from bets) and make 60 dollars if team B wins (20 × $2 − $100). Is this position desirable for you? Of course you will break even on average if you play the game several times, but you will
bear risk for any individual game. Ideally, a bookmaker would like to charge a small amount in excess of the original $1 and lock that profit in without risk.

Let us consider what happens when you change the odds in the following way: For every dollar bet on team A you will pay $1.25 if team A wins and nothing otherwise; and for every dollar bet on team B the payoff will be $5 in case team B wins and zero if it loses. Now, if team A wins, your P/L is $0.80 \times $1.25 - $100 = $0; if team B wins, your P/L is $20 \times $5 - $100 = $0. You now have a riskless position. In fact, by changing “the odds,” you imposed on the gamblers a set of artificial risk neutral probabilities; which are prescribed by the relative supplies of bets on the two teams reflecting in this way the “market consensus.” These risk neutral probabilities of victory are 4/5 for team A and 1/5 for team B.\(^\text{12}\)

Spot and derivatives prices contain information about the risk neutral probabilities because all relevant or available information about market participants’ beliefs about the future and their risk preferences are compacted into current market prices. The payoff on the bet therefore does not depend on your personal view—or the view of any individual gambler—of the real probabilities. Under the risk neutral probabilities, you assign the payoffs based on the collective views and risk preferences of the gamblers. Market participants, like the bookmaker, can take on or lay off risk at prevailing market prices and rates. If a market participant’s view of the future or risk appetite differs from what is expressed in current market prices and risk neutral probabilities, he can tailor his position accordingly at prevailing prices.

It is important to remember that the study of the information content of current market prices and rates is still under study by researchers, with much work remaining to be done. In addition, certain markets are illiquid or non-existent, so market price data is not as abundant as one would desire, placing a constraint on the universal applicability of this method. Despite the efficient markets hypothesis, many researchers support the idea that macroeconomic variables and past asset price data have some forecasting power. Therefore, a large part of our efforts to forecast long-term market price and rate distributions will rely on a methodology based on econometric models of asset price determination.

B FORECASTS BASED ON ECONOMIC STRUCTURE

Forecasting models based on the economic structure involve parametric models based on historical information. These forecasting models rely heavily on econometric techniques and time series analysis, which incorporate macroeconomic variables into the models.

The data required by the econometric models used in LongRun falls into two general categories: financial and economic time series. Financial time series data includes spot and forward rates and prices of financial assets, whereas economic time series data includes macroeconomic variables such as measures of output (e.g., industrial production) and money supply. Time series of prices and rates are the most widely available type of data.

In our context, we are essentially concerned with constructing a tractable model that describes as accurately as possible the evolution of the variables of interest. One usually defines a model that is consistent with economic theory and the empirical behavior of the time series, and fits the data to the chosen model. The first step in the construction of a model is to define its functional form, that

\(^{12}\) An intuitive way to look at the differences between the real and the risk neutral probabilities is to consider the different methods used by casinos and bookies to assign payoffs. For example, a roulette game in a casino is priced using the real probabilities of the ball landing on a particular box (plus a spread earned by the casino); whereas a bet in a horse race is priced using the risk neutral probabilities by looking at the relative supply and demand for bets (plus a spread earned by the bookie), and not the actual probability of a specific horse winning the race (except for the subjective views embedded in the supply and demand).
is, we need to write an equation describing how the asset prices relate to all the possible economic variables and how those prices evolve over time. The second step is to choose the economic variables we want to include in our model. In practice, this process is done iteratively since the quality of our choices can only be assessed empirically.

A widely used time series model in finance and economics is the autoregressive (AR) model. The idea behind an autoregressive process is that the current value of the time series depends on its immediate past values together with a random error. For example, we can model the price of a stock tomorrow as the price of the stock today plus some random noise. However, since we are concerned with modeling multiple assets that are related to one another and depend on a number of factors, we need a model that takes into account the co-movement in financial and economic variables. In the Vector Autoregressive Model (VARM), each variable depends on its own past values as well as on the past values of all the other variables in the system, allowing for joint forecasts of future values.\(^\text{13}\)

Jointly forecasting a set of related financial variables is often complicated because some properties of the time series, such as mean and variance, are not stable and keep constantly changing through time. Most of the time series theory is built around stable time series, and the analysis often requires one to transform an unstable time series into a stable one.

A popular approach to addressing the non-stationarity (instability) of individual time series is known as co-integration. Economic theory conjectures that there are long-run equilibrium forces which prevent some economic series from drifting too far apart. The basic notion behind co-integration is the existence of a stable relationship between variables, so that even if the individual time series appear to move in an incoherent manner, we can use the long-term stability in their co-movements to make forecasts. This co-integrating relationship is said to provide an error-correction mechanism and gives the foundation for what are called Error Correction Models (ECM).

In order to make the best use of the macro fundamental variables and their long-term cointegration relationship, we construct our model under the VARM and ECM framework to be the Vector Error Correction Model (VECM).

Forecasts provided by econometric and time series models reflect the properties of the historical period used to estimate the model’s parameters. Sometimes the historical data set used to estimate the model is too short to represent the future economic regime (i.e., monetary policy and business cycles regimes), and some other times the history is so long that too many regimes are represented in the data. Regime-switching/detection models are used to capture changes in structural regimes, and the idea behind them is that the model changes in time according to the prevailing economic regime. In LongRun we discuss regime switching/detection models where we either try to forecast regimes using historical information or simply identify historical regimes.

All the models based on the economic structure proposed in LongRun apply to foreign exchange rates, interest rates, equity indices and commodity prices.

In summary, besides random walk models, LongRun employs parametric models that rely on historical as well as current market data. The models that use historical data are based on vector autoregressive or ‘error-correction’ models. In this case, financial returns are assumed to follow a normal distribution. The models that rely on current market information use option prices to extract parameter estimates of the entire (risk neutral) distribution. In this case asset prices are not necessarily normally distributed. Table 1.1 summarizes the forecasting methodologies described in this publication.

\(^\text{13}\) In econometrics, VAR is used as an acronym for “vector autoregressive.” To avoid confusion with VaR (Value-at-Risk) we use VARM as an acronym for “vector autoregressive model.”
The historical market and macroeconomic data necessary for implementing econometric forecasting techniques can be obtained from the DataMetrics service offered by the RiskMetrics Group. In addition, implied volatility and forward rate information is available on a daily basis to facilitate forecasting using current market information. A Web site providing access to DataMetrics, selected long-horizon forecasts, and methodologies is available at http://www.riskmetrics.com.

1.2.2 Scenario simulation

Our goal is to obtain the price—or return—distribution for a portfolio which, in general, contains instruments with different maturities, and use that distribution to calculate a risk measure (e.g., VaR). The reader should note that, at this point, we would have already obtained the distribution of individual prices and rates using any of the two forecasting procedures outlined above, but the distribution of changes in value of the portfolio as a whole would remain unknown. At this point we are missing two components that we need in order to obtain the distribution of changes in value of a portfolio:

1. Correlations between underlying prices and rates at different horizons.
   The forecasting procedures introduced in this document provide forecasts of the mean and variance at multiple forecast horizons. However, the procedures differ from each other, thus making it impossible to infer correlations between prices at different horizons. A procedure for constructing correlation matrices that are consistent with LongRun’s forecasts is presented in Chapter 5 together with the simulation techniques.

2. The distribution of the portfolio’s changes in value.
   In general, we do not have an explicit formula for the distribution of the portfolio’s changes in value, so we must infer it from the joint distribution of the underlying prices and rates. To do so, we first simulate price paths by using Monte Carlo methods, LongRun’s forecasted distributions, and the historical correlation among prices. We then value the positions along each of the simulated price paths and thus obtain the desired distribution of the portfolio’s changes in value.

LongRun supports two types of Monte Carlo simulations:

- **Level I simulation** involves simulating prices at each of the forecast dates. Chart 1.3 shows three hypothetical simulated Level I price paths.

### Table 1.1

<table>
<thead>
<tr>
<th>Methodology</th>
<th>Model</th>
<th>Forecasted mean</th>
<th>Forecasted volatility</th>
</tr>
</thead>
<tbody>
<tr>
<td>Current market</td>
<td>Random walk</td>
<td>Zero</td>
<td>Historical</td>
</tr>
<tr>
<td>information</td>
<td></td>
<td>Forward premium</td>
<td>Historical</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Forward premium</td>
<td>Implied</td>
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<tr>
<td></td>
<td>General implied</td>
<td>Forward premium</td>
<td>Implied</td>
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<tr>
<td></td>
<td>distribution</td>
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</tr>
<tr>
<td>Economic</td>
<td>VECM</td>
<td>Estimated mean</td>
<td>Estimated standard</td>
</tr>
<tr>
<td>structure</td>
<td></td>
<td></td>
<td>deviation</td>
</tr>
</tbody>
</table>

The historical market and macroeconomic data necessary for implementing econometric forecasting techniques can be obtained from the DataMetrics service offered by the RiskMetrics Group. In addition, implied volatility and forward rate information is available on a daily basis to facilitate forecasting using current market information. A Web site providing access to DataMetrics, selected long-horizon forecasts, and methodologies is available at http://www.riskmetrics.com.
Sec. 1.2 Scenario generation

**Chart 1.3**
Level I simulation: Simulated prices at three forecast dates

- **Level II simulation** “fills in the gaps” between Level I forecast dates such that simulated daily prices (1) coincide with the simulated Level I prices at each forecast date and (2) maintain the correlation structure of historical daily prices. Simply put, we use a technique that forces daily prices and rates to ‘go through’ each of the prices at the forecast dates, as illustrated in Chart 1.4.\(^{14}\) It is important to note that Level II simulation may be skipped when scenarios between forecast dates are not needed.

**Chart 1.4**
Level II simulation: Simulated daily prices

Both Level I and II simulations require that we simulate from multivariate distributions taking into account the aforementioned correlation structure between variables.

After the simulation has been completed, all positions must be valued at future points in time using the simulated prices. Based on a cashflow map that specifies the relationship between cash flows

\(^{14}\) This approach is based on a Brownian Bridge model.
and market prices, the future value of cash flows can be estimated by forecasts of future prices. Finally, we get a portfolio’s distribution of potential changes in value from today up to and including the last forecast date.

1.3 Summary

LongRun is a methodology that provides a framework to generate long-term market price and rate scenarios for risk management purposes. LongRun’s scenario generation procedure is done in two separate steps: forecasting and simulation. LongRun consists of two alternative forecasting procedures based on different data sets. The first is based on current market data and relies on implied (risk neutral) parameters to obtain forecasts, while the second uses historical financial and economic data and is constructed around econometric modeling. Once we obtain price and rate forecasts, we can choose to simulate scenarios either at the same horizons for which we obtained forecasts (Level I simulation), or at any arbitrary date between forecast horizons (Level II simulation). Chart 1.5 outlines LongRun’s scenario generation procedure.

The remainder of this document presents the forecasting methodologies and the scenario simulation procedure in detail, as well as backtesting results over a broad range of market prices and rates. We conclude the document with a review of some practical applications.
Chart 1.5
LongRun’s scenario generation procedure

| Input: Financial time series and macroeconomic data and/or current market information |
| Econometric models and/or implied (risk neutral) parameters |
| LongRun data set (forecasted mean and volatilities) |
| Simulation of prices and rates |
| Level I simulation (at forecast horizons) |
| Level II simulation (between forecast horizons) |
| Generated scenarios (input to a risk management system) |
Chapter 2. Forecasts based on current market prices

As discussed in Chapter 1, one of the key assumptions made in RiskMetrics is that the expected return on any asset is zero. In the short-term forecasting context of RiskMetrics, this is an innocuous assumption. Unless the expected return is very large, it is unlikely to affect short-term value-at-risk calculations significantly. When we address longer-term value-at-risk calculations, the issue of expected return forecasting becomes unavoidable.

LongRun provides two basic approaches to long-term forecasting of future asset returns. This chapter shows how long-term forecasts can be drawn from current prices of marketable cash and derivative assets. The next chapter presents an econometric forecasting methodology based on historical data, including both historical prices of the assets and data on other macroeconomic variables. The two approaches have different methodological bases, so they provide two independent methods for generating the long-term asset price forecasts that are a necessary element of risk management at longer horizons.

We begin by surveying those derivatives markets that contain exploitable information regarding future asset prices and assessing their forecasting performance. We then proceed to a detailed account of how LongRun uses derivative prices to generate forecasts. One of the limitations of the LongRun approach to forecasting is the difficulty of anticipating extreme market events. In the final section of this chapter, we present an overview of new research exploiting option prices for information about large price changes. Chapter 3 contains an analogous discussion of how the ability of econometric forecasts to anticipate large events can be improved.

2.1 Forecasts using futures, forwards, and options

The long-term return forecasts provided by LongRun consist, for each asset and each forecast horizon, of the mean or expected value of the return and the variance or confidence interval of the return. The methodology of this chapter will draw on the prices of cash, futures and forward assets to infer expected asset returns, and on the prices of options to infer asset return variances.

Forward and futures prices are commonly treated as a market estimate of the future cash price because—abstracting from transactions costs and other market imperfections—a long or short forward position will break even if the future cash price is equal to the current forward price. Analogously, option implied volatilities can be treated as estimates of expected volatility over the life of the option because—under similar ideal conditions—a long or short position in an option is likely to break even if the volatility of asset returns over the life of the option is equal to the current option implied volatility.

In LongRun, we use prices as proxies for market expectations because

- asset prices are closely related to market forecasts of future asset prices, even if they are not identical with market forecasts;
- forecasts inferred from asset prices are market-driven and thus objective; and
- current price data are readily and cheaply available for a wide range of marketable assets, so forecasts based on asset prices are eminently practical.

The advantages of asset prices as proxies for market expectations mirror their chief limitation. Asset prices are not forecasts, but the prices at which market participants can speculate and hedge, that is, take on or lay off exposures to market risks. Investors’ willingness to bear risk and market liquidity therefore also influence asset prices. The influences of risk and liquidity are meshed inextricably in asset prices with information about expectations.
2.1.1 Cash and derivatives contracts and markets

In order to understand how to derive and interpret forecasts based on market data, some familiarity with cash and derivative assets and with the markets in which they trade is needed.¹ We will begin by outlining the characteristics of forward, futures, and option contracts, and then identify the most liquid markets for cash and derivative assets.

A FORWARDS AND FUTURES CONTRACTS

In a forward contract, one party agrees to deliver a specified amount of a specified marketable asset (the underlying asset) to the other at a specified date in the future (the maturity date of the contract) at a specified price (the forward price). The price of the underlying asset for immediate (rather than future) delivery is called the cash or spot price. The percentage difference between the forward price and the cash price is called the forward premium.

The party obligated to deliver the commodity is said to have a short position. The party obligated to take delivery of the commodity and pay the forward price for it is said to have a long position. A party with no obligation offsetting the forward contract is said to have an open position. A party with an open position is sometimes called a speculator. A party with an obligation offsetting the forward contract is said to have a covered position. A party with a covered position is sometimes called a hedger.

The market sets forward prices so there are no cash flows—no money changes hands—until the maturity date. The payoff to the owner of a long forward position at maturity is the difference between the forward price, which is set at initiation, and the future cash price, which is learned at maturity. Chart 2.1 illustrates with a gold forward against the dollar, initiated at a forward price of $285 per ounce.

Chart 2.1
Payoff at maturity on a forward
Gold forward locked in at $285 per oz.

There is an important set of relationships among the cash and forward prices of an asset, the money market interest rate with the same maturity as the forward price, and the dividends or interest earned

¹ See Hull (1997) and Smithson, Smith and Wilford (1995) for more information on the structure of derivatives contracts and markets.
on the asset during the time to maturity. Given any three of these pieces of information, we can infer the fourth. In interest rate markets, these relations underpin the derivation of forward interest rates from spot interest rates. In foreign exchange markets, these relations are referred to as the **covered interest parity hypothesis**, which states that the forward premium of a currency pair equals the differential between the money market rates in the two currencies. For convenience, we will call the analogous relationships in the interest rate, equity and commodity markets “covered parity” relations. Appendix 2.A details these relationships for the foreign exchange, fixed income, equity and commodity markets.

**Futures** differ from forwards in two related respects. First, futures trade on organized commodity exchanges such as the Chicago Mercantile Exchange (CME) or the London International Financial Futures and Options Exchange (Liffe). Forwards, in contrast, trade **over-the-counter** (OTC), that is, as simple bilateral transactions, conducted as a rule by telephone, without posted prices. Second, a forward contract involves only one cash flow, at the maturity of the contract, while futures contracts generally require interim cash flows, called **initial** and **variation margin**, prior to maturity.

Margining induces a potential difference between the prices of futures and forwards: the value of the futures, in contrast to that of a forward, will depend not only on the expected future price of the underlying asset, but on expected future short-term interest rates and on their correlation with future prices of the underlying asset. The price of a futures contract may therefore be higher or lower than the price of a congruent forward contract. In practice, however, the differences are very small.\(^2\)

Several other differences between exchange-traded and OTC contracts can be important in extracting information about market sentiment from asset prices:

- Exchange-traded contract specifications are completely standardized and are specified in every detail. OTC contracts may have a typical baseline contract design, but are often tailored quite specifically to the counterparties’ particular wishes.

- Futures prices are expressed in currency units, with a minimum price movement called the **tick size**. Futures prices cannot be any positive number, but must be rounded off to the nearest tick.

- Exchange-traded contracts typically expire on fixed dates, while OTC contracts typically have standard times to maturity.

Forward and futures prices are often interpreted as market-adjusted forecasts of the future price of the underlying asset because the forward or futures price is the price of an asset delivered on a date in the future, and forwards and futures are used by hedgers and speculators to lock in a future exposure. We will adopt this interpretation in *LongRun* and use the forward or futures price as a forecast, setting the forecast horizon equal to the time to maturity of the forward.

*LongRun* will use forward prices in preference to futures where liquid forward markets and publicly posted forward prices are available. Forward prices are more convenient because of their standard times to maturity. For example, suppose we wish to forecast an asset price 1 month hence and that both forward and futures prices are available. The 1-month forward price can serve as a forecast with no additional data manipulation. We can use the futures price directly only if there happens to be a futures contract with exactly 1 month remaining to expiry on the day we make the forecast. Otherwise, interpolation or some other transformation of the futures price is needed to create a 1-month forecast.

\(^2\) Williams (1986) provides an overview of the institutional setup of futures markets and Cox, Ingersoll, and Ross (1981) discuss the relationship between forward and futures prices.
B. Option Contracts

Options markets have grown rapidly in the past few decades in response to market participants’ need to tailor market exposures more precisely than is possible with forwards and futures alone. In LongRun, we will draw estimates of the variance of the future asset price or the confidence interval around the forecasted expected value from option prices.

A call option is a contract giving the owner the right, but not the obligation, to purchase at expiration an amount of a commodity at a specified price called the exercise price. A put option is a contract giving the owner the right, but not the obligation, to sell at expiration an amount of a commodity at the exercise price. The amount of the underlying asset is called the notional principal or underlying amount.

The relationship between an option’s exercise price and the current cash or forward price of the underlying asset is called the option’s moneyness. A call with an exercise price below the current cash price is called in-the-money: if it were exercised today, it would have a positive payoff. A call with an exercise price above the current cash price is called out-of-the-money. A call or put with an exercise price equal to the current cash price is called at-the-money. A call or put with an exercise price equal to the current forward price is called at-the-money forward.

The issuer of the option contract is said to have the short position. The owner of the option is said to be long. The price of the option contract is called the option premium. A European option can be exercised only at expiration. An American option can be exercised at any time between initiation of contract and expiration.

Chart 2.2 illustrates with the payoff at maturity on a European call option on gold with an exercise price of $285 per ounce. In contrast to the payoff on a forward contract, if the gold price at maturity is below $285, the option payoff is zero.

Chart 2.2  
Payoff at maturity on a forward and on European call and put options
Gold forward locked in at $285 per oz.; option exercise price $285 per oz.

Currency options have an added twist: A domestic currency call is also a foreign currency put. For example, the purchaser of the right to buy one dollar for DEM 1.60 in three months has also purchased the right to sell DEM 1.60 at USD 0.625 per mark.
C THE RANDOM WALK MODEL AND IMPLIED VOLATILITY

Just as forward or futures prices are often interpreted as a market estimate of the mean future asset price, option prices are often interpreted as containing market estimates of the variance of the future asset price. To understand better why this interpretation has taken hold, we need to define implied volatility.

The notion of implied volatility emerged from the Black-Scholes option pricing model. The Black-Scholes model starts from a simplified picture of real-world asset price behavior and asset markets in which

- the underlying asset price fluctuates randomly in such a way that percent changes in the asset price (asset returns) are normally distributed;
- the domestic money market interest rate is default-free and both it and the cash yield on the underlying asset—e.g., bond coupon interest, an equity dividend, a gold lease rate, or a foreign money market rate—are constant and known in advance; and
- transactions costs and taxes are absent, the market is perpetually open, and traders can put on unlimited short positions.

The statistical model corresponding to the first assumption is called the random walk model. It states that asset returns are independent and normally distributed. This means that at the beginning of each period, the as-yet unknown asset return over the period is a normally distributed random variable that is independent of returns in earlier periods.

The random walk hypothesis is widely used in financial modeling. In the random walk model, asset prices are continuous: they move in small steps, but do not jump. Over a year, they may wander quite a distance from where they started, but they do it by moving a little bit each day. A random walk has no tendency to revert to the historical mean or other “correct” level. To the extent that the random walk assumption holds, chartist technical analysis is irrelevant.

The random walk is an example of a stochastic process or time series model: it describes the behavior of a fluctuating financial variable on a sequence of dates. The behavior of a random walk over a particular time interval, even though it is a sequence of many numbers, is still but a single observation on the time series. To describe the statistical properties of a time series model, one must imagine repeating, many times over, the experiment of observing its fluctuations over a given time interval. Chart 2.3 illustrates the properties of a random walk with six possible time paths over a year of an asset price, dollar-mark, with a starting value of DEM 1.60, an annual volatility of 12%, and a drift rate or expected rate of return of zero.

An asset price following a random walk can be thought of as having an “urge” to wander away from any starting point, but not in any particular direction. The volatility parameter measures that urge to wander. The longer the time interval, the more uncertain the asset price will be and thus the higher will be its variance. That is, the further into the future we look, the less likely the asset price is to be close to its current level. In the Black-Scholes model, volatility is a parameter, usually denoted \( \sigma \), which quantifies the likelihood that the asset price will wander far from its current level in a given time interval. When the volatility parameter is large, it is more likely that the asset price will move far from its current level in a short time.

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3 The model was developed by Black and Scholes (1973) and Merton (1973).
4 For an intuitive introduction to the properties of a random walk, see Dixit and Pindyck (1994), Chapter 3. Merton (1982) provides a rigorous treatment that avoids advanced mathematics. The treatments in Ingersoll (1987), Dornhan (1990), and Duffie (1996) are a step up in difficulty. The standard rigorous presentation is Karatzas and Shreve (1991).
Specifically, in the Black-Scholes model the asset return over a time interval $\tau$ is normally distributed with a standard deviation $\sigma \sqrt{\tau}$. Typically, $\sigma$ is expressed as an annual rate, so $\tau$ is measured in years. For example, suppose the volatility parameter for an equity index is $\sigma = 0.216$. Then the standard deviation of percent changes in the index over the next week, according to the model, is $\sigma \sqrt{\tau} = 0.216 \times \sqrt{52} = 0.03$ or 3%.$^5$

The Black-Scholes model assumes that volatility can be different for different asset prices, but is a constant for a particular asset. For example, the IBM Corporation common share price and the price of gold each have their own characteristic but unchanging volatility. That implies that asset prices are *homoscedastic*, showing no tendency towards “volatility bunching.” A wild day in the markets is as likely to be followed by a quiet day as by another wild day. We will discuss this hypothesis and alternative views of how asset prices actually behave in Section 2.2.

The Black-Scholes model implies that a hedge amount of the underlying asset, called the *delta hedge*, can be continuously adjusted over time so as to exactly mimic the changes in value of a call option. In other words, a perfect hedge can be maintained. We noted above that the model assumes that the asset return is normally distributed with a standard deviation $\sigma \sqrt{\tau}$, but did not say anything about the mean asset return. The asset may have a positive or negative expected return. Because the option can be perfectly hedged, it can be priced as though the drift or expected return were equal to the risk-free rate at which the hedge is financed. This type of reasoning is called *risk neutral valuation*.

The model thus sets the expected return on the asset equal to the risk-free rate and the variance of the asset return to $\sigma^2 \tau$. This specifies the entire probability distribution of the asset and makes it possible to derive formulas for the values of European put and call options, *given the model assumptions*. There are six inputs to the formulas: the current asset price $S_0$, the exercise price $X$, the option's time to maturity $\tau$, the asset price volatility $\sigma$, the money market rate $r$, and the dividend or interest yield $d$ on the underlying asset. Of the arguments, all except the volatility are either observable market prices or part of the terms and conditions of the option contract.

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$^5$ This “scaling” procedure is valid only if one assumes, as does the Black-Scholes model, that the volatility parameter is constant. One can use any time unit, say daily, as long as the $\sigma$ and $\tau$ units match.
One can replace the call option value provided by the Black-Scholes formula with an observed option price \( v_t \), and interpret the valuation function as stating the volatility as an implicit function of \( v_t \), the variables \( S_t, r, \) and \( d \), and the contractually-specified parameters \( X \) and \( \tau \). Essentially, we are then using option prices to estimate the volatility \( \sigma \), which in this context is called the **Black-Scholes implied volatility**. The Black-Scholes call valuation function increases as \( \sigma \) increases (see Chart 2.4), so the implied volatility corresponding to an observed option price \( v_t \) is easy to find (see Chart 2.5).6

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**Chart 2.4**

**Black-Scholes call value as a function of implied volatility**

*At-the-money forward, premium as percent of forward price, 1-month maturity, volatility in percent*

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**Chart 2.5**

**Implied volatility as a function of option premium**

*At-the-money forward, premium as percent of forward price, 1-month maturity, volatility in percent*

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6 Methodologies for estimating implied volatility were originally developed by Latané and Rendleman (1976) and Schmalensee and Trippi (1978), among others.
In the Black-Scholes world, the implied volatility has a straightforward statistical interpretation as the annualized standard deviation of asset returns over the life of the option. It is a forward-looking measure of volatility, in contrast to historical volatility, which measures returns over a recent past period.

**D  TERM STRUCTURE OF IMPLIED VOLATILITY**

The Black-Scholes model assumes that all options on the same asset have identical implied volatilities, regardless of time to maturity and exercise price. However, the pattern of market option prices is usually quite different from what the model predicts. These differences are often called pricing biases or anomalies. This is a convenient if somewhat misleading label, since the phenomena in question are biased only from the point of view of the Black-Scholes model, which neither dealers nor academics consider an exact description of reality.

One such anomaly is that options with the same exercise price but different maturities often have different implied volatilities, giving rise to a term structure of implied volatility. A rising term structure indicates that market participants expect short-term implied volatility to rise or that they are willing to pay more for protection against longer-term asset price volatility. Chart 2.6 illustrates with a plot of the implied volatilities of options on a 10-year U.S. dollar swap (swaptions) with option maturities between 1 month and 5 years.

*Chart 2.6  
**Term structure of implied volatility of 10-year swaptions**  
*U.S. dollar, option maturities from 1 month to 5 years, as of September 15, 1998, volatility in percent*

Shorter-term implied volatilities may be below the longer-term volatilities, giving rise to an upward-sloping term structure, or above the longer-term volatilities, giving rise to a downward-sloping term structure. Downward-sloping term structures typically occur when shocks to the market have abruptly raised volatilities across the term structure. Short-term volatility responds most rapidly, since shocks are usually expected to abate over the course of a year. Upward-sloping term structures are usually observed in markets that have been quiescent for some time but are expected eventually to return to a higher level of volatility.

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7 Implied volatilities are generally expressed at an annual rate. If implied volatilities were constant for options with different maturities, but not annualized, longer-term volatilities would, of course, be higher than shorter-term ones.
The mean reversion in implied volatility also leads longer-term implied volatilities to vary less over time than shorter-term volatilities on the same asset. As a result, there are typically only small differences among the historical averages of implied volatilities of different maturities. Longer-term volatilities are therefore usually closer to the historical average of implied volatility than shorter-term implied volatilities. 8

E CASH AND DERIVATIVE MARKETS

In basing asset price forecasts on market data, it is best to use the most liquid markets, so as to reduce distortions in the data arising from transient imbalances of supply and demand. For many currencies and interest rates, the most liquid derivatives trade in OTC markets with special market conventions which the forecast user should understand. This section surveys the most liquid cash and derivatives markets for the four asset classes we focus on, highlighting OTC markets with special market conventions. 9

A special convention encountered quite frequently in the OTC foreign exchange and interest rate option markets is the use by option dealers of the Black-Scholes implied volatility as a “metric” or yardstick of option prices. This is possible because of the simple mapping between implied volatility and the option premium in currency units illustrated in Charts 2.4 and 2.5. In markets employing this convention, implied volatilities are not derived quantities, but raw market price data. In contrast, prices of exchange-traded options and options on futures are always expressed in ticks or currency units. The implied volatility, if needed, must be calculated from the prices in ticks.

OTC market dealers do not affirm a belief in the Black-Scholes model by trading implied volatilities. Rather, they detach implied volatility from the Black-Scholes model and treat it as a mere parameter. The implied volatility is a convenient metric for option prices because it enables option dealers to avoid repricing their option portfolios and recalibrating their bid and ask prices whenever the underlying asset price moves, if that move is seen as a random fluctuation unrelated to market views on volatility.

Foreign exchange

Most trading in spot and forward foreign exchange as well as foreign exchange options takes place in the OTC market. In addition, futures and options on futures trade on a number of exchanges worldwide, but account for only a small portion of foreign exchange derivatives turnover.

The OTC foreign exchange forward and option markets are quite highly developed, with liquid markets in contracts for a wide range of currency pairs and standardized maturities up to 1 year. Beyond 1 year, and for some currency pairs, good price quotes can be more difficult to obtain. In LongRun, forecasts of all exchange rate means and variances will be based on forward prices and OTC implied volatilities.

OTC currency options are traded with prices expressed in implied volatility terms. Most options are at-the-money forward, with exercise prices set equal to the current forward rate. Trading in out-of-the-money options is highly liquid for some currency pairs. When out-of-the-money options are traded, it is frequently in the form of risk reversals, in which the counterparties exchange an out-of-the-money call for an equally out-of-the-money put. One also frequently encounters the strangle, in which one counterparty buys an equally out-of-the-money call and put from the other.

8 See Campa and Chang (1995), Xu and Taylor (1994), and Stein (1989) for a more detailed description of the volatility term structure in the markets for foreign exchange options and options on stock index futures, as well as studies of the predictive power of the term structure for future implied volatility.

9 Information on trading volumes in derivatives markets and other institutional details can be obtained from Bank for International Settlements (1996) as well as in the sources cited in footnote 1.
Equities and commodities
Cash equities of individual companies trade both on exchanges and OTC. Futures are much more common than forwards in equity markets. Equity index futures and options on equity index futures are among the most heavily traded equity contracts. There are smaller but significant and growing markets for OTC equity derivatives. There are also old and well-established, albeit small, markets in options on shares of individual companies. LongRun focuses on major equity indexes and uses exchange-traded futures and implied volatilities calculated from options on futures to forecast means and confidence intervals for equity indexes.

Most spot commodity trading takes place OTC, while the bulk of derivatives trading is on exchanges. One exception is the gold market, in which a significant gold forward and related gold leasing market exist alongside substantial trading in gold futures and options on futures on the exchanges. Similarly, while most commodity options are options on futures, there are smaller parallel markets in OTC options, which are frequently components of highly structured transactions. The OTC gold options market is also quite active and is structured in many ways like the foreign exchange options markets.

LongRun uses exchange-traded futures and implied volatilities calculated from options on futures to forecast means and confidence intervals for all commodities except gold. In the case of gold, LongRun uses OTC gold forward prices and option implied volatilities.

Interest rates
Trading of fixed-income investments accounts for a large portion of financial market transactions worldwide. The government bond and bill markets, which for most major currencies are the most liquid fixed-income markets, are one large component. The interbank credit markets, which include time deposits and swaps, are another. Plain vanilla interest-rate swaps are cash-settled agreements to exchange the interest payments on a fixed-rate security for those on a floating-rate security. For some major currencies, such as the Swiss franc and Italian lira, interest rate liquidity and price discovery are centered in the deposit and swap market rather than the government bond markets.

Exchange-traded futures and options on futures have been introduced for most significant government bond markets. Government bonds are often traded forward, and active OTC markets for some government bond options also exist. Futures and options on futures are traded on interbank time deposits. Prominent examples are eurodollar and short sterling futures on the CME and Liffe.

Interbank time deposits and swaps are also traded forward. In fact, the market for forwards on time deposits, called forward rate agreements (FRAs), is quite well-organized and liquid. In a FRA, one party agrees to pay a specific interest rate on a eurodeposit of a specified currency, maturity, and amount, beginning at a specified date in the future. FRA prices are defined as the spot rate the buyer agrees to pay on a notional deposit of a given maturity on a given settlement date. Usually, the reference rate is Libor. For example, a 3X6 (spoken “3 by 6”) Japanese yen FRA on ¥100,000,000 can be thought of as a commitment by one counterparty to pay another the difference between the contracted FRA rate and the realized level of the reference rate on a 3-month ¥100,000,000 deposit commencing in 3 months and terminating in 6 months. Details of the calculations are provided in Appendix 2.A.

The prices, payoffs, and exercise prices of interest rate options can be expressed in terms of bond prices or interest rates, and the convention differs for different instruments. The terms and conditions of all exchange-traded interest rate options and some over-the-counter interest rate options are expressed as prices rather than rates. The terms and conditions of certain types of over-the-counter interest rate options are expressed as rates. A bond call expressed in terms of interest rates is identical to a bond put expressed in terms of prices.
Caplets and floorlets are over-the-counter calls and puts on interbank deposit rates. The exercise price, called the cap rate or floor rate, is expressed as an interest rate rather than a security price. The payoff is thus a number of basis points rather than a currency amount. A contract containing a series of caplets or floorlets with increasing maturities is called a cap or floor.

- In the case of a cap, the payoff is equal to the prevailing rate on the maturity date of the cap minus the cap rate, or zero, whichever is larger. For example, a 3-month caplet on 6-month U.S. dollar Libor with a cap rate of 5.00% has a payoff of 50 basis points if the 6-month Libor rate six months hence ends at 5.50%, and a payoff of zero if the 6-month Libor rate ends at 4.50%.

- In the case of a floor, the payoff is equal to the floor rate minus the prevailing rate on the maturity date of the cap, or zero, whichever is larger. For example, a 3-month floorlet on 6-month U.S. dollar Libor with a floor rate of 5.00% has a payoff of 50 basis points if the 6-month Libor rate six months hence ends at 4.50%, and a payoff of zero if the 6-month Libor rate ends at 5.50%.

Chart 2.7
Payoff on a Forward Rate Agreement, caplet, and floorlet
FRA locked in at 5%, cap and floor rates 5%

A caplet or a floorlet also specifies a notional principal amount. The obligation of the writer to the option owner is equal to the notional principal amount times the payoff times the term of the underlying interest rate. For example, for a caplet or floorlet on 6-month Libor with a payoff of 50 basis points and a notional principal amount of USD 1,000,000, the obligation of the option writer to the owner is USD \(0.0050 \times (1/2) \times 1,000,000 = 2,500\).

To see the equivalence between a caplet and a put on a bond price, consider a caplet on 6-month Libor struck at 5%. This is equivalent to a put option on a 6-month zero coupon security with an exercise price of 97.50% of par. Similarly, a floor rate of 5% would be equivalent to a call on a 6-month zero coupon security with an exercise price of 97.50.

A collar is a combination of a long cap and a short floor. It protects the owner against rising short-term rates at a lower cost than a cap, since the premium is reduced by approximately the value of the short floor, but limits the extent to which he benefits from falling short-term rates.

Swaptions are options on interest rate swaps. The exercise prices of swaptions, like those of caps and floors, are expressed as interest rates. Every swaption obligates the writer to enter into a swap at the initiative of the swaption owner. The owner will exercise the swaption by initiating the swap
if the swap rate at the maturity of the swaption is in his favor. A **receiver swaption** gives the owner the right to initiate a swap in which he receives the fixed rate, while a **payer swaption** gives the owner the right to initiate a swap in which he pays the fixed rate.

There are two maturities involved in any fixed-income option, the maturity of the option and the maturity of the underlying instrument. To avoid confusion, traders in the cap, floor and swaption markets will describe, say, a 6-month option on a 2-year swap as a “6-month into 2-year” swaption, since the 6-month option is exercised “into” a 2-year swap (if exercised).

As in the foreign exchange markets, caps, floors and swaptions are traded in implied volatility terms. However, implied volatility is expressed as a **yield volatility** rather than the **price volatility** on which we have focused. The relationship between yield volatility and price volatility corresponds to the relationship between considering the option as written on an interest rate or on a bond price.

By assuming that the interest rate on which the option is written behaves as a random walk, we can apply the Black-Scholes formulas with interest rates substituted for bond price levels. The yield volatility of a fixed-income option, like the price volatility, is thus a Black-Scholes implied volatility. As in the case of other options quoted in volatility terms, this practice does not imply that dealers believe in the Black-Scholes model. It means only that they find it convenient to use the formula to express prices.

There is a useful approximation that relates yield and price volatilities:

\[
\frac{\text{yield volatility}}{\text{duration} \cdot \text{yield}} \approx \frac{\text{price volatility}}{
\]

To use the approximation, we must express the yield as a decimal. Note that when yields are low, yield volatility tends to be higher.

There is generally a significant and variable spread between the interest rates on government-issued and bank-issued obligations in the same currency and with the same time to maturity. In the U.S. dollar money markets, it is known as the TED spread, since it is measured by the spread between the money market rates implied by Treasury bill futures and eurodollar futures prices. In the bond markets, the spread is called the swap-Treasury spread. *LongRun* will calculate forecasts for either or both of the government and interbank interest rate curves, provided liquid markets for cash and derivatives assets exist.

### 2.1.2 The performance of derivatives prices in forecasting cash prices

The common interpretation of forward and futures prices as market forecasts of future cash prices has been examined in many academic studies. More recently, analogous studies have been made of the forecasting performance of implied volatility. Little agreement has been reached on the core questions of whether forwards, futures and options are good predictors, and if not, why not. This section sketches some of the issues involved.

A **FORECASTING PERFORMANCE OF FORWARD AND FUTURES PRICES**

The forecasting performance of one forecasting technique can only be assessed relative to others. *LongRun* adopts as a benchmark the forecast of “zero change,” that is, the forecast technique which assumes the expected change in the asset price is zero. In *LongRun*, we call the benchmark zero change forecast the **random walk with zero expected return** (RWZ) method, since it corresponds to the random walk model with an expected return or drift rate equal to zero. It serves as a natural
benchmark because it is so simple and because it is the forecasting technique used in the short-term context of RiskMetrics.

It is more convenient to express this model in terms of the logarithm of the asset price. If \( S_t \) denotes the time-\( t \) asset price, we will let \( s_t \) denote its logarithm.\(^{10}\) In the RWZ approach, the logarithmic change in the asset price is assumed to be zero:

\[
E_t [s_{t+\tau} - s_t] = 0
\]

for any forecast horizon \( \tau \), where \( E_t [\cdot] \) denotes an expected value as of time \( t \).

For an asset price following a random walk with a zero drift rate, the change in price over the next time interval is independent of both the change over the last time interval and the level of the asset price. For this reason, the random walk with zero drift is sometimes described as “memoryless.” There is no trend, and thus no tendency for an up move to be followed by another up move, or by a down move.

If we observe a zero-drift random walk process over a given time interval, however, its tendency to wander away from its starting point may make it appear to follow a trend. (In contrast to the thought experiment illustrated in Chart 2.3, in real life we can observe only one realization of the asset price’s random evolution over time.) This is illusory: eventually, if we observed it long enough, the random walk would move closer to its initial level—and wander far away in either direction—over and over again. In Chapter 3, we will study these apparent trends generated by random walks, called stochastic trends, in more detail.

When we use the forward or futures price to forecast the future asset price, we assume that the asset return follows a random walk with a non-zero drift equal to the forward premium. In LongRun, we call the technique of basing forecasts on the forward premium the random walk with expected return equal to the forward premium (RWF) method. In the RWF model, the asset has a non-random trend and is more likely to move in one direction than in the other.

The covered parity relationship among the forward and cash asset prices, the money market interest rate, and the cash yield on the asset, which we discussed in Section 2.1.1, is helpful here. The general covered parity relationship is\(^{11}\)

\[
F_{t,t+\tau} = \left[ 1 + (r_{t,t+\tau} - d_{t,t+\tau})\tau \right] S_t,
\]

where \( F_{t,t+\tau} \) is the time-\( t \) price of a forward or futures expiring at time \( t + \tau \). The quantity \( (r_{t,t+\tau} - d_{t,t+\tau})\tau S_t \), where \( r_{t,t+\tau} \) is the \( \tau \)-period money market or financing rate at time \( t \) and \( d_{t,t+\tau} \) is the asset’s \( \tau \)-period cash yield at time \( t \), is the \( \tau \)-period forward premium at time \( t \), expressed in currency units. The quantity \( (r_{t,t+\tau} - d_{t,t+\tau}) \tau \) is the forward premium expressed as a percentage rate. The financing rate is roughly the same for all assets,\(^{12}\) while the cash yield depends on the type of asset.

---

\(^{10}\) The lower-case notation for logarithms will be used throughout the LongRun Technical Document.

\(^{11}\) If time is measured in years, both yields are at an annual rate, with a compounding interval of \( \tau \). The derivation of Equation [2.2] and its application to particular asset classes is discussed in Appendix 2.A.

\(^{12}\) The general collateral repo rate is a good proxy for the financing rate. If no repo rate exists, the short-term government bill rate is the next best proxy. If neither a repo rate nor a government bill rate are observable, the interbank deposit rate is an acceptable proxy for the financing rate.
In the RWF approach, the logarithmic change in the asset price is assumed to equal the forward premium:

\[ 2.3 \]

\[ E_t[s_{t+\tau} - s_t] = (r_{t,t+\tau} - d_{t,t+\tau})\tau. \]

The covered parity relationship Equation [2.2] shows that the forward price will forecast changes in the cash price better than current cash price only if the forward premium contains meaningful information about the future cash price which has not already been embedded in the cash price through the market process. This necessary condition of forecasting power is met by some assets at some forecast horizons.\(^\text{13}\)

Asset prices often follow apparent trends that can last months or years. Regardless of asset class, forwards and futures, in common with many forecasting techniques, do a poor job of predicting turning points, the times at which a trend is broken and a new trend sets in. We will return to this important issue in our discussions of the statistical behavior of asset prices and of market expectations concerning large asset price moves in Section 2.2.

**Foreign exchange**

In foreign exchange markets, the common market view of forecasting performance is couched in terms of the **open interest parity hypothesis**, which states that the forward exchange rate is equal to the market expectation of the future spot exchange rate. The role of the asset yield \( d_{t+t} \) is played by the foreign money market interest rate, since one earns interest on a foreign currency position (see Appendix 2.A for details). The forward foreign exchange premium is equal to the differential between the domestic and foreign interest rates. If depreciation of the domestic currency is expected, the domestic rate will tend to be higher than the foreign rate.

When the forward premium (interest rate differential) is high, forward foreign exchange rates may predict exchange rate changes better than spot exchange rates. This is typically the case for exchange rates of developing countries against the dollar. Nominal interest rates and inflation rates are high in emerging markets compared to industrial economies and are associated with steady currency depreciation. At short horizons the predictive power of the forward premium may be masked by short-term fluctuations in spot and forward rates, so forecasting performance is better at longer forecasting horizons.

When the forward premium is low, as is the case for most industrial economies’ exchange rates against one another, forward rates generally do not outperform spot exchange rates in forecasting future spot exchange rates except over very long forecast horizons. Typically, selling pressure on the exchange rate impacts equally and simultaneously on the spot and forward rate, leaving the forward premium unchanged. However, even for some currency pairs with low forward premiums, the forward rate has some predictive power in the very long term. For example, the yen has usually traded at a forward premium against the dollar, and 10-year forecasts of a weaker dollar based on 10-year interest-rate differentials have usually proved accurate.

**Interest rates**

In fixed-income markets, the common market view of forecasting performance is called the **expectations hypothesis of the term structure**. The forward interest rate is the price at which a future

\(^{13}\) For a survey of the issues and the methodologies used to address them, see Campbell, Lo and MacKinlay (1997). On foreign exchange, Lewis (1995) and Engel (1996) provide up-to-date surveys with additional references. On fixed income, see Shiller and McCulloch (1990), and chapter 10 of Campbell, Lo and MacKinlay (1997). On equities, see chapters 7 and 10 of Campbell, Lo and MacKinlay (1997), and the references cited therein. On commodities, see Pindyck (1993).
interest rate exposure can be locked in, and is interpreted as a forecast of the future spot interest rate. Equivalently, the term premium or the slope of the term structure—the spread of a long-term rate over a short-term rate—can be interpreted as a forecast of changes in future short-term rates. The interpretation of forward rates as forecasts implies that an increase in the spread between long- and short-term rates predicts a rise in both short- and long-term rates.

To see the implications of the expectations hypothesis for future short-term rates, consider the relationship among spot interest rates with two different terms \( \tau_1 \) and \( \tau_2 \), with \( \tau_1 < \tau_2 \). Under the expectations hypothesis, the \( \tau_2 - \tau_1 \)-month interest rate currently expected to prevail at time \( t + \tau_1 \) is given by

\[
(1 + r_{t,t+\frac{\tau_2}{12}})^\frac{\tau_1}{12} = (1 + r_{t,t+\frac{\tau_1}{12}})^\frac{\tau_1}{12} (1 + E_t[r_{t+\tau_1, t+\tau_2}])^\frac{\tau_2-\tau_1}{12}.
\]

For example, if the current 9-month rate is 4.8% and the 1-year rate is 5.0%, the 3-month rate expected to prevail 9 months hence is given by

\[
1.050 = (1 + 0.048)^\frac{3}{12} (1 + E_t[r_{t+9, t+1}])^\frac{1}{4}.
\]

implying, under the expectations hypothesis, that \( E_t[r_{t+9, t+1}] \) is 5.60%.

To see the implications of the expectations hypothesis for future long-term rates, consider the relationship among 1-, 2- and 3-year spot rates, with \( r_{t,t+12} < r_{t,t+36} \) and \( r_{t,t+24} < r_{t,t+36} \) (the 3-year rate is higher than either of the shorter-term rates). The expectation hypothesis implies that the 2-year rate will rise from \( t \) to time \( t + 12 \). Under the expectations hypothesis, the expected future 2-year rate is given by

\[
(1 + r_{t,t+36})^3 = (1 + r_{t,t+12})(1 + E_t[r_{t+12, t+36}])^2.
\]

Since \( r_{t,t+24} < r_{t,t+36} \)

\[
(1 + r_{t,t+24})^2 < (1 + r_{t,t+12})(1 + E_t[r_{t+12, t+36}])^2.
\]

Since \( r_{t,t+12} < r_{t,t+36} \)

\[
(1 + r_{t,t+24})^2 < (1 + r_{t,t+12})^2 < (1 + E_t[r_{t+12, t+36}])^2 < (1 + E_t[r_{t+12, t+36}])^2,
\]

implying

\[
r_{t,t+24} < E_t[r_{t+12, t+36}].
\]

The forecasting performance of forward rates with respect to short-term rates has generally been better than that for long-term rates. Central banks in industrialized countries generally adopt a short-term interest rate as an intermediate target, but they also attempt to reduce short-term fluctuations in interest rates. Rather than immediately raising or lowering interest rates quickly by a large amount to adjust them to changes in economic conditions, they change rates in small increments over a long period of time. This practice, called interest rate smoothing, results in protracted periods in which the direction and likelihood, but not the precise timing, of the next change in the target
interest rate can be guessed with some accuracy, reducing the error in market predictions of short-term rates.  

Central banks’ interest rate smoothing improves the forecasting power of short-term interest rate futures and forwards at short forecasting horizons. This is in contrast to forwards on foreign exchange, which tend to predict better at long horizons. At longer horizons, the ability of forward interest rates to predict future short-term deteriorates. Forward rates have less ability to predict turning points in central banks’ monetary stance than to predict the direction of the next move in an already established stance.

**Equities and commodities**

For equity indexes, \( d_{t, t+\tau} \) is the dividend yield. The spread between the futures price and the cash index is equal to the short-term interest rate minus the dividend yield. Generally, the futures price is higher than the cash index because dividend yields are generally lower (e.g., on the order of 2% for the United States) than money market yields. Dividend yields do not fluctuate significantly over short time intervals, so changes in the spread between futures and cash index levels are due primarily to changes in money market rates. These have only limited forecasting power for equity index levels. Changes in expectations about the future level of the index are reflected much more rapidly in changes in the cash level of the index. The futures and cash index level are therefore more or less equally accurate predictors of future cash prices.

For many commodities, there is an observable asset yield. For example, physical gold commands a lease rate in the market. In other cases, there is no observable yield, so a “convenience yield” must be imputed to the market by solving Equation [2.2] for \( d_{t, t+\tau} \). The gold lease rate tends to be small, on the order of 2% per annum, although it occasionally abruptly soars for short periods. Imputed convenience yields for other commodities are larger and vary more. As in the case of equities, changes in expectations about future commodity prices are reflected more rapidly in changes in cash prices than in the forward or futures premium, so current forward and cash prices are more or less equally accurate predictors of future cash prices.

**B FORECASTING PERFORMANCE OF IMPLIED VOLATILITIES**

In analogy to the market interpretation of forward and futures prices as forecasts of future cash prices, implied volatility is commonly interpreted as a forecast of future asset price volatility. The analogy to forwards is not perfect, however. The forward price is unambiguous: it is observed in the market and not tied to a model. An implied volatility, in contrast, always emerges from a particular pricing model. We noted above that dealers use the Black-Scholes implied volatility as a metric for option prices without necessarily believing in the Black-Scholes model. In the present context, however, we are going beyond dealers’ use of implied volatility as a price metric and attempting to use it as an estimate of future volatility.

Any test of the forecasting properties of implied volatility is therefore a joint test of both the forecasting power of implied volatility and a particular option pricing model. If implied volatility is drawn from a correctly specified model, and implied volatility is a useful forecasting tool, then it should be a good predictor of future volatility over the life of the option.

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15 See Pindyck (1993).
16 See Bates (1996c).
To date, most work in this area has been done using stock index and OTC currency option prices. Most researchers find that

- implied volatility has at least some significant predictive power for future volatility;
- the predictive power of implied volatility appears to be best in the very short term and to decline at longer forecast horizons; and
- implied volatility tends to overstate future volatility.

### 2.1.3 Efficient markets theory

Forward, futures and option prices are not forecasts, but prices at which market participants can lock in exposures to future asset price fluctuations. *LongRun* uses forward, futures and option prices as proxies for market forecasts because they are practical, objective and market-driven. The markets’ common interpretation of forward, futures and option prices as forecasts is borne out empirically in some cases, but not others, as we saw in the previous section. The common interpretation also receives some support from finance theory.

Efficient markets theory implies that all available information regarding future asset prices is impounded in current asset prices and suggests that prices of forwards, futures and options are closely related to expected future asset prices. If, therefore, we can take it as roughly true that markets are efficient, current asset prices should provide useful long-term forecasting tools.

In this section, we review efficient markets theory by way of presenting in detail the methodology of employing forward, futures and option prices as forecasts. Our survey of efficient markets theory will also help us understand the limitations of forward, futures and option prices in forecasting future asset prices and volatility.

#### A Asset prices in efficient markets

The efficient markets hypothesis implies that market prices contain implicit forecasts of future market prices. In efficient markets, forward and futures prices are estimates of the markets’ perceived expected value of the future asset price. The expectations hypothesis of the term structure and the open interest parity hypothesis in foreign exchange markets, discussed in Section 2.1.2, are applications of efficient markets theory.

The efficient markets hypothesis follows from so-called no-arbitrage conditions. There are two types of arbitrage that must be ruled out:

- The first type of arbitrage that must be ruled out is violations of the “single-price law of markets,” that is, the requirement that the prices of two asset portfolios with identical cash flows be the same. If this type of arbitrage is possible, then covered parity, which we dis-
discussed in Section 2.1 above, fails to hold and market participants can execute a set of transactions—buying or selling cash and forward assets and borrowing or lending in the deposit market—which have zero net cash outlays now but lock in a positive profit when the contracts mature without risk of loss.

In arbitrage operations involving options and other contingent liabilities, rather than locking in a profit with certainty, a market participant may find an opportunity to lock in a positive probability of a profit, without risk of loss. Such opportunities, too, should exist only fleetingly in an efficient market.

- For the efficient markets hypothesis to hold, we must also rule out a second form of arbitrage: there cannot be a portfolio of assets which has a zero net cost now and a positive expected value in the future. Otherwise the current value of the portfolio would be bid up. In this type of arbitrage, there is a positive probability of loss.

The expected value of the future profit in this weaker form of arbitrage depends on some model of how asset prices behave. For example, in the expectations hypothesis of the term structure, a rise in the slope of the term structure might indicate the possibility of arbitrage profits from purchasing, say, a 2-year bond and financing it by rolling over a short position in a 1-month treasury bill or by lending the bond as collateral in the repo market. The expectations hypothesis implies that the arbitrageur will pay out less interest than earned on the bond. The arbitrage will lead to a loss if the term structure continues to steepen and the long bond position suffers a capital loss.

The covered parity relations are independent of any model. Frequently, several different traded assets have closely related payoffs and therefore contain the same information. The covered parity conditions tell us which portfolios of financial instruments are essentially “the same commodity” and are thus particularly useful for inferring forward prices of assets for which explicit forward markets do not exist. Most analysts accept covered parity as an accurate description of actual markets, both because it has strong empirical support and because it is difficult to imagine a world in which market participants ignore riskless profit opportunities.

If transactions costs were zero, the covered parity relationships would hold exactly at all times, because any momentary deviation could be exploited by alert market participant to lock in a profit at the maturity date of the forward with no cash outlay now and no risk. In real-life markets, covered parity holds almost, but not quite exactly, as transactions costs are not zero and even alert traders may miss momentary departures from covered parity if prices are changing rapidly.

There are also differences among asset classes in the extent to which covered parity can be violated. Violations are very rare in the foreign exchange markets, but less so in the equity index futures markets. It is more difficult, slow and costly to assemble the long or short basket of cash equities that mimics the index than to buy and sell spot foreign exchange or borrow and lend in the interbank market. The percentage spread between the equity index futures price and the level of the cash index therefore deviates from the covered parity condition more often and for longer than in the foreign exchange and fixed-income markets. The most persistent and frequently observed arbitrage opportunities arise from international differences in tax, regulatory, and accounting regimes which can be exploited by only a subset of market participants.

In contrast to covered parity, the efficient markets hypothesis remains controversial, partly because, even if true, it is very difficult to support empirically, and partly because one can easily imagine a world in which market participants disagree about the correct model of future returns or fail to react quickly to uncertain prospects for profit. In order to bring empirical evidence to bear on the efficient markets hypothesis, researchers must use a specific model, which will necessarily be controversial, of the fundamental determinants of returns, including market participants’ preferences. It is there-
fore difficult to test the hypothesis; any test is in addition a test of a particular theory of asset price
determination.

Another important assumption underlying the efficiency hypothesis is rational expectations, the
notion that market participants’ subjective expectations of future economic variables, including fi-
nancial asset prices, are equal to optimal forecasts or to mathematical expected values given “all
available information” on those variables. In one form or another, rational expectations has become
a standard part of the methodology of economic and financial research because it is difficult to for-
mulate alternative hypotheses about market participants’ expectation formation that are consistent
with optimizing behavior.\textsuperscript{20}

Throughout this discussion, we will assume that there are no transactions costs or taxes, that mar-
kets are in session around the clock, that nominal interest rates are positive, and that unlimited short
sales are possible.\textsuperscript{21} These assumptions are fairly innocuous for most of the instruments from which
we will be gathering market information. In the international financial markets, transactions costs
typically are quite low for most standard financial instruments, and most of the transactions dis-
cussed here are not taxed, since they are conducted in the euromarkets or on organized exchanges.

The efficient markets approach to explaining asset prices views them as the present values of the
income streams they generate. In a simple version of the present value relation, we can imagine
there are \( W \) possible states of the world, each with probability \( \pi_w, w = 1, \ldots, W \). Assume that the
states are distinguished only by the constellation of asset prices realized in that state, and that ev-
everyone agrees on what states are possible. Which state will prevail is unknown now. We divide time
into discrete moments \( t, t + 1, t + 2, \ldots \) at which dividends or other cash flows from an asset are
received. We denote future cash flows from the asset by \( d_{t+\tau} \), \( \tau = 1, 2, \ldots \), expressed as a percent
of the contemporaneous asset price, which we assume for convenience has no term structure.

Each \( d_{t+\tau} \) is treated now as a random variable, dependent on which state of the world prevails, so
we denote the actual realization of the cash flows in state \( w \) by \( d_{w,t+\tau} \). Similarly, price realizations
at each time period and in each state are denoted \( S_{w,t+\tau} \). A simple form of the present value relation
states that the expected total one-period return on the asset is equal to a “well-behaved” rate \( \rho_t \):

\[
[2.10] \quad \frac{E_t[S_{t+1} + d_{t+1}S_t]}{S_t} - 1 = \rho_t.
\]

Much of the efficient markets approach has been embedded in the operator \( E_t[\cdot] \), which rep-
sents investors’ rational expectations (optimal) forecast of the uncertain future quantity \( X_{t+1} \) given
the information they have at time \( t \). Equation [2.10] implies that the current cash price is a fore-
cast of future dividends and the future asset price:

\[
[2.11] \quad S_t = \frac{E_t[S_{t+1} + d_{t+1}S_t]}{1 + \rho_t}.
\]

\textsuperscript{20} Nonetheless, there remain puzzling anomalies in real-world asset prices which indicate the importance of habit
and myopia in financial decision-making. This is the subject matter of behavioral economics, surveyed in

\textsuperscript{21} Nominal interest rates can be slightly negative, as has recently occurred for some favored borrowers in the Japa-
nese yen deposit market. These rates are bounded below by the cost and inconvenience of storing cash in the form
of banknotes in a vault. Real interest rates, that is nominal interest rates less the correctly-measured price level
inflation expected over the term of the interest rate, may be (and, for example, in Japan, currently are) negative.
If Equation [2.11] holds, the current asset price equals the expected value of all future cash flows, discounted to the present at the asset’s required rate of return $\rho_t$:

$$[2.12] \quad S_t = \sum_{\tau = 1}^{\infty} \left( \frac{1}{1 + \rho_t} \right)^\tau E_t \left[ d_{t + \tau} S_{t + \tau - 1} \right].$$

We can also express Equation [2.12] in terms of the probabilities of the states:

$$[2.13] \quad S_t = \sum_{\tau = 1}^{\infty} \sum_{w = 1}^{W} \left( \frac{1}{1 + \rho_t} \right)^\tau d_{w, t + \tau} S_{w, t + \tau - 1}. $$

Equations [2.12] and [2.13] do not have a lot of empirical content and cannot be used directly to test the efficient markets hypothesis, because given any hypothesis about the probability distribution of the dividends or any ex post time series of dividends, we could conjure up a discount factor or probabilities that would make the formula work. The point, rather, is that if we maintain the hypothesis of efficient markets, we can represent asset prices this way.

B Why don’t forwards forecast better? The role of risk preferences

In efficient markets, not only expectations about the future, but risk factors determine asset prices. In the efficient markets model, risk factors are embedded in the expected rate of return $\rho_t$. Investors holding an asset feel fully compensated for the risks they take by the difference between $\rho_t$ and $r_t$, the risk-free rate of return. The difference $\rho_t - r_t$ is called the risk premium. The expected rate of return and the risk premium are influenced by such factors as market participants’ attitude towards risk, outstanding asset supplies, the riskiness of the return, the rates of return on alternative investments, the anticipated time profile of returns, and market participants’ preferred time profile of consumption.

Risk premiums are believed to be one reason that forward and futures prices are not better predictors of future asset prices. The adverse impact on the forecasting power of forward prices is particularly great if risk premiums fluctuate over time and if their fluctuations are correlated with errors in market forecasts. For example, suppose the term premium rises both because the market believes that a rise in short-term interest rates is imminent and because it has a greater desire to protect exposures to short-term rates. If the short-term rate fails to rise as anticipated, the forecasting error will appear even greater because the risk premium rose, too.

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22 One more assumption is still needed, that so-called “rational bubbles” are excluded. Here is a set of assumptions that is sufficient for Equation [2.12]: The representative investor’s utility is time additive and returns are independent of the marginal utility of consumption, so the stock market doesn’t tend to crash when you want to go on vacation. The asset price will then adjust so as to equalize the expected rate of return to the investor’s subjective discount rate. We assume, as we did for the dividend rate, that the expected rate of return may fluctuate over time, but has no term structure.

23 The role of risk in creating a spread between forward and anticipated future cash prices has been studied since the turn of the century, as surveyed in Williams (1986). For example, the spread was explained by Keynes (1930, p. 143) as "the amount which the producer is ready to sacrifice to ‘hedge’ himself, i.e., to avoid the risk of price fluctuations during his production period.” The downward-sloping term structure of forward and futures prices implied by this theory is called normal backwardation.

Ideally, we would like to forecast expected values and confidence intervals using the market’s true probability beliefs $\pi_w$. However, we can only use market asset prices to identify risk neutral probabilities. These are defined by a present value formula for the asset price which looks much like Equation [2.12], but with some key differences:

$$S_t = \sum_{\tau = 1}^{\infty} \left( \frac{1}{1 + r} \right)^\tau E_t^\tau \left[ d_t e^{S_t + \tau} \right] = \sum_{\tau = 1}^{\infty} \left( \frac{1}{1 + r} \right)^\tau \sum_{w = 1}^{W} q_w d_{w, t + \tau} e^{S_{w, t + \tau}} ,$$

where $r_t$ is the riskless rate of return, and $q_w, w = 1, \ldots, W$ are the risk neutral probabilities.

The risk neutral probabilities are the probabilities of future prices that are embedded in asset prices as they now stand. The structure of Equation [2.14] is similar to Equation [2.12], but discounting is done at the risk-free rate rather than the asset rate of return, and expectations are taken with respect to the risk neutral rather than the true probabilities. The risk neutral probabilities are defined as the set of $q_w$ that make Equation [2.14] “work,” that is, let’s observed asset prices appear as the present values of their future payoffs, discounted at the risk-free rate.

Empirically, $\rho_t$ (or the risk premium) is just about impossible to identify, but we have replaced it with the easily observed risk-free rate of return $r_t$, subsuming the risk premium in the risk neutral probabilities. Whether or not we can actually identify the $q_w$, Equation [2.14] will be “legitimate” in the sense that there actually exists a set of numbers $q_w$ that behave like proper probabilities, i.e., $q_w \geq 0$, for all $w$, and $\sum q_w = 1$.

The intuition behind Equation [2.14] is similar to that for Equation [2.12]. However, in this case, the appropriate discount rate is the risk-free rate $r_t$. That means that the $q_w$ are the probability beliefs we would attribute to the market if we believed it was composed of a single risk neutral agent (hence the name “risk neutral probability distribution”). In such a model, there are no risk premiums: all changes in an asset price are due to changes in the expected value of the payoff or in the risk-free interest rate, and not to changes in risk appetites or in outstanding stocks of assets.

The risk neutral probabilities are identical, in this simple setup, to state prices, the price of a dollar’s worth of consumption in a particular state. The same cash flow may be more valuable in one state than in another because asset returns generally are expected to be low in that state, affecting the perceived expected value of returns. It might, however, be because the desire to ensure a minimum level of consumption will be particularly high in that state, affecting the risk premium embedded in the rate of return.

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25 By “true expectations,” we mean the actual subjective expectations “in the heads” of market participants. We cannot observe these, but we can imagine them as the result of accurate polling of the market consensus regarding the future price. We do not mean to suggest that the market, the financial analyst, or anyone else knows what price will actually be realized in the future.

26 This will be true under very general conditions, that is without being wedded to any specific model of asset price determination. These general conditions are more or less equivalent to covered parity. Astonishingly, the risk neutral probabilities are identical for all assets. The risk neutral measure is unique if markets are complete, that is, any desired state-contingent payoff can be constructed from some combination of existing assets. Dreze (1970) presents a simple model that clarifies the relationship between risk neutral probabilities and market prices. The application to asset pricing dates back to Cox and Ross (1976) and was formalized by Harrison and Kreps (1978). Harrison and Pliska (1981) present a detailed formal exposition of the conditions for the existence of risk neutral probabilities and their role in calculating derivative prices. Several advanced textbooks cover the material in this section: Ingersoll (1987), Huang and Litzenberger (1988), Dothan (1990), and Duffie (1996).
Example 2.1
State prices in a two period model
Suppose there are two goods, shoes and grain, and two possible future states, “feast” (state 1) and “famine” (state 2), and that the prices of shoes and grain in the two states, called state-contingent prices, are known. These future prices can be presented in a state-space tableau:

<table>
<thead>
<tr>
<th></th>
<th>feast</th>
<th>famine</th>
</tr>
</thead>
<tbody>
<tr>
<td>shoes</td>
<td>100</td>
<td>95</td>
</tr>
<tr>
<td>grain</td>
<td>100</td>
<td>150</td>
</tr>
</tbody>
</table>

Let the current prices be $90 for a pair of shoes and $100 for a sack of grain. The riskless rate of interest is then 10%.\(^{27}\) The risk neutral probabilities of feast and famine can then be inferred as 80% and 20%, which can be verified by noting that the current prices are the expected values of the future payoffs, discounted at the risk-free rate (shoes: \(90 = \frac{0.8 \times 100 + 0.2 \times 95}{1.10}\); grain: \(100 = \frac{0.8 \times 100 + 0.2 \times 150}{1.10}\)).

In the efficient markets model, the forward price is the risk neutral estimate (expected value) of the future spot price:\(^{28}\)

\[
F_{t,T} = E_t^*[S_T].
\]

Whatever the true, but unobservable, market expectation, the forward price is the risk neutral expectation. In the example here, the forward prices are the undiscounted risk neutral expected values of the assets: 99 = \(0.8 \times 100 + 0.2 \times 95\) for shoes and 110 = \(0.8 \times 100 + 0.2 \times 150\) for grain.

Suppose the true probability of a famine is only 2% rather than 20%. If the forward prices were to reflect the true probabilities rather than the risk neutral ones, as would be the case in a world in which agents were indifferent to risk, they would be 99.9 = \(0.98 \times 100 + 0.02 \times 95\) for shoes and 101 = \(0.98 \times 100 + 0.02 \times 150\) for grain. Shoes would be costlier and grain cheaper in a risk neutral world. Agents’ aversion to the risk of famine and their demand to insure against it generates a higher risk neutral probability of famine.

Expected returns on grain are low, 1%, since it has a high payoff in famine, when it is relatively highly valued. It has a negative risk premium of 9%. Conversely, starving agents will find shoes relatively uninteresting, so the expected return is high: 11%, of which 1% is a positive risk premium.

\(^{27}\) Denote the state-space tableau by \(D\) and the vector of current prices by \(q = (90,100)\). The vector of state prices, the price of one dollar’s worth of commodities in the second period in a particular state, is then \(D^{-1}q = (0.787, 0.182)\). One dollar next period if there is a feast costs $0.787 today, and one dollar next period if there is a famine costs $0.182 today. The cost of one dollar next period, regardless of which state prevails, is the sum of the two state prices, $0.909. The risk-free interest rate is thus \(1/0.909\).

\(^{28}\) This notion was developed by Samuelson (1965, 1973), Rubinstein (1976), and Cox, Ross, and Rubinstein (1979).
Example 2.1 is easy to compute, but somewhat abstract. A more pertinent example is the following:

**Example 2.2**

**Risk neutral and true distributions**

Suppose the Dow Jones Industrial Average is at a level of 10000 and has an implied volatility of 22.5%. Suppose further the public believes that the Dow is a bit shaky and that there is a 1% chance that the market will drop 1000 points (10%) and volatility will rise 50% over the next month. The public is somewhat anxious about this possibility, having shifted its asset allocation heavily towards equities in recent years. It is therefore willing to pay up for protection against this contingency as though the likelihood were 10% rather than only 1%. In Chart 2.8, this is illustrated by a risk neutral density which assigns a much higher probability than the market’s true probability beliefs to the event that the Dow closes lower than 9000 in 1 month’s time.

**Chart 2.8**

**Risk neutral and true probability distributions**

The terminology “risk neutral probability” and “risk neutral” can be somewhat misleading, since they suggest probability measures from which risk preferences have been purged. In fact, the terms mean the opposite: risk neutral probabilities are risk-compensated, that is, adjusted higher or lower to reflect market participants’ willingness to hold the asset given the range of prices it might fetch in the future. Risk neutral probabilities are a mixture of both the market’s true expectations and its risk preferences, so they are affected by both changes in what the market believes will actually happen in the future and by factors related to the market’s risk appetite, such as its asset holdings and its aversion to asset price volatility.

It is perhaps less misleading to call the forward price the **market-adjusted certainty equivalent** of the future cash asset price. A certainty equivalent is the certain prospect which an individual considers to be of equal value to an uncertain prospect. The forward price is the price at which risk can be taken on or laid off in the market. It is the answer to the question, what funds does one have to commit now to acquire the asset on a future date, rather than, what does the market believe the asset price will be on a future date?

In LongRun, we will use market prices as proxies for market forecasts, which is tantamount to assuming that risk premiums are equal to zero. This does not mean that we assume that market par-

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29 This characterization of forward prices is due to Fama (1984).
participants are indifferent to risk, but rather that we recognize that risk factors as well as expectations are compacted into the market forecasts.

C SOME FURTHER CAVEATS IN THE INTERPRETATION OF MARKET PRICES

Asset prices may reflect influences other than the risk neutral probability distribution of the future exchange rate. One set of influences are the risk factors just discussed. Another set of influences which may drive a wedge between asset prices and expectations are transient supply and demand factors other than those relating to “fundamental” factors such as expectations and risk appetites. These are frequently observed even in liquid markets: for example, large buy or sell orders can temporarily move asset prices.

Legal, regulatory and institutional constraints may also influence prices. As mentioned in Section 2.1.3, such constraints may lead to persistent arbitrage opportunities and obscure the information about expectation in observed asset prices. A more dramatic example is the impact of capital and exchange controls.

Regulatory constraints can be particularly important in the options markets. Some market participants may be legally barred from using options, or may need to adhere to accounting practices which make their use unattractive. If the clientele of option end-users is homogeneous, there may be a lasting propensity for clients to be “on the bid” or “on the offer,” forcing option dealers to skew prices to make markets.

Option hedging may cause temporary order flow imbalances in the cash markets in which the options are hedged. The impact of option hedging techniques has become more powerful as options markets have grown and as new instruments are introduced. New types of options, so-called “exotics,” permit market participants to tailor their option protection to a particular view on the development of future exchange rates. Dealers often hedge bought and sold exotic options with standard options. Because of the sensitivity of the payoffs of many exotic options to particular exchange rate levels, these option hedges can be large multiples of the face value of the exotic option. This can lead to feedback effects both on option prices and on spot prices.\(^{30}\)

2.1.4 Risk neutral forecasts from derivative asset prices

A MEAN FORECASTS FROM FORWARD PRICES

To turn the efficient markets hypothesis into a statistical model, we must make assumptions about the probability distribution of the asset price. In LongRun, we adopt the random walk model and assume that asset returns are normally distributed. As noted above, we will use two variants of the random walk model:

- In the RWZ (random walk with zero expected return) model, the statistically optimal point forecast—or estimator of the expected value—of the future value of a random walk with zero drift is just its current value, since the change in the random walk between now and the future date is random, and is as likely to be positive as negative.

\(^{30}\) For an example of such an episode, see Malz (1995).
• In the RWF (random walk with expected return equal to the forward premium) model, the statistically optimal point forecast is the forward price.\textsuperscript{31}

The relationship between the efficient markets and the random walk hypotheses may seem contradictory. On the one hand, all currently available information relevant to future returns is already impounded in current returns. On the other hand, the return sequence is random and unpredictable. The riddle’s solution is that, if current prices contain all available information, only new information can have an impact on next period’s prices. New information is by nature random and unpredictable—until it arrives: we can as easily imagine news provoking a price decline as a rally.

For convenience, we write the present value relation Equation [2.10] in logarithms:

\[ E[t]_t[s_{t+1}] - s_t + E[t]_t[d_{t,t+1}] = \rho_t. \]

The assumption that the returns follow a random walk is equivalent to maintaining three hypotheses:

i. Known dividends

The sequences of dividends and the one-period risk-free rate, while random, are known by the beginning of the period over which they accrue.\textsuperscript{32} This assumption is less strong than appears: money market yields and bond coupons are contractually specified at the start of the investment period and stock dividends are generally known well in advance. Since it is known at the beginning of period \( t \), \( d_{t,t+1} \) can be treated as a constant from that moment on, so \( E[t]_t[d_{t,t+1}] = d_{t,t+1} \) for all \( t \), and Equation [2.16] becomes

\[ E[t]_t[s_{t+1}] - s_t + d_{t,t+1} = \rho_t. \]

Equation [2.17] says that the expected capital gains portion of the return, \( E[t]_t[s_{t+1}] - s_t \), is equal to the total expected return minus the dividend rate.

ii. Normally distributed returns

In the random walk model asset returns are normally distributed, so asset prices are lognormally distributed. Charts 2.9 and 2.10 illustrate by showing the 1 standard deviation confidence interval of a random walk at a given point in time (Chart 2.9) and how the confidence interval widens in proportion to the square root of time (Chart 2.10).

The error in predicting next period’s asset return is assumed to be an identically and independently normally distributed (i.i.d. normal) random variable. In other words, the distribution of the error is the same normal distribution each period and each period’s error is independent of earlier errors.

The expected return is \( \rho_t \) and is given in Equation [2.17]. The realized return is

\[ s_{t+1} - s_t + d_{t,t+1}. \]

The error is the difference \( s_{t+1} - s_t + d_{t,t+1} - \rho_t \), which we denote by \( \varepsilon_t \) and assume to be normally distributed with the same mean of zero and standard deviation \( \sigma \) in each period \( t \). The time series of errors \( \varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots \) is called a Gaussian (because normally distributed) white noise process (because mean-zero and i.i.d.). We write this assumption \( \varepsilon_t \sim N(0, \sigma^2) \) and with it, Equation [2.17] becomes the forecasting equation

\textsuperscript{31} The expected value of the asset return is zero (or the forward premium), but the expected value of the asset price is its current cash (or forward) price minus the so-called Jensen’s inequality term \( 0.5 \sigma^2 t \), where \( \sigma \) is the random walk’s volatility. The difference arises from defining asset returns as the logarithmic rather than the percent difference in successive asset prices. We will ignore Jensen’s inequality terms in what follows.

\textsuperscript{32} In academic parlance, the dividend process is adapted or previsible.
Equation [2.18] says that the change in log future asset price is equal to the expected return minus the dividend rate, plus a normally distributed random fluctuation. We now also have a precise expression for the probability distribution of logarithmic changes in the asset price $s_{t+1} - s_t$:

$$s_{t+1} = \rho_t - d_{t+1} + s_t + \varepsilon_t.$$  

[2.18]  

**Chart 2.9**  
**Confidence interval for a random walk**  
*Probability density function of dollar-mark exchange rate with ±1 standard deviation bounds*  

**Chart 2.10**  
**Confidence interval for a random walk over time**  
*Six realizations of zero-drift random walk with ±1 standard deviation confidence interval*  

Note that logarithmic changes in the asset price are independently normally, but not identically, distributed, since the expected return and dividend rate can fluctuate.
We can also express Equation [2.18] in terms of the forward premium using the covered parity formula Equation [2.2], which in logarithmic form is

\[ f_{t,t+\tau} = \ln[1 + r_{t,t+\tau} - d_{t,t+1}] + s_t = s_t + r_{t,t+\tau} - d_{t,t+1}. \]

Substituting into Equation [2.18], we have

\[ s_{t+1} = \rho_s - r_{t,t+1} + f_{t,t+1} + \varepsilon_t. \]

Equation [2.21] states that the forecast of the log asset price is equal to the log forward price plus the risk premium \( \rho_s - r_{t,t+1} \).

**iii. Risk premium equals zero**

The third and last hypothesis is that the risk premium equals zero \( \rho_s = r_{t,t+1} \), so Equation [2.21] becomes

\[ s_{t+1} = f_{t,t+1} + \varepsilon_t. \]

Equation [2.22] states that the future asset price is equal to the current forward price of the asset, plus a white noise error term.

With the risk premium set to zero, the forward premium is the expected percent change in the asset price:

\[ E_t[s_{t+1}] - s_t = \mu_{t,t+1} = r_{t,t+1} - d_{t,t+1} = f_{t,t+1} - s_t \]

so we can write Equation [2.18] as a random walk with a time-varying drift equal to the forward premium:

\[ s_{t+1} = \mu_{t,t+1} + s_t + \varepsilon_t. \]

While the drift term \( \mu_{t,t+1} \) may be a random variable, it is known by time \( t \). Equation [2.24] implies that first differences in the logarithmic asset price are equal to the drift plus a Gaussian white noise process. Focusing on this property, the time series of these first differences is said by statisticians to possess a **unit root** or to be **integrated**.

The expected value of the forecast error is zero:

\[ E_t[s_{t+1} - s_t - \mu_{t,t+1}] = E_t[\varepsilon_t] = 0. \]

Summing up, in the RWF model,

- logarithmic changes in the asset price follow a random walk with a drift equal to the forward premium. The drift is random and time-varying, but known by time \( t \); and
- asset returns in excess of the risk-free rate \( s_{t+1} - s_t + d_{t,t+1} - r_{t,t+1} \), which are identical to forecast errors using the RWF model, follow a random walk with zero drift.

To arrive back at a forecast of the level of the future cash price, take the exponential of the logarithmic forecast:

\[ E_t[S_{t+1}] = S_t e^{\mu_{t,t+1}}. \]
Chapter 2. Forecasts based on current market prices

The variance of the forecast error is $\sigma^2$:

\[ \text{Var}[s_{t+1} - s_t - \mu_{t,t+1}] = E_t[(s_{t+1} - s_t - \mu_{t,t+1})^2] = E_t[\varepsilon_t^2] = \sigma^2, \]

which can be estimated by the average squared difference between the current log forward price and the one-period ahead log cash price it forecasts:

\[ \hat{\sigma}^2 = \frac{1}{T} \sum_{i=0}^{T-1} (f_{t-i-1,t-i} - s_{t-i})^2. \]

In markets in which forward price data are available only for the more recent past, one can instead use the estimator

\[ \hat{\sigma}^2 = \frac{1}{T} \sum_{i=0}^{T-1} (s_{t+1-i} - s_{t-i})^2, \]

the average squared difference between the current log cash price and the one-period ahead log cash price. There is unlikely to be a significant difference between the two estimators.

The discussion thus far has focused on one-period ahead forecasts. This method is applicable to any horizon, as long as forward prices of the desired horizon are observable. For example, if time is measured in years, then for the 2-month horizon, we estimate $s_{t+2}$ as $f_{t,t+2} = s_t + f_{t,t+2} - d_{t,t+2}$, and so on.

However, forward rates at many maturities are not available for many asset prices. Only the prices of frequently traded commodities as well as the foreign exchange rates, interest rates, and equity indices of some developed countries have rich sets of forward rates. If forward prices are unavailable for a desired forecast horizon lying between two available forward maturities, we can interpolate. Assume for example that the 1-month forward premium is $\mu_{t,t+1} = f_{t,t+1} - s_t = 0.04$ while the 3-month premium is $\mu_{t,t+3} = f_{t,t+3} - s_t = 0.05$. If we wish to calculate a 2-month ahead forecast, we can calculate it as $E_t[S_{t+2}] = S_t e^{0.045}$.

Alternatively, if data is quite limited, we can assume that the drift is constant and equal to, say, the 1-month forward premium, i.e., $\mu_{t,t+\tau} = f_{t,t+1} - s_t$ for all $\tau$. Then the forecast of the $\tau$-period-ahead log price $s_{t+\tau}$ is

\[ E_t[s_{t+\tau}] = s_t + \tau \mu_{t,t+1} = s_t + \tau (f_{t,t+1} - s_t). \]

The variance of the $\tau$-period-ahead forecast error is

\[ \text{Var}[s_{t+\tau} - s_t - \mu_{t,t+\tau}] = E_t[(s_{t+\tau} - s_t - \mu_{t,t+\tau})^2] = E_t[\varepsilon_t^2] = \tau \sigma^2, \]

and can be estimated by $\tau \hat{\sigma}^2$, with $\hat{\sigma}^2$ calculated from Equation [2.28] or [2.29].

We can simply scale the one-period variance by the forecast horizon $\tau$ because under the i.i.d. normal assumption underpinning the random walk forecasts, the sampling interval is immaterial. If we are unsure that the i.i.d. normal is a good approximation, we can take a sampling interval equal to $\tau$. If $\tau$ is long, however, for many assets there will not be enough historical data to estimate the variance accurately. For example, if $\tau$ is two years, we would need 40 years of data to have even
20 independent (i.e., non-overlapping) observations on the squared asset return. Alternatively, where available, we can use implied volatility to forecast the variance.

**Example 2.3**  
**U.S. dollar-Mexican peso exchange rate**  
Chart 2.11 displays the evolution of the spot exchange rate of the Mexican peso (pesos per U.S. dollar) together with two sets of monthly forecasts running over 24 months as of the end of September 1996.

**Chart 2.11**  
**Forward and spot forecasts of USD-MXP**  
*Forecasts as of end-September 1996. Confidence interval based on historical volatility.*

The two sets of forecasts are based on the random walk model with the drift rate set to zero (RWZ) for one set (thin lines in the graph) and with the drift rate set to the forward premium (RWF) for the other set (heavy lines in the graph). Each set of forecasts consists of the exchange rate path with the 90% confidence interval based on historical volatility over the previous five years (that is, the standard deviation of monthly returns September 1991 to September 1996). Any vertical slice through the cone on a given date between September 1996 and September 1998 will intersect the forward outright rate (mean forecast) for that date, the 90% confidence interval (5% and 95% quantiles) around the forward rate, the September 1996 spot rate and the 90% confidence interval around the September 1996 spot rate for the given date, and the realized future spot rate on that date.

The reader may find details of the forecast calculations useful. For the 1-month horizon, for instance, the drift rate in Equation [2.24] is estimated as \(\mu_{t+1} = -0.0172\), since the 1-month forward discount of the peso against the dollar was 1.72%. The forecasted level of the exchange rate at the end of October 1996 is then

\[
E_t[S_{t+1}] = 7.535 e^{-0.0172},
\]

since the spot rate at end-September 1996 was MXP 7.535.

Forward rate data was available for the 1-, 2-, 3-, 6-, 9-, and 12-month maturities. The forward rate for monthly forecast horizons under 1 year for which no forward rate was observed (4, 5, 7, 8, 10, and 11 months) were interpolated as follows. The observable forward premiums were calculated either from the forward points or from the forward outright rates, and the missing horizons linearly interpolated. All the forward outright rates were then calculated by taking the exponents, as in Equation [2.32]. The monthly forecasts for the 13- to 24-month horizons were estimated by
applying the 12-month forward premium, so that the 13-month forward premium equals $\frac{13}{12}$ times the 12-month forward premium, and so on.

The annualized historical volatility of dollar-peso was 23% at the end of September 1996. Multiplying by the square root of the forecast horizon gives the appropriate volatility $\hat{\sigma} \frac{\sqrt{\tau}}{\sqrt{12}}$ for that horizon. Thus, for example, the 1-month volatility is $\hat{\sigma} \frac{0.23}{\sqrt{12}} = 0.066$.

To calculate the bounds of the 90% confidence interval for a forecast horizon of $\tau$ months, we first calculate the 5th and 95th percentiles of the normal distribution with mean $\mu_{t, t + \tau}$ and standard deviation $\hat{\sigma} \frac{\sqrt{\tau}}{\sqrt{12}}$. Let us denote these by $z_{0.05}(\tau)$ and $z_{0.95}(\tau)$. The confidence interval is then $[S_t e^{\mu_{t, t + \tau} + z_{0.05}(\tau)}, S_t e^{\mu_{t, t + \tau} + z_{0.95}(\tau)}]$.

The peso has depreciated rapidly since the collapse of the adjustable exchange rate peg in December 1994. Forward peso rates have exhibited a large forward discount of the peso vis-à-vis the dollar, reflecting the general tendency toward depreciation and capturing the behavior of the peso more accurately than simply extrapolating the spot rate into the future. This can be seen in Chart 2.11 for peso forecasts made on September 1996: the path of forecasted rates using the forward rate captures the upward trend of the exchange rate more accurately than extrapolating the September 1996 spot rate out.

Example 2.4
U.S. dollar-German mark exchange rate

Chart 2.12 displays the spot exchange rate of the German mark (marks per U.S. dollar) together with monthly forecasts, as of the end of May 1994 and July 1996, based on the random walk model with zero drift (thin lines in the graph) and with the drift rate set to the forward premium (heavy lines in the graph). The example is set up similarly to Example 2.3 and Chart 2.11.

The main difference between the peso and mark examples is that dollar-mark lacks the pronounced trend of the peso over the past decade. The forward premium of the mark is therefore considerably smaller in magnitude than that of the peso. The forward rate, in such cases, is less likely to outperform the spot rate as a forecast of the future spot rate.

In forecasting fixed-income asset prices, we must be careful to keep in mind whether we are working with interest rates or asset prices. For most interbank fixed-income assets, interest rates rather than prices are quoted. It is preferable to treat the quoted interest rate as a lognormally distributed asset price rather than a normally distributed return in order to exclude the possibility of negative interest rates. The standard deviation of the quoted rate is then a yield volatility.

The next example pertains to Japanese interest rates and illustrates the motivation for treating interest rates as lognormally rather than normally distributed.

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33 An equivalent way to calculate the confidence interval would be to find the 5th and 95th percentiles of the lognormal variate with parameters equal to the mean and standard deviation of the normal distribution governing the return.
Chart 2.12
Forward and spot forecasts of USD-DEM
Forecasts as of end-May 1994 and end-July 1996. Confidence interval based on historical volatility.

Chart 2.13
Forward and spot forecasts of Japanese yen Libor rates

Example 2.5
Short-term Japanese yen interest rates
Chart 2.13 displays 3-month Japanese yen Libor rates together with three sets of monthly forecasts over 1 year, formulated at end-February 1995, end-April 1996 and end-June 1997, based on the random walk model with zero drift (thin lines in the graph) and with the drift rate set to the forward premium (heavy lines in the graph).

The example is set up similarly to Examples 2.3 and 2.4, except that the forecasts reach out only 1 year rather than 2. The forward rate forecasts are set equal to FRA rates on the dates the forecasts were made. The standard deviations are the historical yield volatilities of the Libor rate.
Several features of Chart 2.13 are striking. The first is the extraordinarily low level to which Japanese money-market rates have fallen since January 1995. The proximity to zero of Japanese rates mandates the assumption that the rates are lognormally distributed, since a lognormally distributed random variable is always positive. If the rates were assumed to be normally rather than lognormally distributed, a high probability would be assigned to the outcome of significantly negative interest rates, an outcome we generally consider impossible.34

The second, and related, striking feature is the extreme width of the confidence intervals surrounding the forward-rate forecasts. The reason is, again, the extremely low level of Japanese rates. For a given variance parameter, a lognormally distributed variate that is far from zero looks much like a normally distributed one, but as its value falls, it appears skewed as it is “squeezed” more and more against the zero “wall.” Japanese rates are so low that the upper limit of the 90% confidence interval is much farther away from the mean of the distribution—the forward rate—than the lower limit.

Finally, Chart 2.13 shows that the markets did not anticipate the abrupt decline in Japanese rates. At all forecast dates, the term structure of spot Japanese yen interest rates has been upward-sloping, leading to rising 3-month rates along the forecast paths. At the end of April 1996, in fact, the forecast path slopes upward even more steeply than in early 1995, indicating an expectation that rates would return to more historically typical levels, or at least a desire by market participants to protect against that possibility. Only when rates have actually persisted below 1% for some years, as at the end of July 1997, does the expectation of a rebound begin to dissipate.

**B VOLATILITY FORECASTS FROM OPTION PRICES**

In the previous section, we used current market data to generate estimates of the mean or expected future asset price, while relying on statistical methods and historical price data to estimate the variance of the future asset price. In this section, we will show how current market prices of options can be used in place of historical price data to estimate variances.

We continue Examples 2.3 and 2.4 above of the exchange rate of the Mexican peso and the German mark vis-à-vis the U.S. dollar. Our estimate of the standard deviation is now drawn from option implied volatilities. As with the forwards, we observe implied volatility quotes for options with maturities of 1, 2, 3, 6, 9, and 12 months.35 We apply straight-line interpolation between the 3- and 6-month, 6- and 9-month, and 9- and 12-month vols, apply the 12-month to all horizons between 13 and 24 months, and multiply the vol for each maturity \( \tau \) by the scale factor \( \frac{\tau}{\sqrt{12}} \).

We then match each interpolated and scaled vol to the \( \tau \)-month forward rate with the same maturity, giving us the two parameters of the normal distribution of the logarithm of the spot rate in \( \tau \) months. This creates a confidence interval “cone” similar to that based on historical volatility.

Implied volatility generally has a term structure, so we can vary the width of the cone at each horizon to reflect not only the time scaling of volatility, but the market-assessed level of volatility for that term. The term structure of implied volatility is evidence that the volatility of returns may vary for different return horizons. When we use historical data to estimate the one-period variance and scale up to longer horizons, we ignore this possibility. It may be impractical to estimate longer-term

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34 As noted in footnote 21 above, nominal interest rates may become slightly negative, since the substitute for a bank deposit, storing cash in the form of banknotes in a vault, is costly and inconvenient.

35 The raw data are in the form of bid and offered vols in percent, so we take the midpoint and divide by 100 to get vols as decimals.
return variances directly because insufficient historical return data is available. Implied volatility estimates of future volatility are easy to calculate and are a practical complement to historical volatility estimates.

Example 2.6 (continued from Example 2.3)

**U.S. dollar-Mexican peso exchange rate**

Chart 2.14 displays the end-September 1996 forecast based on forward peso rates with two 90% confidence intervals. The heavy lines show the option-based confidence interval and the thin lines show the confidence interval based on historical volatility. For both confidence interval estimates, the expected value forecast is set equal to the forward outright rate. A vertical slice through the cone at a given date will provide the forward rate (mean forecast), the 90% confidence interval (5% and 95% quantiles) based on implied volatility for that date, the 90% confidence interval based on historical volatility for that date, and the realized future spot rate.

Historical volatility was higher and gave rise to considerably wider confidence intervals than implied volatility at the end of September 1996. This is not always the case; implied volatility is frequently higher than historical, as can be seen from Chart 2.15, which plots the 1-month confidence interval based on implied volatility (heavy lines), the 1-month confidence interval based on historical volatility (thin solid lines), and the spot rate observed 1 month later (dashed line). (The mean forecast, which is not plotted, is exactly midway between the two bounds of the confidence intervals.) The chart displays several occasions on which the confidence interval based on implied volatility was wider than that based on historical volatility.

Some notion of the performance of these forecasts can also be gained from Chart 2.15. Under the Black-Scholes model assumptions which underpin this technique, one should observe a realized spot rate in excess of the 95th percentile or below the 5th percentile about 10% of the time. In the case of dollar-peso over the 39 months from July 1995 to September 1998, we observe two such occasions, which is below the expected value of 4, but within acceptable statistical bounds. There are no exceptions outside the historical volatility confidence interval.\(^{36}\)

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\(^{36}\) The 90% confidence interval for the exceptions is [1, 7]. The mean and confidence interval for the exceptions are based on the binomial distribution with 39 trials and a success probability of 10%.
**Chart 2.15**

Option-implied confidence interval of dollar-peso

As shown in Chart 2.17, in the case of dollar-mark over the 44 months from February 1995 to September 1998, we observe three excessions outside the implied volatility confidence interval, well within the 90% confidence interval [1,8]. There is only one excession outside the historical volatility confidence interval.

**Example 2.7 (continued from Example 2.4)**

**U.S. dollar-German mark exchange rate**

Chart 2.16 displays the end-May 1994 and end-July 1996 forecasts based on forward dollar-mark rates with two 90% confidence intervals. As in Chart 2.14, the heavy lines show the option-based confidence interval and the thin lines show the confidence interval based on historical volatility. Each vertical slice through one of the cones will provide the forward rate (mean forecast), the 90% confidence intervals based on implied and historical volatility, and the realized spot rate.
The reader may find useful a summary of the three forecast techniques based on current market data used in *LongRun*:

- **mean asset price equal to spot rate (mean return equal to zero)**
- **mean asset price equal to forward rate (mean return equal to forward premium)**
- **standard deviation equal to historical volatility** - random walk with zero expected return (RWZ)
- **standard deviation equal to implied volatility** - random walk with expected return equal to forward premium (RWFH)
- **random walk with expected return equal to forward premium (RWF)**

### C IMPLIED PROBABILITY DISTRIBUTIONS

A useful extension of *LongRun*’s forecasting technique is to calculate implied probability distributions of future asset prices and rates. Implied probability distributions enable market analysts to ascribe probabilities to a wide variety of market events and to easily visualize the forecasts implied by current market data. Like forecasts of asset price means and confidence intervals using current market data, implied probability distributions are risk neutral, meaning that the probabilities are compensated for risk.

We can estimate an implied probability distribution for a given time horizon based on the Black-Scholes model, that is, assuming asset returns are normally distributed. The data are the asset’s forward price, representing the estimated mean, and implied volatility, representing the estimated standard deviation. Some examples will make the technique clear.

**Example 2.8**

**Probability distribution of dollar-yen**

Suppose the dollar is currently trading at ¥145 spot, the 1-month forward outright rate is ¥144.35, and 1-month implied volatility is 14% per annum. What is the probability that dollar-yen will end higher than ¥150 in 1 month? We can answer this question under the assumption that the Black-Scholes model holds, that is, 1-month dollar-yen returns are normally distributed.
To do the calculations, we convert all the market data to a 1-month return basis. For the forward rate, that means converting the swap rate of 144.35 − 145 = −65 points into a forward discount of the dollar of \( \frac{65}{145} = 0.45 \% \) per month (5.25 per year). For the implied volatility, that means scaling down the standard deviation of dollar-yen returns over the next month at an annual rate into a standard deviation of dollar-yen returns over the next month at a monthly rate by the square root of time: \( \frac{13.5}{\sqrt{12}} = 0.039 \) or 3.9%. Thus our market data and maintained hypothesis about the exchange rate’s return distribution imply that 1-month dollar-mark returns are normally distributed with a mean of −0.45% and a standard deviation of 3.9%.

Chart 2.18 depicts this distribution. (Note that the mean of the distribution, the forward premium, is slightly smaller than zero.) To hit 150, the spot rate would have to move \( \frac{150 - 144.35}{144.35} = 3.9 \% \) higher than the current forward rate. That is one standard deviation away from the mean. According to the normal distribution tables, the likelihood of a realization one standard deviation or more above the mean is 15.8%. That corresponds to the area under the normal curve to the right of the grid line in Chart 2.18.

**Example 2.9**

**Probability distribution of swap rates**

Suppose the 10-year deutsche mark swap rate is currently 4.75%. What is the probability that it will rise to 5.0% or more in a year? We need two more pieces of market data: the current rate on a 1-year by 10-year forward swap, which we will assume is also 4.75%, and the implied yield volatility on a 1-year by 10-year swaption, which we will assume to be 11.5%. Percent changes in the deutsche mark swap rate vis-à-vis the current forward rate are then normally distributed with a mean of zero and a standard deviation of 11.5%.

A rise of 25 basis points from 4.75% is an increase of about 5.25% in the rate. That is one-half of a standard deviation. The probability of the swap rate ending 0.5 standard deviations above the mean or higher is 31%. Thus there is a 31% likelihood that the swap rate will be higher than 5% in 1 year.
Note something interesting about Examples 2.8 and 2.9: the cash rate was not central to the calculations. In the currency example, we used the spot rate to calculate the forward premium and express the distribution in terms of exchange rate returns rather than levels. In the swaption example, the spot swap rate was entirely dispensable, since we assumed that the swap par rate itself is the lognormally distributed random variable.

It is important to understand that we are estimating the distribution of the asset price (or return) at the end of the horizon, rather than the distribution of the asset price at some intermediate point. In the interest rate example, for instance, we estimated the likelihood the 10-year swap rate would be 5.0% or higher 1 year from today, not at any intermediate point during the year.

2.2 Forecasts of extreme moves in asset prices

Up until now, we have used current market data as forecast proxies under the assumption that the market believes that cash or underlying asset returns follow a random walk, that is, are independently and identically normally distributed (i.i.d. normal). While this is a perfectly serviceable first approximation to the actual behavior of asset prices, it is only an approximation. Even though most widely-traded cash asset returns are close to i.i.d. normal, they display small but important “non-normalities.” In particular, the frequency and direction of large moves in asset prices, which are very important in risk management, can be quite different in real-life markets than the i.i.d. normal model predicts. Moreover, a few cash assets behave very differently from i.i.d. normal.

Option markets contain much information about market perceptions that asset returns are not i.i.d. normal. We focused in Section 2.1 on volatility forecasts based on implied volatilities of at-the-money options, that is, options whose exercise prices are equal to or nearly equal to the current cash price of the asset. Many options are in- or out-of-the-money: their exercise prices are above or below the current cash or forward asset price. The relationship between their prices and those of at-the-money option contains a great deal of information about the market perception of the likelihood of large changes, or changes in a particular direction, in the cash price.

2.2.1 Statistical behavior of asset prices

The random walk hypothesis on which the Black-Scholes model is based is a good first approximation to the behavior of most asset prices most of the time. However, even nominal asset returns, which are quite close to normally distributed, display small but important deviations from normality. The option price patterns discussed in Section 2.2.2 below reveal how market participants perceive the distribution of future asset prices. Empirical studies of the stochastic properties of nominal returns focus on the behavior of realized asset prices. The two approaches largely agree.

i. Kurtosis

The kurtosis or leptokurtosis (literally, “fat tails”) of a distribution is a measure of the frequency of large positive or negative asset returns. Specifically, it measures the frequency of large squared deviations from the mean. The distribution of asset returns will show high kurtosis if asset returns which are far above or below the mean occur relatively often, regardless of whether they are mostly above, mostly below, or both above and below the mean return.

Kurtosis is measured in comparison with the normal distribution, which has a coefficient of kurtosis of exactly 3.0. If the kurtosis of an asset return distribution is significantly higher than 3,

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37 We stress cash asset returns because option returns are not normally distributed even if the returns on the underlying asset are normal.

it indicates that large-magnitude returns occur more frequently than in a normal distribution. In other words, a coefficient of kurtosis well over 3.0 is inconsistent with the assumption that returns are i.i.d. normal. A kurtotic distribution is compared with a normal distribution with the same variance in Chart 2.19.

Chart 2.19
A normal and a kurtotic probability distribution
Distribution of 1-month returns vis-à-vis forward price

ii. Skewness
The skewness of a distribution is a measure of the frequency with which large returns in a particular direction occur. An asset that displays large negative returns more frequently than large positive returns is said to have a return distribution skewed to the left or to have a “fat left tail.” An asset that displays large positive returns more frequently than large negative returns is said to have a return distribution skewed to the right or to have a “fat right tail.” The normal distribution is symmetrical, that is, its coefficient of skewness is exactly zero. Thus a significantly positive or negative skewness coefficient is inconsistent with the assumption that returns are i.i.d. normal. A skewed distribution is compared with a normal distribution with the same variance in Chart 2.20.

Table 2.1 presents estimates of the kurtosis and skewness of some widely-traded assets. All the assets displayed have significant positive or negative skewness, and most also have a coefficient of kurtosis significantly greater than 3.0.

The exchange rates of the Mexican peso and Thai baht vis-à-vis the dollar have the largest coefficients of kurtosis. They are examples of intermittently fixed exchange rates, which are kept within very narrow fluctuation limits by the monetary authorities. Typically, fixed exchange rates are a temporary phenomenon, lasting decades in rare cases, but only a few years in most. When a fixed exchange rate can no longer be sustained, the rate is either adjusted to a new fixed level (for example, the European Monetary System in the 1980s and 1990s and the Bretton Woods system until 1971) or permitted to “float,” that is, find a free-market price (for example, most emerging market currencies). In either case, the return pattern of the currency is one of extremely low returns during the fixed-rate period and extremely large positive or negative returns when the fixed rate is abandoned, leading to extremely high kurtosis.
The return patterns of intermittently pegged exchange rates also diminishes the forecasting power of forward exchange rates for these currencies, a phenomenon known as regime-switching or the peso problem. The term “peso problem” has its origin in experience with spot and forward rates on the Mexican peso in the 1970’s. Observers were puzzled by the fact that forward rates for years “predicted” a significant short-term depreciation of the peso vis-à-vis the U.S. dollar, although the peso-dollar exchange rate was fixed. One proposed solution was that the exchange rate peg was not perfectly credible, so that market participants expected a switch to a new, lower value of the peso with a positive probability. In the event, the peso has in fact been periodically permitted to float, invariably depreciating sharply.

The peso problem and the statistical methods which have been developed to identify peso problems are intended to address economic crises, particularly speculative attacks on pegged currencies and interest rates, and their effects on asset prices. Regime switching, in contrast, refers to shifts in policy which, while not responses to immediate crises, also have important effects on asset prices.

To the extent regime shifts can be predicted, they affect asset prices today. We described a dramatic example of regime switching—the rapid and sustained decline in Japanese interest rates—in Example 2.5. Regime switching is the more relevant concept for asset prices that display infrequent turning points or changes in trend, such as the exchange rates and interest rates of industrialized countries, rather than a succession of extreme moves at random times.

iii. Autocorrelation of returns

The distribution of many asset returns is not only kurtotic and skewed. The return distribution may also change over time and successive returns may not be independent of one another. These phenomena will be reflected in the serial correlation or autocorrelation of returns. Table 2.1 displays evidence that asset returns are not typically independently and identically distributed. The right-most column displays a statistic that measures the likelihood that there is serial corre-

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40 See, for example, Hamilton (1988) and Engel and Hamilton (1990). Chapter 3 of this document contains an extended discussion of regime switching, which also affects econometric forecasts of asset prices.
lation between returns on a given day and returns on the same asset during the prior five trading
days. High values of this statistic indicate a high likelihood that returns are autocorrelated.

2.2.2 The volatility smile in option markets

Under the Black-Scholes model and in the forecasting methodology of Section 2.1, volatility is a
constant. The Black-Scholes model predicts that all options on the same asset have identical im-
plied volatilities, regardless of time to maturity and moneyness. In Section 2.1.1, we discussed the
term structure of implied volatility, a so-called pricing bias or discrepancy between the actual pat-
tern of market option prices and the Black-Scholes model predictions. This section discusses two
features of implied volatility that pertain to options with different exercise prices but with the same
maturity:

• Out-of-the-money options often have higher implied volatilities than at-the-money options,
indicating that the market perceives the return distribution to be kurtotic, (that is, large asset
price moves in either direction are likelier than is consistent with the Black-Scholes model),
or that market participants are willing to pay extra for protection against sharp price moves.

This phenomenon is known as the volatility smile because the plot of implied volatilities of
options with a given maturity against the different exercise prices is curved. Typically, the
lowest implied volatility is that of the at-the-money option. In the Black-Scholes model, the
plot is a horizontal line at the level of the unique implied volatility of all options on that asset.

• Out-of-the-money call options often have implied volatilities that differ from those of
equally out-of-the-money puts, indicating that the market perceives the return distribution to
be skewed or that market participants are willing to pay more for protection against sharp
asset price moves in one direction than in the other. When this is the case, the volatility
smile is skewed rather than symmetrical about the current spot or forward price, and is occa-
sionally referred to as the volatility “smirk.”

The volatility smile generally displays both curvature and skewness, as in the case of the U.S. Treas-
sury bond implied volatilities in Chart 2.21.
Chart 2.21
Volatility smile for options on U.S. Treasury bond futures
Implied volatilities in percent per annum. Diamonds indicate observed prices.

The implied volatilities of puts and calls with the same exercise price are identical, so Chart 2.21 can be viewed as either a plot of call volatilities or put volatilities. Either way, it shows that out-of-the-money calls, which have exercise prices above the futures price, have higher implied volatilities than out-of-the-money puts, which have exercise prices below the futures price.

In this example, options that pay off if there is a particularly large move in Treasury bond prices have higher implied volatilities than options which pay off for smaller moves. Also, options that pay off if prices rise by a given amount—that is, interest rates fall—over the subsequent life of the options are more highly valued than options that pay off if prices fall by the same amount. That could be due to a strong market view that rates are more likely to fall than to rise or to market participants’ desire to protect against losses from falling rates. At the time the options displayed in Chart 2.21 traded, U.S. government debt prices were strong in consequence of the Russian debt crisis, so market participants may also have bid up out-of-the-money call option prices to protect against losses in markets correlated with U.S. interest rates.

Occasionally, the skewness in the volatility smile is so strong as to dominate the appearance of the graph, as in Chart 2.22, depicting the volatility smile for Italian government bonds.

Chart 2.22 depicts a situation in which implied volatility falls monotonically as the exercise price rises. In other words, options that have large payoffs if prices fall (long-term Italian interest rates rise) trade at a premium over options that have large payoffs if prices rise by the same amount. That could be due to a strong market view that rates are more likely to rise than to fall or to market participants seeking to protect against losses from falling prices.

41 Due to put-call parity.
Different asset classes have different characteristic volatility smiles. In some markets, the typical pattern is highly persistent. For example, the negatively sloping smile for options on S&P 500 futures illustrated in Chart 2.23 has been a virtually permanent feature of U.S. equity index options since October 1987, reflecting market eagerness to protect against a sharp decline in U.S. equity prices. Out-of-the-money calls have much lower implied volatilities relative to out-of-the-money puts than would be the case if the market believed equity returns over the option maturity were normally distributed. In contrast, the volatility smiles of many currency pairs such as dollar-mark have been skewed, depending on market conditions, against either the mark or the dollar.  

The volatility smile may differ for options with different maturities. The volatility surface plots the volatility smile against the option maturity, capturing the smile and the term structure of implied volatility in one graph:

**Chart 2.24**

**Volatility surface**

Dollar-yen, option maturities from 1 month to 1 year, as of March 1, 1999.

Annualized implied volatilities in decimal format. Option maturity indicated by $\tau$.

The volatility smile is evidence not only that the Black-Scholes model does not hold exactly, but that market participants are perfectly aware of this. The market often believes that large asset price moves or large moves in a particular direction are likelier to occur than would be the case if asset returns followed a random walk. The market nonetheless contentedly continues using the terminology of the Black-Scholes model, and raises or lowers the implied volatility or prices of options with particular exercise prices to equate their supply and demand.

### 2.2.3 Interpreting the volatility smile

The departures from the i.i.d. normal return model, which the behavior of asset prices as well as option pricing patterns exhibit, have motivated a search for alternative asset return models. We will discuss here two widely-adopted alternatives, the jump-diffusion and stochastic volatility models. These models are used to explain deviations from the random walk model and to develop more accurate option pricing techniques.

#### A THE JUMP-DIFFUSION MODEL

We saw in Section 2.2.1 that intermittently pegged exchange rates display high kurtosis, because their return history is composed of low returns when the peg is in place and high returns on days when the peg has been abandoned. In the past, other asset prices, such as U.S. government bond rates prior to the 1951 Federal Reserve-Treasury Accord and the price of gold between 1933 and the breakdown of the Bretton Woods system, have also been fixed. The range of direct application today is narrow. While few exchange rates are fully freely floating, very few are officially fixed.
Some examples which survived into this decade include the old European Monetary System with its narrow $\pm2.25$ fluctuation limits, abandoned in August 1993, Mexican peso’s crawling peg, which collapsed in December 1994, and the currency pegs in southeast Asia which fell in the summer of 1997.

Other asset returns display analogous peso problems or regime-switching patterns even when their prices are not formally fixed. For example, stock prices experience rare daily fluctuations of 10% or more. A stock index may be lower than it would otherwise be because the market sees the possibility of a crash. Bond prices often move sharply in response to currency crises and when the stance of monetary policy changes. The term structure of interest rates may be unusually positively-sloped because there is a significant likelihood that the monetary authorities will tighten monetary policy.

When regime switching is relevant, the asset price can be thought of as a weighted average of two values, one in the event of a switch to a new regime (say, devaluation, major creditor default, or monetary tightening) and one if the current regime continues. The first value is weighted by the probability $\pi$ of a regime switch and the second by the probability $1-\pi$ of no switch:

$$E_t[s_{t+1}] = \pi E_t[s_{t+1} | \text{switch}] + (1-\pi)E_t[s_{t+1} | \text{no switch}].$$

For example, if the market believes there is a 1% chance that the S&P500 average could abruptly fall by 20% at any time, then the S&P500 average should be 0.2% lower today than it would be if no abrupt fall were thought possible.

One way of modelling asset returns in a peso problem environment, in which the distribution of returns does not change after an extreme price move, is the jump-diffusion model.\(^{43}\) Earlier (see Section 2.1.1), we referred to the random walk model as a stochastic process, a sequence of random variables associated with dates. The jump-diffusion model builds on the random walk model by introducing a second type of stochastic process called a jump process. In contrast to the random walk, in which the asset price is extremely unlikely to move very far in a short time period, in a jump process the asset price level may rise or fall abruptly (a sharp asset price decline is also a “jump”). The timing of the jumps, which we can associate with regime switches, is random.

A jump process has two aspects: the probability distribution of the jump frequency, that is, the random “instants” at which jumps occur, and the probability distribution of the jump size.

- One widely adopted approach to the jump frequency is to assume that the number of jumps occurring in a specified time interval follows a Poisson distribution. The Poisson distribution is used to model rare but recurrent phenomena in the physical world such as radioactive decay. In its application to asset price behavior, the Poisson distribution parameter $\lambda$ is the expected frequency with which a discontinuous jump occurs and $q_t$ is the number of jumps that have occurred up to time $t$. For example, if $\lambda = 0.5$ and time is measured in years, then one would expect on average to witness an extreme move in an asset price following a Poisson distribution once in 2 years, that is, $\lambda$ is the expected value of $q_{t+1} - q_t$. One might observe zero sharp moves, or two, in a given year, but one would be very surprised to see five in a year.

- The jump size can be modelled as a normally distributed logarithmic change in the asset price with a mean $k$ and a standard deviation $\sigma_k$. In other words, if a jump occurs, the logarithm of the asset price will on average rise by $k$ percent (or fall, if $k$ is negative), but the exact jump size is not known with certainty. We can simplify somewhat without losing any

essential insights by assuming the jump size to be a constant $k$, so that if a jump occurs, the asset price moves instantaneously by $k$ percent.

The jump-diffusion approach models asset returns as a sum of random walk (the diffusion) and jump components. Recall that in Equation [2.24], which we repeat here for convenience, we wrote the random walk model as

$$s_{t+1} = \mu_{t,t+1} + s_t + \epsilon_t.$$  \[2.34\]

The jump-diffusion model can be represented as

$$s_{t+1} = \mu_{t,t+1} + s_t + \epsilon_t + k(q_{t+1} - q_t),$$  \[2.35\]

where $q_{t+1} - q_t$ is the number of jumps occurring from time $t$ to time $t + 1$.\(^\text{44}\) In Section 2.2.4 below, we will present a method of estimating the parameters of the jump-diffusion model.

Chart 2.25 displays sample paths of an asset price following the jump-diffusion model of Equation [2.35]. The random walk component is assumed, as in Chart 2.3, to have no drift term and an annual volatility of 12%. The jump process is assumed to have an expected annual frequency of one and an expected jump size of 10%. In other words, the asset price is expected to follow a random walk, but on average once per year it is expected to rise discontinuously by 10%. The occasional jumps are clearly visible.

**Chart 2.25**

**Realizations of a jump diffusion over a year**

*Expected jump frequency 1 per annum, jump size 10%, diffusion volatility 12%*

The contrast between the random walk and the jump-diffusion models is easier to see if we focus on returns. Chart 2.26 displays returns on an asset following a random walk with zero drift and an annual volatility of 12% over 1000 consecutive trading days (a bit less than four years). The 99% confidence interval is indicated by horizontal lines. The returns are distributed normally, so about the right number of outliers (one expects five) are to be found above the 99.5% and below the 0.5% quantiles.

\(^{44}\) We assume that the random walk volatility does not change if a jump occurs. More general jump-diffusion models loosen this assumption.
Returns on an asset following a random walk with 99% confidence interval
Sample of 1000 realizations, returns in percent, expected return 0% p.a., volatility 12% p.a.

Returns from a jump-diffusion look markedly different, as shown by the sample in Chart 2.27. In this case, all the outliers from the 99% confidence interval are positive, revealing the skewness of the distribution. They are also extremely large: returns from a jump-diffusion display high kurtosis. This helps explain the high kurtosis seen for the peso and baht in Table 2.1.

Returns on an asset following a jump-diffusion with 99% confidence interval
Sample of 1000 realizations, expected return 0% p.a., volatility 12% p.a.

In the model shown in Charts 2.25 and 2.27, the jump size $k$ was fixed at 10%. If the jump size is random and if jumps in either direction are equally likely, then jump-diffusion returns will be kurtotic, that is, the frequency of large changes will be high, but not necessarily skewed.

B THE STOCHASTIC VOLATILITY MODEL

In the jump-diffusion model, the mean of the asset return can abruptly take on a new value. Many asset returns exhibit volatility “clustering,” that is, large-magnitude returns often occur one after the other. This return pattern suggests that the volatility rather than the mean of the asset return is
changing. A time-varying return volatility can be represented by a stochastic volatility or GARCH model.45

Chart 2.28 illustrates one version of stochastic volatility, in which volatility switches at random times between low-volatility and high-volatility states. Returns from the stochastic volatility model display too many outliers from the 99% confidence interval to be normally distributed. Also, the return variance is not constant, but varies over the sample, with distinct high- and low-variance phases in succession. The return pattern of the random walk illustrated in Chart 2.26, in contrast, is uniform throughout the 4-year observation interval, since the returns are independently and identically distributed.

Chart 2.28

**Returns on an asset with stochastic volatility**

*Sample of 1000 realizations, expected return 0% p.a., volatility 12% p.a.*

A related phenomenon, which occasionally can have a strong influence on the level of implied volatility and on the shape of the volatility smile, is an actual or perceived correlation between the level of the asset price and the level of implied volatility. For example, the implied volatility of an equity index may tend to rise when the index level falls, or the implied volatility of an exchange rate may rise when the rate approaches the edges of its recent trading range.

Researchers have attempted to fit the jump-diffusion, stochastic volatility, and other alternative models to the random walk to actual asset returns, with varying degrees of success. In the end, these models, like the random walk, are simplified views of actual asset price behavior. They can be practically useful in helping to construct better option pricing models and in helping us to understand asset behavior intuitively.

### 2.2.4 Implied probability distributions using the volatility smile

In a relatively new area of research, methods have been devised to capture the information about skewness and kurtosis in the expected distribution of future asset prices from prices of options on the asset. As explained in Appendix 2.B, if we had sufficient data—options on the same asset and with the same maturity but with a continuous, or at least closely spaced, series of exercise prices—

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45 See, for example, Hull and White (1987), Stein and Stein (1991), Ball (1993), and Dupire (1993) and the papers collected in Engle (1995).
we could trace the entire implied risk neutral density by taking second differences of the option price with respect to the exercise price.

In practice, not many options on a given asset with a given maturity and different exercise prices trade in the marketplace at any one time. The sparseness of the available option data often makes the estimation of risk neutral distributions difficult:

- For most options, even those traded on organized exchanges, the few options which trade actively are all more or less at-the-money, so that very little of the price axis is covered and thus only a small portion of the risk neutral distribution can be recovered. Options which are only modestly out-of-the-money do not lend themselves to estimating skewness and kurtosis, which reveal themselves most clearly in the tails of the distribution.

- The exchanges generally provide closing prices for all options in which trading is open, even rarely traded far out-of-the-money exercise prices. However, these prices are not quotes, but indications. They are also rounded off to the nearest tick, and for option prices close to zero, the rounding error can be a significant fraction of the price. As a result, official closing prices are often ill-conditioned in that they admit arbitrage (of the first type). For example, two far out-of-the-money options with different exercise prices may be assigned the same low option price. This leads to nonsensical probability distributions, for example negative probabilities of some ranges of asset prices.

One solution to the “shortage” of option data adds structure by assuming that the risk neutral probability distribution of asset returns belongs to a particular parametric family other than the normal, for example a jump-diffusion or stochastic volatility. The distributional assumption leads to option pricing formulas in which the parameters of the distribution appear. The parameters can be estimated by fitting the option data to the pricing formulas.46

An alternative solution to the sparseness of option data is to interpolate between observed option prices.47 In the remainder of this section, we will explain both approaches and provide sample estimates.

### A. A Technique Based on Estimated Jump Diffusion Parameters

As an example of how to derive the risk neutral probability distribution from a particular distributional hypothesis, consider a simplified version of the jump-diffusion model of Section 2.2.3. We assume that only zero or one jump, but not many jumps, can take place during a time interval $\tau$. The number of jumps $q_{t+\tau} - q_t$ during the time interval have a simple probability distribution:48

$$q_{t+\tau} - q_t = \begin{cases} 1 & \text{with probability $\lambda \tau$} \\ 0 & \text{with probability $1 - \lambda \tau$} \end{cases},$$

where $\lambda$ is the jump frequency parameter. We continue to assume that $k$ is fixed rather than random.

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47 Examples of the interpolation approach include Shimko (1993) and Malz (1997a and 1997b).

48 The number of jumps over the interval are said to be Bernoulli distributed with parameter $\lambda \tau$. See Ball and Torous (1983).
There is a well-established formula for the value of a call option when the asset price follows the model in Equation [2.35]. In the simple case of zero or one jump, the \( \tau \)-period call pricing formula is a weighted average of two Black-Scholes values:

\[
[2.37] \quad v^{JD}(S_0, \tau, X, \sigma, \lambda, k) = \lambda \tau \cdot v\left(\frac{S_0 (1 + k)}{1 + \lambda k \tau}, \tau, X, \sigma\right) + (1 - \lambda \tau) \cdot v\left(\frac{S_0}{1 + \lambda k \tau}, \tau, X, \sigma\right),
\]

where \( v(S_0, \tau, X, \sigma) \) is the Black-Scholes value of a call option with an exercise price \( X \) when the cash price is \( S_0 \) and the random walk volatility is \( \sigma \).

The unobserved parameters \( \sigma, \lambda, \) and \( k \) can be estimated via non-linear least squares from observed option prices with the same maturity and different exercise prices. In this simplified version of the jump-diffusion, the asset price’s risk neutral probability distribution is a mixture of lognormals, that is, a weighted sum of (in this case) two lognormal variates. Given estimates of the parameters \( \sigma, \lambda, \) and \( k \), we can calculate the distribution.

The product \( \lambda k \) is the expected value of a jump. It can be interpreted as the percent amount by which the current level of the asset price is higher (if \( k > 0 \)) or lower (if \( k < 0 \)) because a jump is anticipated.

**Example 2.10**

**Probability distribution of S&P index futures**

As an example, let us apply the technique to the S&P index futures option data displayed in Chart 2.23. The estimated parameters are \( \sigma = 0.14, \lambda = 0.40, \) and \( k = -0.16 \). This results in the bi-modal probability distribution of Chart 2.29. The jump probability is very high, consistent with the prevailing condition of market stress, so the distribution appears to be nearly equally divided between two possibilities: If a jump occurs, there will be a sharp decline to a level around 900 and if no jump occurs, there will be a sharp recovery to a level around 1100.

**Chart 2.29**

**Implied probability distributions for S&P index futures**

*One-month options, September 14, 1998*

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49 For details of the estimation procedure, see Malz (1996).
**Example 2.11**

**Probability distribution of Italian bond prices**

We can also apply the technique to the Italian bond futures option data displayed in Chart 2.22. The estimated parameters are $\sigma = 0.045$, $\lambda = 0.28$, and $k = -0.07$. This results in the negatively skewed probability distribution of Chart 2.30. The jump probability is reflected in the small hump on the left side of the distribution and is associated with market anxiety regarding a sudden drop in Italian bond prices. The estimated jump-diffusion model implies a considerably higher probability of a sharp drop in bond futures prices below 105 than does the at-the-money Black-Scholes implied volatility.

**Chart 2.30**

**Implied probability distributions for Italian bond futures**

*Three-month options, September 14, 1998*

A disadvantage of estimates using the jump-diffusion model is that the estimation algorithm will generally find it difficult to distinguish between the parameters $\lambda$ and $k$, which almost always appear as a product in Equation [2.37]. A somewhat higher estimate of $\lambda$ and a somewhat lower estimate of $k$ would fit the data almost as well. The shape of the distribution, however, is determined largely by the product $\lambda k$, which is well identified.

**B A TECHNIQUE BASED ON THE INTERPOLATED VOLATILITY SMILE**

Surprisingly, price data for over-the-counter options, while less easily available to most market participants than exchange-traded data, may lend itself more readily to the computation of risk neutral distributions. Increasingly, the over-the-counter markets, particularly in foreign exchange, trade options on a given asset and with a given maturity with exercise prices covering a wide segment of the price axis.

For these markets, a simple interpolation method is available. The great advantage of the over-the-counter option data is that the exercise prices are nicely spaced over the price axis. However, relatively few options are traded. Three options are generally traded: an at-the-money call, an out-of-the-money call and an out-of-the-money put. For many currency pairs, in addition, a far out-of-the-money put and call are traded. In essence, the technique is to pass an interpolating polynomial function through the three or five option prices. The interpolating function provides an estimated option
price for any exercise price. We can then estimate the risk neutral distribution by computing the second differences of the estimated option prices with respect to the exercise price. 50

Chart 2.31 illustrates the results for a typical recent observation on the dollar-mark exchange rate.

**Chart 2.31**

**Implied probability distributions for dollar-mark**

*One-month options, December 11, 1998*

On this particular date, out-of-the-money dollar put prices were higher than those of equally out-of-the-money calls, indicating that the market believed—or feared—a sharp depreciation of the dollar against the mark to be more likely than an equally sharp depreciation. As a result, the risk neutral distribution of the exchange rate is skewed to the left: it has a long fat tail on the left-hand, weak dollar, side. The risk neutral distribution is compared with a lognormal distribution with a standard deviation equal to the at-the-money volatility. The forward rate, indicated by a vertical line, is common to both distributions.

We can use implied volatilities for options with different exercise prices and different times to maturity to show how the risk neutral distribution changes over time, from the standpoint of a fixed forecasting date. For example, using the volatility surface depicted in Chart 2.24, we can estimate the probability distribution of the dollar-yen exchange rate at the end of each of the next 12 months. The result is displayed in Chart 2.32. There is a slight skew in favor of a stronger dollar. Note that the variance of the distribution increases with the forecast horizon, as one would expect.

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50 Appendix 2.B and Malz (1997a and 1997b) provide a more detailed discussion. Campa, Chang and Reider (1997) use a similar data set, but spline rather than polynomial interpolation, to estimate risk neutral distributions for European cross-rates against the deutsche mark. Their interpolation technique, in contrast to the one used here, may under some circumstances misprice far out-of-the-money options (see Malz [1997a] for details).
Example 2.12
The Mexican peso 1997–1998

Chart 2.33 illustrates how risk neutral distributions may change over time, reflecting changing market sentiment regarding an asset price. In common with many emerging market currencies, the Mexican peso came under severe pressure in the second half of 1997, eventually giving way and depreciating sharply over the next year (albeit less sharply than many Asian currencies, since the peso had left its peg years earlier, in December 1994). The evolution of its risk neutral distribution illustrates several typical features of option-based asset price forecasts.

As in Chart 2.31, the risk neutral distribution is derived from an interpolating polynomial passed through option prices for five different exercise prices. The option maturity and forecast horizon are 1 month in each panel. The thin curve represents a lognormal (random walk with drift) distribution with a standard deviation equal to the at-the-money volatility and the thin grid line represents the 1-month forward rate at the time of the forecast. Both the risk neutral distribution and the lognormal distribution have the same mean, equal to the current forward rate. The heavy gridline represents the realized exchange rate 1 month later.

On September 30, 1997, as depreciation fears grew pronounced, the risk neutral distribution is bimodal. The right “hump” represents the probability of a sharp depreciation. The markets at this point appear to have believed that the peso would either fall sharply, or the pressure would dissipate and the peso would bounce back modestly, with few alternatives in between. This is the classic jump or event risk situation, in which either a large drop in price occurs, with low probability, or a small rise in price, with high likelihood. The forward rate appears as an average of these two possibilities. Note that the dispersion of the peso’s value in the event of a devaluation is greater than that in the event of a recovery, as one would expect.
In the event, the peso fell far more sharply than the market’s worst expectations. This led to the next phase, illustrated by the risk neutral probability on October 31, 1997. The general level of implied volatility has now skyrocketed from under 10% to over 30% per annum. Uncertainty now dominates the risk neutral distribution, as at these levels of implied volatility, skewness and kurtosis retreat into the background, swamped by the high dispersion (note the wide range of exchange rates with a positive probability). The graph depicts a state of extreme anxiety and uncertainty, with the market protecting itself against a sharp further depreciation, of which examples abounded in the exchange markets, and a sharp rally back to “fundamental” values. Both extremes, and any intermediate outcome, seemed plausible. The realized rate on November 30, 1997 was in fact quite close to its forward rate expected value.

For the time being, the peso depreciation had run its course, and the markets calmed a bit, while remaining wary. The pace of depreciation began to pick up again in mid-1998, in tandem with other emerging market currencies. This situation is reflected in a risk neutral distribution with a more typical volatility of about 10%, but with a large right-hand tail, as on June 30, 1998. The market appears to have been protecting itself against some significant peso depreciation over the ensuing month, without having a strong view on whether it would be a sharp jump or a continuation of the peso’s slow grind lower. And, in the event, the peso was little changed a month later.

The final panel depicts the risk neutral distribution at the end of July 1998. Market anxiety, the dispersion of the expected future value of the peso, and the skewness of the distribution have all abated somewhat. In the following month however, contagion from the largely unanticipated Russian debt and currency crises brought the peso sharply lower, far exceeding its forward rate forecast.
It is important to distinguish between the procedure outlined here and the random walk approach to forecasting confidence intervals. In those earlier sections, we posited the stochastic process of the asset, that is, a statistical model of how the asset price moves over time. Here, we infer the risk neutral distribution of the asset price at a particular point in time, the maturity date of the options. This is less information than contained in a stochastic process: the process will always imply a terminal risk neutral distribution, but knowledge of the terminal distribution alone is not sufficient to infer the stochastic process, that is, the statistical model of the entire time path of the asset price.51

51 Rubinstein (1994) and Derman and Kani (1994) estimate the entire stochastic process of the asset price from option prices. Recovering the stochastic process rather than merely the terminal distribution is particularly valuable in the pricing and risk management of path-dependent options such as barrier options.
Appendix 2.A  Covered parity in the currency, interest rate, commodity and equity index markets

2.A.1 Cost-of-carry and the mechanics of forward prices

Covered parity is often useful in identifying forward prices. The mechanics are somewhat different in different markets, depending on what instruments are most actively traded. The simplest case is that of a fictitious commodity that has no convenience value, no storage and insurance cost, and pays out no interest, dividends, or other cash flows. The only cost of holding the commodity is then the opportunity cost of funding the position.

The payoff on a long (short) forward position is \( S_T - F_{t,T} - (F_{t,T} - S_T) \), where \( F_{t,T} \) is the current forward price agreed at time \( t \) for delivery of the commodity at time \( T \), and \( S_T \) is the future cash price. The term of the forward is \( \tau \equiv T - t \). For example, \( \tau = 1/12 \) for a 1-month forward.\(^{52}\)

Now, imagine an alternative way to create a long forward payoff. It might be needed by a dealer hedging a short forward position:

- Buy the commodity with borrowed funds, paying \( S_T \) for one unit of the commodity borrowed at \( r_{t,t+\tau} \), the \( \tau \)-month annually compounded spot interest rate at time \( t \). Like a forward, this set of transactions has a net cash flow of zero.
- At time \( T \), repay the loan and sell the commodity. The net cash flow is \( S_T - (1 + r_{t,t+\tau})S_T \).

This strategy is called a synthetic long forward.

Similarly, in a synthetic short forward, you borrow the commodity and sell it, lending the funds at rate \( r_{t,t+\tau} \); the net cash flow now is zero. At time \( T \), buy the commodity at price \( S_T \) and return it: the net cash flow is \((1 + r_{t,t+\tau})S_T - S_T \).

The payoff on this synthetic long or short forward must equal that of a forward contract:

$$ [2.A.1] \quad S_T - F_{t,T} = S_T - (1 + r_{t,t+\tau})S_T. $$

If it were greater (smaller), one could make a riskless profit by taking a short (long) forward position and creating a synthetic long (short) forward. This implies that the forward price is equal to the future value of the current spot price, i.e., the long must commit to paying the financing cost of the position:

$$ [2.A.2] \quad F_{t,T} = (1 + r_{t,t+\tau})S_T. $$

Two things are noteworthy about this cost-of-carry formula. First, the unknown future commodity price is irrelevant to the determination of the forward price and has dropped out. Second, the forward price must be higher than the spot price, since the interest rate \( r_{t,t+\tau} \) is positive.

Short positions can be readily taken in most asset markets. However, in some commodity markets, short positions cannot be taken and thus synthetic short forwards cannot be constructed in sufficient volume to eliminate arbitrage entirely. Even, in that case, arbitrage is only possible in one direction. Equation [2.A.2] becomes an inequality:

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\(^{52}\) All time rates (interest rates, volatilities, cash flows, and so on) in this document will be expressed as annual rates, while time intervals will be expressed in months to avoid notational clutter.
If the commodity pays dividends or a return $d_{t, t + \tau}$ (expressed as a percent per period of the commodity price, discretely compounded), which is known in advance, the analysis becomes slightly more complicated. You can think of $d_{t, t + \tau}$ as the dividend rate per “share” of the asset: a share of IBM receives a dividend, an equity index unit receives a basket of dividends, $100$ of par value of a bond receives a coupon, and so on. The $d_{t, t + \tau}$ may be negative for some assets: you receive a bill for storage and insurance costs, not a dividend check, on your 100 ounces of platinum. The amount of dividends received over $t$ periods in currency units is $d_{t, t + \tau} S_t \tau$.

The synthetic long forward position is still constructed the same way, but in this case the accrued dividend will be received at time $T$ in addition to the commodity price. The net cash flow is $S_T + d_{t, t + \tau} S_T \tau - (1 + r_{t, t + \tau}) S_t$. The no-arbitrage condition is now

\[ S_T - F_{t, T} = S_T - (1 + (r_{t, t + \tau} - d_{t, t + \tau}) \tau) S_t. \]

The forward price will be lower, the higher the dividends paid:

\[ F_{t, T} = (1 + (r_{t, t + \tau} - d_{t, t + \tau}) \tau) S_t. \]

The forward price may be greater than, less than or equal to than the spot price if there is a dividend. The long’s implied financing cost is reduced by the dividend received. The remainder of this Appendix discusses how cost-of-carry and forward prices are calculated in the currency, interest rate and other markets.

### 2.A.2 Foreign exchange

**Forward foreign exchange** is foreign currency deliverable in the future.\(^{53}\) Its price is called **forward exchange rate** or the **forward outright rate**, and the differential of the forward minus the spot exchange rate is called the **swap rate** (not to be confused with the rate on plain-vanilla interest rate swaps).

We can easily apply the general mechanics of a forward transaction to this case. Let $r_{t, T}$ ($r^*_{t, T}$) represent the domestic (foreign) interest rate. To create a synthetic long forward,

- **Borrow** $\frac{S_t}{1 + r^*_{t, T} \tau}$ domestic currency units at rate $r$ and **buy** $\frac{1}{1 + r^*_{t, T} \tau}$ foreign currency units. Deposit the foreign currency proceeds at rate $r^*_{t, T}$. There is no net cash outlay now.

- **At time $T$, the foreign currency deposit has grown to one foreign currency unit, and you must repay the borrowed** $\frac{S_t}{1 + r^*_{t, T} \tau}$, including interest.

This implies that the forward rate is

\[ F_{t, T} = \frac{1 + r_{t, T} \tau}{1 + r^*_{t, T} \tau} S_t. \]

\(^{53}\) That is, with a value date later than the two-day standard for spot foreign exchange.
To relate $d_{t, t+\tau}$ in the generic forward presentation of Section 2.A.1 to $r^*_t$, note that $r^*_t$ is expressed as a percent of the foreign currency unit, while $d_{t, t+\tau}$ is a percent of the time-$t$ asset price, expressed in domestic currency units. At time $t$, $\frac{1 + r^*_t \tau}{1 + r^*_t \cdot T}$ domestic currency units are paid for each time-$T$ foreign currency unit, so the analogue of $d_{t, t+\tau}$ is $r^*_t \cdot T$.

2.A.3 Equities and commodities

One use of equity index futures is to extract an estimate of the dividend yield on stocks. Let $F_{t, t+\tau}$ represent an equity index futures. Using Equation [2.A.5], we can estimate the dividend yield, at an annual rate, by

$$d_{t, t+\tau} = r^*_t \cdot (\frac{F_{t, t+\tau}}{S_t} - 1)^{\frac{1}{\tau}}.$$

Gold is often leased by its owners, primarily central banks but also occasionally mining operations. Leased gold is used by the over-the-counter gold forward market to satisfy the needs of hedgers. Refiners, fabricators, and occasionally mining operations purchase gold forward. Let $g_{t, t+\tau}$ represent the gold lease rate, measured as a percent of the gold price $S_t$. It plays the role of $d_{t, t+\tau}$ in the generic presentation above. Generally, it is about 2%, well below typical U.S. dollar money market rates, so gold for forward delivery is somewhat costlier than spot gold.

Example 2.A.1

Equity forwards

Suppose you have one share of Sunbeam, currently priced at $6.00 per share. Sunbeam pays no dividend, so the 3-month forward price is the cash price plus the cost of borrowing $6.00 for three months. If the 3-month rate is 4.8% per annum, we have a forward price equal to

$$\left(1 + 0.048 \cdot \frac{3}{12}\right)6 = 6.072.$$  

IBM, in contrast, is paying a dividend of $1 or a rate of 0.8% per share. It is currently priced at $125, so the 3-month forward price is equal to

$$\left[1 + (0.048 - 0.008) \cdot \frac{3}{12}\right]125 = 126.25.$$  

Example 2.A.2

Gold leasing

Suppose you are a bank intermediating in the gold market. A miner wants to sell 1000 ounces gold forward for delivery in 6 months. He transacts with you at the market price of $F_{t, t+6} = 279.95$. You now have a long forward position to hedge, which you can do in several ways. You can use the futures market, but perhaps the delivery dates do not coincide with the forward. Alternatively, you can lease gold from a central bank for 6 months at $g_{t, t+6} = 0.02$ or 2% and sell it immediately in the spot market at a price of $S_t = 275$, investing the proceeds ($275,000) in a 6-month deposit at 5.6%. Note that there is no net cash flow now.

In 6 months, these contracts are settled. First, you take delivery of forward gold from the miner and immediately return it to the central bank along with a wire transfer of 2750 = $S_t \cdot g_{t, t+\tau} \cdot \tau \cdot 1000$ dollars. You redeem the deposit, now grown to $282,700$, from the bank and pay $279,950$ to the miner.

It is generally more difficult to short physical commodities than financial assets. The role of the lease market is to create the possibility of shorting gold. Leasing gold creates a “temporary long”
for the hedger, an obligation to divest himself of gold 6 months hence, which can be used to construct the synthetic short forward needed to offset the customer business.

2.A.4 Interest rates

In contrast to other asset classes, interest rates have a “built-in” time dimension. This enables us to calculate implied forward interest rates without having to observe prices in actual forward or futures markets. Since our attention will be drawn to the interbank interest rate markets—those for swaps, FRAs and eurocurrency deposits—we will refer not to “bonds” but to “deposits” whenever we want to describe a generic fixed income investment.\(^ {54}\)

Spot and forward interest rates

We will need a few definitions. The spot rate \( r_{t,T} \) is the constant annual rate at which a fixed income asset’s value must grow starting at time \( t \) to reach \$1 at time \( T \). The term of the spot rate is \( \tau \equiv T - t \). The spot curve or zero-coupon curve relates the spot rate to the maturity date or term.\(^ {55}\)

It is often more convenient to work with discount factors rather than spot rates, since we can then dispense with compounding intervals and annualization. The \( t \)-period ahead discount factor at time \( t \) is related to the spot rate by

\[
 v_{t,\tau} = \frac{1}{1 + r_{t,T}^\tau}.
\]

The forward rate \( f_{t,T_1,T_2} \) from time \( T_1 \) to time \( T_2 \) is the annual interest rate contracted at time \( t \) to be paid from time \( T_1 \) to time \( T_2 \). We now have two time intervals at work: the interval \( t \) to \( T_1 \), when the forward deposit settles, and the interval \( T_1 \) to \( T_2 \), when the forward deposit matures. We call \( \tau_1 \equiv T_1 - t \) the time to settlement and \( \tau_2 \equiv T_2 - T_1 \) the time to maturity. The forward curve relates forward rates of a given time to maturity to the time to settlement or the settlement date. There is thus a distinct forward curve for each time to maturity. For example, the 3-month forward curve is the curve relating the rates on forward 3-month deposits to the future date on which the deposits settle.

Any forward rate can be derived from a set of spot rates via arbitrage arguments by identifying the set of deposits which will lock in a rate prevailing from one future date to another, without any current cash outlay. We assume that there are no transactions costs, i.e., lending and borrowing can be done at the same rates.

If you want to earn interest from time \( T_1 \) to \( T_2 \), but not from time \( t \) to \( T_1 \), you can make a deposit maturing at \( T_2 \) (lend money for \( \tau_1 + \tau_2 \) years) at a rate \( r_{t,T_1} \) and borrow in the deposit market for \( \tau_1 \) years at a rate \( r_{t,T_1} \). If you borrow and lend the same amount you will have no net cash flow now. We will assume that amount is \( \frac{1}{1 + r_{t,T_1}^\tau_1} \) dollars.

\(^{54}\) See Malz (1997c) for additional details and derivations of the interest-rate computations in this section.

\(^{55}\) We will focus here exclusively on discretely compounded interest rates. The mathematics of continuously compounded rates is considerably simpler, but real-life interest rates are always discretely compounded. Different real-life rates often have different compounding intervals which can be a source of great confusion. A distinct spot curve can be calculated for different compounding intervals.
At time $T_1$ there will be a first cash outlay: you pay 1 dollar on the maturing short-term deposit. At time $T_2$ there will be a second cash outlay: you receive $\frac{1 + r_{t, T_2}(\tau_1 + \tau_2)}{1 + r_{t, T_1}}$ dollars on redemption of the long-term deposit. Effectively, you have contracted to lay out 1 dollar at time $T_1$ and get $\frac{1 + r_{t, T_2}(\tau_1 + \tau_2)}{1 + r_{t, T_1}}$ dollars back at time $T_2$. To eliminate arbitrage opportunities, the market must therefore adjust $f_{t, \tau_1, \tau_2}$ so that

$$[2.A.9] \quad 1 + f_{t, \tau_1, \tau_2} = \frac{1 + r_{t, T_2}(\tau_1 + \tau_2)}{1 + r_{t, T_1}}.$$ 

implying

$$[2.A.10] \quad f_{t, \tau_1, \tau_2} = \frac{1}{\tau_2} \left( 1 - \frac{1 + r_{t, T_2}(\tau_1 + \tau_2)}{1 + r_{t, T_1}} \right) = \frac{r_{t, T_2}(\tau_1 + \tau_2) - r_{t, T_1}}{1 + r_{t, T_1}}.$$ 

Equation [2.A.10] can be applied directly to any discount instrument, such as U.S. Treasury bills, zero-coupon bonds (stripped government bonds), or FRAs.

We can get further insight into Equation [2.A.10] by rearranging its terms as follows:

$$[2.A.11] \quad \frac{r_{t, T_2}}{1 + r_{t, T_1}} = \frac{1}{\tau_1 + \tau_2} \left( f_{t, \tau_1, \tau_2} + \frac{r_{t, T_1}}{1 + r_{t, T_1}} \right).$$

The term $1 + r_{t, T_1}$ is fairly close to unity, so we see that the market sets cash and forward rates so that the longer-term interest is more or less the maturity-weighted average of the shorter-term and forward rates. Viewed somewhat differently, if the term structure of money-market rates is upward (downward) sloping, the FRA rate will tend to be higher (lower) than both the longer- and shorter-term cash rates.

To fit the foregoing discussion into the generic framework of Section 2.A.1, think of the commodity as $1 available at time $T_2$, that is, discount paper maturing at time $T_2$. The price of a $\tau$-year discount bond with a yield to maturity of $r$ is $\frac{1}{1 + r \tau}$. We can treat $S = \frac{1}{1 + r_{T_2}(\tau_1 + \tau_2)}$ as the cash price and $F_{t, \tau_1} = \frac{1}{1 + f_{t, \tau_1, \tau_2}}$ as the $\tau_2$-year forward price of discount paper maturing at time $T_2$.

Equation [2.A.2] then applies with $r_{t, \tau_1}$ as the cost of carry.

**Example 2.A.3**

**Forward rate agreements**

FRAs are generally cash settled by the difference between the amount the notional deposit would earn at the FRA rate and the amount it would earn at the realized Libor or other reference rate, discounted back to the settlement date.

Suppose the 3-month and 6-month German mark Libor rates are respectively 4.55 and 4.45%. We would expect that the 3X6 FRA rate will be below 4.45%. In fact, applying Equation [2.A.10], it is 4.3011%. Say Bank A takes the long side and Bank B takes the short side of a DEM 10,000,000
3X6 FRA on January 1 at a rate of 4.30%, and suppose 3-month DEM Libor is 4.50%. If the FRA were settled by delivery, Bank A would place a 3-month deposit with Bank B at a rate of 4.30%. It could then close out its position by taking a deposit at the going rate of 4.50%, gaining $0.002 \times \frac{90}{360} \times 10000000 = 5000$ marks when the deposits mature on June 1. The FRA is in fact cash settled by Bank B paying Bank A the present value of that amount on March 1. With a discount factor of $1.045 \times \frac{90}{360} = 1.01125$, that comes to DEM 4,944.38.

**Cash and forward swap rates**

A **plain vanilla interest rate swap** is an agreement between two counterparties to exchange a stream of fixed interest rate payments for a stream of floating interest rate payments. Both streams are denominated in the same currency and are based on a notional principal amount. The notional principal is not exchanged. The design of a swap has three features which determine its price: the maturity of the swap, the maturity of the floating rate, and the frequency of payments. We will assume for expository purposes that the latter two features coincide, e.g., if the swap design is fixed against 6-month Libor, then payments are exchanged semiannually.

At initiation, the price of a plain-vanilla swap is set so its current value—the net value of the two interest payment streams, fixed and floating—is zero. The swap can be seen as a portfolio which, from the point of view of the payer of fixed interest (called the “payer” in market parlance) is long a fixed-rate bond and short a floating-rate bond, both in the amount of the notional principal. The payer of floating interest (called the “receiver” in market parlance) is long the floater and short the fixed-rate bond.

The price of a swap is usually quoted as the swap rate $r_{t,T}^{S,h}$, that is, the yield to maturity with compounding interval $h$ on a notional par bond. What determines this rate? A floating-rate bond always trades at par at the time it is issued. The fixed-rate bond which represents the payer’s commitment in the swap must then also trade at par if the swap is to have an initial value of zero. In other words, the swap rate is the market-adjusted yield to maturity on a par bond and satisfies

$$1 = \frac{\sum_{i=1}^{\tau_h} \frac{1}{1 + r_{t,T}^{S,h} i/h}}{1 + \frac{1}{1 + r_{t,T}^{S,h} i/h}}.$$  \[2.A.12\]

The swap rate must also be consistent with the prevailing term structure of interest rates, implying that it must satisfy

$$1 = \frac{\sum_{i=1}^{\tau_h} v_{t+ih} + v_{t,T}}{1 + \sum_{i=1}^{\tau_h} v_{t+ih} + v_{t,T}}.$$  \[2.A.13\]

A forward swap is an agreement between two counterparties to commence a $\tau_2$-year swap in $\tau_1$ years. The rate $f_{t,T_1,T_2}^{S,h}$ on a forward swap can be calculated from forward rates or spot rates. As in the case of a cash swap, the forward swap rate is the market-adjusted par rate on a coupon bond issued at the initiation of the swap at time $T_1$. 

---

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The forward swap rate must also be consistent with the term structure of forward interest rates at the time the swap counterparties enter into the contract. This requirement implies

\[
\frac{\tau_2}{h} = \sum_{i = 1}^{\frac{\tau_2}{h}} \frac{1}{(1 + f_{t, T_i, T_2} h)^i} + \frac{1}{(1 + f_{T, T_2} h)^{\frac{\tau_2}{h}}}.
\]

The forward swap rate must also be consistent with the term structure of forward interest rates at the time the swap counterparties enter into the contract. This requirement implies

\[
\frac{\tau_2}{h} = \sum_{i = 1}^{\frac{\tau_2}{h}} \frac{\nu_{t, T_i} + \nu_{t, T_2} + i h}{\nu_{t, T_i}} + \frac{\nu_{T, T_2}}{\nu_{t, T_i}}.
\]
Appendix 2.B  Why do option prices contain so much information?

It may at first appear unusual that such arcane financial instruments as option contain complete information on the risk neutral distribution of asset prices. The reason is that the market value of an option, like that of any other asset, can be stated as the present value of its expected payoff at maturity, calculated under the risk neutral probability distribution.

The payoff at maturity to a European call option with an exercise price \( X \) maturing at time \( T \) is \( \max(S_T - X, 0) \), where \( S_T \) represents the terminal, or time-\( T \), asset price, since an option is either in-the-money or worthless at expiration. We can thus express the probability distribution of the option payoffs as a function of the risk neutral probability density function \( q(x) \) of the terminal asset price \( S_T \). Letting \( c(t, X, T) \) denote the time-\( t \) market value of a European call with an exercise price \( X \) maturing at time \( T \), we have

\[
c(t, X, T) = e^{-rT}E^{*}_{t}\left[ \max(S_T - X, 0) \right]
\]

[2.B.1]

\[
e^{-rT}\int_{0}^{\infty} \max(S_T - X, 0)q(S_T) dS_T
\]

\[
e^{-rT}\int_{X}^{\infty} (S_T - X)q(S_T) dS_T.
\]

We discount to the present at the risk-free rate \( r \) because \( q(x) \) is the risk neutral density. For example, in Chart 2.B.1, the probability of exercise of a 124 Treasury bond call equals the area under the risk neutral distribution to the right of the exercise price.

The second derivative of the option price is

\[
\frac{\partial^2}{\partial X^2}c(t, X, T) = e^{-rT}q(X).
\]

[2.B.2]

Equation [2.B.2] states that the second derivative of the current market price of a European call option with respect to the exercise price equals the risk neutral density of the future asset price (times the present value of one currency unit deliverable at time \( T \)). In other words, given prices of enough options on the same asset and with the same maturity, but with different exercise prices, we could trace out the entire risk neutral probability distribution of the asset.\(^{56}\)

---

\(^{56}\) This result is due to Breeden and Litzenberger (1978).
To show this, we construct *elementary contingent claims* on an underlying asset from options. Imagine that the asset price followed a discrete probability distribution. An elementary claim pays the holder $1 if $S_T \geq X$ and zero otherwise. The price of each elementary claim is denoted $\varepsilon(X, t, T)$. Elementary contingent claims are closely related to the state prices we discussed in Section 2.1.3. Here, we have identified each discrete state of the world with a discrete value the asset price might take on.

Suppose we could construct elementary claims over the entire asset price axis, or at least for all $X$ such that $\varepsilon(X, t, T) \geq 0$. The prices of the elementary claims constitute a probability measure; each $\varepsilon(X, t, T)$ gives the probability that the terminal asset price is $X$. The value of a portfolio containing all of these claims would be $1$, and we could then plot the discrete probability distribution of the asset price.

An elementary claim can be constructed out of European call options. Suppose, for example, the dollar-mark exchange rate can only take on values with a step size of DEM 0.01. Consider an elementary claim on the dollar, denominated in German marks, which pays DEM 1.00 if the exchange rate in $\tau$ years is DEM 1.60. This claim can be replicated by purchasing 100 dollar calls struck at DEM 1.59 and 100 dollar calls struck at DEM 1.61, and selling 200 dollar calls struck at DEM 1.60. This type of option portfolio is called a long *butterfly spread* and its construction is illustrated in Chart 2.B.2.
The price of the contingent claim must be equal to the price of the butterfly spread:

$$\varepsilon(1.60, t, T) = \frac{100[c(t, 1.59, T) + c(t, 1.61, T) - 2c(t, 1.60, T)]}{0.01}$$

Otherwise a riskless arbitrage opportunity would present itself. We could make the same calculation for every other possible price as well, so we can represent the probability measure $\varepsilon(X, t, T)$ by the prices of butterfly spreads centered at each step on the price axis. Chart 2.B.3 illustrates this by representing the market value of each elementary contingent claim/butterfly spread as the area of a bar in a histogram centered at $X$:

$$[2.B.3] \quad \varepsilon(X, t, T) = \frac{\left[c(t, X - \Delta X, T) - c(t, X, T)\right] - \left[c(t, X + \Delta X, T) - c(t, X, T)\right]}{\Delta X}$$

where $\Delta X$ is the step size, in this case DEM 0.01.
This argument can be extended to smaller and smaller step sizes. The height of each histogram bar can be written

\[
\frac{\varepsilon(X, t, T)}{\Delta X} = \frac{[c(t, X - \Delta X, T) - c(t, X, T)] - [c(t, X + \Delta X, T) - c(t, X, T)]}{\Delta X^2}.
\]

Taking this expression to the limit as $\Delta X \to 0$, the elementary claims divided by the step size converge to the risk neutral probability density function, evaluated at $X$. This is shown in Chart 2.B.4 as the density function which passes through the upper edge of each histogram bar. As the bars grow narrower, their upper edges converge to points on the density.

**Chart 2.B.4**

**Risk neutral probability distribution**

The right side of Equation [2.B.4] converges to the second derivative of the call value with respect to the exercise price:

\[
\lim_{\Delta X \to 0} \frac{\varepsilon(X, t, T)}{\Delta X} = \frac{\partial^2}{\partial X^2} c(t, X, T) = e^{-rT} q(X).
\]
Chapter 3. Forecasts based on economic structure

3.1 Introduction

The forecasting techniques implemented in LongRun are intended for use in Value-at-Risk calculations by corporate treasurers, asset managers, and other market participants requiring measures of longer-term financial risk. Long-term forecasting techniques are not distinguished for predictive accuracy, nor is there universal agreement on what techniques are appropriate. The responsible financial manager will therefore wish to consider forecasts generated from different techniques. The methodologies employed should be independent of one another and of widely recognized validity.

Based on these considerations, we developed two different forecasting techniques for LongRun. Chapter 2 presented one technique based on current market data, which enjoys widespread if not unanimous acceptance. Chapter 3 presents an alternative approach with a very different starting point, but with an equally strong underpinning in forecasting theory and practice. The approach uses historical data on the economic fundamentals in order to estimate the parameters of a forecasting model, which determines asset prices. Economic theory determines the selection of the historical data that is most relevant to the asset prices we need to forecast. This approach to forecasting dates back to the earliest days of econometric modeling and remains in widespread use in the financial services industry and in central banking.

What kinds of models can be used to make forecasts based on economic structure? Models that are used to make forecasts based on economic structure typically specify three properties of the variables that we want to forecast: (1) their evolution (time series analysis), (2) their relationship to other variables (economic structure), and (3) the statistical distribution of forecast prices at any point in time (econometric estimation).

We classify these models into two broad categories: (A) parametric models, which define a specific form for the evolution of some time series as well as its distribution, and (B) nonparametric models, which attempt to produce forecasts without imposing any assumptions on the shape of the time series’ distribution. A common problem associated with nonparametric methods is that they require relatively large amounts of data. The models we select to develop for long-horizon forecasting must be not only general enough to apply to various asset prices across foreign exchange rates, interest rates, equity indices, and commodity prices, but also simple enough to avoid the need for large amounts of historical data. Therefore, we consider several parametric models in our investigation into long-horizon forecasting.

Historically, the parametric econometric forecasting has relied primarily on either structural models of relevant aspects of the economy (large-scale structural econometric model) or on pure time-series methods (Univariate Autoregressive-moving Average [ARIMA] model). The structural econometric modeling attempts to construct a set of equations that capture the relevant interactions among the economic variables of interest. This approach dates back to the League of Nations models of the 1930s and is today epitomized by the large-scale econometric forecasting models maintained by the Federal Reserve Board and the Deutsche Bundesbank, to name but two. The pure time series method capitalizes on the information about future values contained in the past values of the variables of interest, and has drawn more interest as confidence in large-scale structural econometric models declined in the wake of the forecasting disasters of the 1970s.

In the early 1980s, the vector autoregressive model was introduced as a synthesis of the large-scale structural econometric model and the univariate ARIMA model. The VARM is a multivariate model in which each variable is explained by its own past values and the past values of all the other variables in the equation system. The VARM is less restrictive than the large-scale structural econometric model, since all variables in the VARM are determined within the equation system and are not structured by specific economic theory. In contrast to the univariate ARIMA model, cross-variable link-
ages are automatically incorporated in the VARM because it includes lags of all variables in the equation system.

The models we select to develop for long-horizon forecasting must be flexible enough to accommodate the historical path of a large number of macroeconomic time series as well as the financial asset price which we should forecast. Hence, we restrict the basis for selecting our best forecasting model to VARM. Throughout, the conditions are as follows:

- All the models that we present in this document rely on time series of financial and/or economic data.

- The natural logarithm of each variable that we want to forecast (i.e., the forecast variable) is assumed to be normally distributed.

- We specify a time series model for the mean of each forecast variable, but not the volatility or correlation with other variables.¹

- The mean of each forecast variable may depend on (1) its own past values, (2) past values of some other financial variables, and/or (3) past fundamental (macroeconomic) variables.

- All parameter estimates are obtained from time series data.

More recently, a family of new techniques has been proposed that blends elements of all the earlier approaches. The techniques start with the related notions of cointegration and error correction. Cointegration refers to variables that tend to move together in an equilibrium relationship over long periods of time. Over shorter periods, the variables may drift quite far from the values posited by this long-term relationship, but then converge to equilibrium on longer time scales. The dynamics of this slow convergence to equilibrium are described by an error correction term. We thus concentrate on two types of models for long-horizon forecasting: (1) VARM and (2) models based on cointegrating relationships, such as ECM.

### 3.2 Constructing an econometric ‘forecasting system’

We define a forecasting system in terms of (1) a functional form of the model, (2) a set of explanatory variables, and (3) a structural regime, which establishes the relationship between the variables we want to forecast, the model, and the explanatory variables.

For a given structural regime, the functional form of the model defines how the forecast variables change over time, how they relate to explanatory variables, and how they are distributed statistically. For example, suppose we want to model some variable \( x \). Equation [3.1] specifies a potential model for \( x \):

\[
 x_t = a + bx_{t-1} + c x_{t-1} + e_t, 
\]

where

the subscripts “\( t \)” and “\( t - 1 \)” denote the current and last-period value, respectively. \( a, b \) and \( c \) are fixed parameters, also known as coefficients.

¹ We do not specify the time-series model directly to forecast variance-covariance structure itself like Auto-Regressive Conditional Heteroskedasticity (ARCH) model, but we can estimate the variance-covariance matrix from the vector type time-series model for the mean forecasts using estimated error terms. For a detailed discussion of variance-covariance structure, see Chapter 5.
Sec. 3.2 Constructing an econometric ‘forecasting system’

$z_{t-1}$ represents the last period’s value of some explanatory variable.

$e_t$ is a normally distributed variable with mean 0 and standard deviation $\sigma_t$.

It follows from Equation [3.1] that $x_t$ is normally distributed with a (conditional) mean

$$a + bx_{t-1} + cz_{t-1}$$

and standard deviation $\sigma_t$. If time is measured in days, then Equation [3.1] simply says that today’s value of $x$ is equal to some fixed number $a$, plus a constant times yesterday’s value of $x$, $(bx_{t-1})$, plus a constant times yesterday’s value of an explanatory variable, $(cz_{t-1})$, plus a randomly distributed error term $e_t$.

Now, assume that $x_t$ represents the current value of some stock price. The true model that describes how $x_t$ changes over time is unknown, so the best we can do is make an educated guess of how the stock price changes over time. Moreover, even if the ‘true’ model were known, it is reasonable to assume that it might change abruptly rather than evolve gradually with $x_{t-1}$ and $z_{t-1}$. In other words, the stock price’s model depends on the economic or structural regime over which the model is defined. For example, suppose that Equation [3.1] is the ‘best’ model for $x$ in a high-inflationary environment, but when inflation is low or falling, a better model for $x$ is

$$[3.2] \quad x_t = \bar{a} + bx_{t-1} + y_{t-1} + \bar{\varepsilon}_t,$$

where $\bar{a}$ is different from $a$, $y_{t-1}$ is an explanatory variable (which is different from $z_{t-1}$) and $\bar{\varepsilon}_t$ is a normally distributed random variable with a much smaller variance than $e_t$. Consequently, a more appropriate model for $x$ is

$$[3.3a] \quad x_t = a + bx_{t-1} + cz_{t-1} + e_t \quad \text{during high inflation}$$
$$[3.3b] \quad x_t = \bar{a} + bx_{t-1} + y_{t-1} + \bar{\varepsilon}_t \quad \text{during low inflation}$$

From this simple example it should not be hard to imagine how difficult it is to specify a realistic model for a stock price, or any financial variable, given the sheer number of potential model specifications (e.g., linear, quadratic), explanatory variables (e.g., macroeconomic time series, financial prices), and economic regimes (e.g., high inflation, low inflation).³

Instead of trying to find the true model or the data generating process (DGP) for $x$, we, like many researchers, take a more modest approach and simply posit some ‘reduced-form’ model for $x$, that is ‘parsimonious’, i.e., does not depend on many parameters. In other words, we choose a simple relationship between the future value of $x$ and the past values of explanatory variables. Moreover, we allow for economic regimes by first specifying the regime that will exist over the forecast horizon, and then estimating the model’s parameter using historical data that coincides with this regime. Once such a model is fitted to the observed data, it could be used to forecast values of $x_t$.

The following four steps summarize how we construct LongRun’s forecasting system based on economic structure.

- **STEP 1:** Define the functional form of the model, which establishes how the variables we want to forecast evolve over time and how they relate to other variables. In this chapter, we develop parametric models that rely on macroeconomic time series and that are based on vector autoregressive models with error correction terms. (See Section 3.3 and Section 3.4.)

² We use the term ‘best’ rather than ‘true’ since we assume we never know the true model but rather just a good approximation of it.

³ In a recent article in the Journal of Applied Econometrics (1996, p. 527), professors Jin-Lung Lin and Ruey S. Tsay remark “all the models used in practice are only approximations, because there are no ‘true’ models for a real-world time series.”
• **STEP 2:** Define the specification of the model, which identifies the explanatory variables we include in the estimation of the model. Based on economic theory that provides the relation between the asset price and economic variables, we select a set of economic variables, for example, CPI, interest rates and net exports, for each class of asset prices. (See Section 3.5.)

• **STEP 3:** Account for economic regimes. To maintain the stability of parameters, the basic model should be adjusted to incorporate economic regimes. We introduce statistical tools that allow us to identify structural breaks, which divide the sample period into structural regimes. We also explain how to estimate a model’s parameters in a way that reflects a particular regime. (See Section 3.6.)

• **STEP 4:** Estimate the defined model. The estimated result is sensitive to such conditions as regression method and lag length included in the model. Based on statistical inference, we set criteria for identifying and testing the conditions that affect the estimate. (See Section 3.7.)

### 3.3 Econometric framework

In Section 3.1, we saw that our survey of forecasting techniques led us to focus on the broad class of VAR(1) for the basic framework of our forecasting system. This framework encompasses a wide variety of models, ranging from “classical” simultaneous equation structural models and time series methods such as autoregressive-moving average (ARIMA) models, to more recently developed techniques for cointegrated time series. We will briefly discuss advantages of the VAR(1) for producing forecasts from various macroeconomic time series in this section. Furthermore, since we will treat the financial time series that we wish to forecast as at least potentially cointegrated with other financial and macroeconomic time series, we will, in this section, review the concepts of cointegration and error correction in some detail.

#### 3.3.1 Advantages of the vector autoregressive model

VAR(1)Ms are, in principle, simple multivariate models in which each variable is explained by its own past values and the past values of all the other variables in the system.\(^4\) A simple VAR(1) for the two variables \(x\) and \(y\) is given by,

\[
\begin{align*}
x_t &= \alpha_0 + \alpha_1 x_{t-1} + \alpha_2 x_{t-2} + \alpha_3 y_{t-1} + \alpha_4 y_{t-2} + \epsilon_{1t}, \\
y_t &= \beta_0 + \beta_1 x_{t-1} + \beta_2 x_{t-2} + \beta_3 y_{t-1} + \beta_4 y_{t-2} + \epsilon_{2t},
\end{align*}
\]

where \(\epsilon_{it}\) are random errors which, in general, will be correlated. For simplicity, only two lagged values for each variable are included in the equations.

To forecast the value of \(x_t\), the VAR(1) uses past information of other variables (\(y_{t-1}\) and \(y_{t-2}\)) as well as its own (\(x_{t-1}\) and \(x_{t-2}\)). This condition provides several advantages for improving forecasts that are based on macroeconomic time series.

* Compared to univariate autoregressive (AR) model, VAR(1) uses additional variables that should be able to forecast asset price under alternative specifications.

• VARM can improve forecasting performance in the long horizon as well as in the short horizon, since it uses various time-horizon autocovariances in forecasting.

• VARM reduces our burden of specifying a model across a large number of asset prices, provided we view VARM as an unrestricted reduced form.\textsuperscript{5} If so, then we are only required to choose a set of explanatory variables based on economic theory and let the data determine the other specification of the model.

3.3.2 Introduction to the error correction model

The forecasting techniques of this chapter are based on the relatively new but already well-researched concepts of cointegration and error correction. Since their formulation in the earlier 1980s, such forecasting techniques have been widely adopted, as they have helped improve the predictive performance of models and proved to be compatible with recent innovations in economic theory that emphasize the importance of market participants’ expectations in determining economic outcomes.

A Cointegration Relationship

Before proceeding, let us return for a moment to the discussion of the random walk in Chapter 2. The random walk is the benchmark example of a nonstationary stochastic process—that is, a stochastic process that has no tendency to return to its starting place once displaced, or even to remain at its current value, but to drift further away from its origin as time goes on. As we saw, most financial time series behave very much, if not precisely, as random walks. One important feature of a nonstationary time series is that either its mean or its variance changes over time. For example, in the simple case of an asset price following a random walk without drift, the mean at each future date is a constant equal to the current level of the asset price. The variance at each future date, however, is not a constant. Since a random walk tends to wander over time from its starting point, the variance grows with the square root of time.

In contrast, a time series that has a tendency to return to some “equilibrium” value, or has no tendency to wander away from its current value, is called stationary. For example, the increments to a random walk, in contrast to the level of the random walk itself, are stationary, with a mean of zero and a constant variance equal to the volatility parameter. Another example of a stationary time series are the residuals from a correctly specified regression equation. Each residual has a mean of zero and the same variance as any other residual.

It is generally much more difficult to forecast a nonstationary than a stationary time series econometrically, since the distribution of a nonstationary series changes over time. While this may seem to be a tremendous limitation—since most financial data appears to be nonstationary—we can overcome it by using our findings that simple combinations of several different financial time series are easier to predict than any individual time series.

Suppose, for example, we are interested in forecasting the future value of some variable $x_1$, which can represent a stock price, a foreign exchange rate, an interest rate, the spot price of flour, and so on. If $x_1$ follows a random walk or close to a random walk, it appears to be unpredictable; $x_1$ apparently does not follow any particular pattern, so that using the past values of $x_1$ is meaningless for predicting the future values of $x_1$.

\textsuperscript{5} See Section 3.5 for a more detailed discussion.
It seems we are rather limited as to which model we can use to forecast values of $x_1$. Suppose though, we also observe some other variable $x_2$ which, similar to $x_1$, also appears to be unpredictable.

$$
x_{1t} = x_{1t-1} + \epsilon_{1t}, \tag{3.6}
$$

$$
x_{2t} = x_{2t-1} + \epsilon_{2t}.
$$

Given information about $x_1$ and $x_2$, is there any way to formulate a model where either or both $x_1$ and $x_2$ are predictable? The answer to this question depends on whether $x_1$ and $x_2$ are “related” in some way. If they are related, then one should be able to use their relationship to forecast $x_1$ and $x_2$.

Suppose that the relationship between $x_1$ and $x_2$ over time is such that these variables never “wander too far apart” from one another. An example of such a relationship is where the difference between $x_1$ and $x_2$, which we represent by a new variable $z_1 = x_1 - x_2$, is relatively stable over time in that it fluctuates around some fixed value without drifting too far in any one direction.

This represents a simple example of a **cointegrating relationship**: while $x_1$ and $x_2$ taken singly are random walks (or at least unpredictable from past information), their difference is stationary. The variables $x_1$ and $x_2$ are said to be cointegrated, or to constitute a **cointegrated system**. The cointegrating relationship is defined by the stationary linear combination of the two variables and can be compactly represented by the vector $\beta = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

If a cointegrating relationship exists, then the behavior of the two variables can be represented by an **ECM**. In an ECM, increments to a set of nonstationary time series are represented as a linear function of past values of the stationary linear combination of the two variables, plus a Gaussian white noise disturbance term. For example, we might have

$$
x_{1t} - x_{1t-1} = 0.5z_{1t-1} + \epsilon_{1t}, \tag{3.7}
$$

$$
x_{2t} - x_{2t-1} = 0.1z_{1t-1} + \epsilon_{2t},
$$

where we have put time subscripts on the variables to signify that the current change in $x_1$ (and $x_2$) is given by 0.5 (and 0.1) times the past value of $z_1$ plus a mean zero random disturbance. Given this information, an estimate of next period’s value $x_1$ is given by

$$
\hat{x}_{1t} = x_{1t-1} + 0.5z_{1t-1}. \tag{3.8}
$$

If we treat $x_{1t}$ in isolation as a random walk, the forecast is simply $x_{1t-1}$. Equation [3.8] adds an error correction term $0.5z_{1t-1}$, which should improve forecast accuracy, as long as $x_1$ and $x_2$ are in fact a cointegrated system. Thus, even if we observe an asset price series that appears to move as a random walk, it may be possible to combine other asset prices with the price we want to predict in order to improve forecasts.

---

6 This property of cointegrated systems is called the Granger Representation Theorem. See Hamilton (1994), pp. 579–582.

7 Equation [3.8] follows from taking the expectation of Equation [3.7]. Also, note that the same argument can be used to describe how to predict $x_2$. 

---
A parable that has proved helpful in conveying the concepts of cointegration and error correction is Professor Murray’s tale of a drunk and her dog (Murray, 1994). The evolution of a random walk is often likened to the steps of a drunk: each step is independent of earlier steps, and has no evident goal, but only a tendency to wander from the initial position. In the same spirit, Murray likens a cointegrated system to the meanderings of a drunk $x_1$ and her dog $x_2$. Each is a random walk, but the meanderings are not independent, since the dog belongs to the drunk:

The drunk sets out from the bar, about to wander aimlessly in random walk fashion. But periodically she intones “Oliver, where are you?”, and Oliver interrupts his aimless wandering to bark. He hears her; she hears him. He thinks, “Oh, I can’t let her get too far off; she’ll lock me out.” She thinks, “Oh, I can’t let him get too far off; he’ll wake me up in the middle of the night with his barking.” Each assesses how far away the other is and moves to partially close that gap.

Now neither drunk nor dog follows a random walk; each has added what we formally call an error correction mechanism to her or his steps. But if one were to follow either the drunk or her dog, one would still find them wandering aimlessly in the night; as time goes on, the chance that either will have wandered far from the bar grows. The paths of the drunk and the dog are still nonstationary.

Significantly, despite the nonstationarity of the paths, one might still say, “If you find her, the dog is unlikely to be very far away.” If this is right, then the distance between the two paths is stationary, and the walks of the woman and her dog are said to be cointegrated of order zero. The dog is not on a leash, which would enforce a fixed distance between the drunk and the dog. The distance between the drunk and the dog is not a random variable, but a stationary one despite the nonstationarity of the two paths.

Mathematically, we can model the woman’s and dog’s cointegrated meanderings as

\begin{align}
[3.9] \quad x_t - x_{t-1} &= c(y_{t-1} - x_{t-1}) + e_t \quad \text{the drunk} \\
[3.10] \quad y_t - y_{t-1} &= d(x_{t-1} - y_{t-1}) + u_t \quad \text{her dog}
\end{align}

The first terms on the right-hand sides of Equation [3.9] and Equation [3.10] are the error correction terms that keep the two wanderers close together and $(x_{t-1} - y_{t-1})$ is a cointegrating relationship between $x_t$ and $y_t$, with weights 1 and $-1$.

How powerful is the error correction mechanism in determining the time series path? To illustrate, we construct an example based on Murray’s parable. We build the steps of the drunk, her dog, and of another player: my dog, who follows a random walk. We draw 100 random numbers each for $e$ and $u$ from a standard normal distribution and calculate the distances between the drunk and her dog and the drunk and my dog if she and her dog follow the error correction equations [3.9]–[3.10] while my dog follows a random walk and does not have the error correction term; we set the cointegration coefficients as small as $c = d = 0.1$. It is worth noting that we use the same random numbers to calculate the paths of her dog and my dog. Thus, the only difference between Equations [3.9]–[3.10] and my dog’s random walk equation is the existence of the error correction mechanism in Equations [3.9]–[3.10].

Charts 3.1a–b show the paths of the drunk and her dog and the distance between them, while Charts 3.2a–b show the paths of the drunk and my dog and the distance between them. Even with small values of the cointegration coefficients, the error correction mechanism is powerful. The distance between the drunk and her dog looks quite stationary, whereas the distance between the drunk and my dog looks much more like a random walk. In other words, the distance between the drunk and her dog does not increase as time goes on, but the distance between the drunk and my dog does.
Chart 3.1
Example of a cointegrated relationship

a. Paths of the drunk and her dog

b. Distance between the drunk and her dog

Chart 3.2
Example of a non-cointegrated relationship

a. Paths of the drunk and my dog

b. Distance between the drunk and my dog

We can now define some general notation for cointegrated systems.\(^8\) Consider an \(n \times 1\) vector time series at time \(t\)

\[
X_t = \begin{bmatrix}
x_{1,t} \\
x_{2,t} \\
x_{3,t} \\
\vdots \\
x_{n,t}
\end{bmatrix}.
\]

\(^8\) This section draws heavily from Hamilton (1994), Chapter 19.
The variable $X_t$ is said to be cointegrated if each of the $x_{i,t}$'s, where $i = 1, 2, \ldots, n$, follow a random walk (i.e., nonstationary with a unit-root), while some linear combination of the series $z_{1,t} = \beta_{11}x_{1,t} + \beta_{12}x_{2,t} + \ldots + \beta_{1n}x_{n,t}$ is stationary. In matrix notation,

\[ z_{1,t} = \beta_1'x_t, \]

where $\beta_1 = (\beta_{11}, \beta_{12}, \ldots, \beta_{1n})'$ is an $n \times 1$ cointegrating vector (i.e., it defines a cointegrating relationship).

Let us study a simple example of a cointegrated process to demonstrate the relationship between cointegration and common trends. Suppose $X_t$ consists of two series, $x_{1,t}$ and $x_{2,t}$ that evolve through time as follows,

\[ x_{1,t} = \beta_{11}x_{2,t} + \epsilon_{1,t}, \]
\[ x_{2,t} = x_{2,t-1} + \epsilon_{2,t}, \]

where $\epsilon_{1,t}$ and $\epsilon_{2,t}$ are uncorrelated stationary random variables, and $\beta_{11}$ is some fixed parameter. Since $x_{2,t}$ follows a random walk and ‘drives’ the movements of $x_{1,t}$, we see that both elements of $X_t$ are nonstationary. Consequently, each series will have the property of wandering arbitrarily far from its starting values $x_{1,0}$ and $x_{2,0}$, respectively.

Next, define a new variable $z_{1,t} = x_{1,t} - \beta_{11}x_{2,t}$, which is simply some linear combination of $x_{1,t}$ and $x_{2,t}$. According to Equation [3.13a], $z_{1,t} = \epsilon_{1,t}$ which, by definition, is a stationary process. Hence, we conclude that $X_t$ is cointegrated with cointegrating vector $\beta_1 = (1, -\beta_{11})$.

In general, for any $n \times 1$ vector of random variables $X_t$, there may exist as many as $h < n$ cointegrating vectors. In other words, we may have as many as $h$ cointegrating vectors, $\beta_1, \beta_2, \beta_3, \ldots, \beta_h$, where each $\beta_i$ is an $n \times 1$ vector of cointegrating parameters. Now, we can stack these vectors column-wise to generate the $n \times h$ cointegrating matrix $\beta$,

\[ \beta = [\beta_1, \beta_2, \beta_3, \ldots, \beta_h], \]

where each column of $\beta$ represents a cointegrating vector. We can use $\beta$ to transform a set of $n$ nonstationary variables (e.g., random walks) in $X_t$ to a set of $h$ stationary variables $Z_t = \beta X_t$. That is, the matrix $\beta$ allows us to convert a set of $n$ nonstationary random variables into a smaller set of $h$ stationary variables. For example, if $n = 4$, and we found $h = 2$ cointegrating relationships, then $\beta$ would be a $4 \times 2$ matrix and $Z_t = \beta'X_t$, where

\[ Z_t = \begin{bmatrix} z_{1,t} \\ z_{2,t} \end{bmatrix} = \begin{bmatrix} \beta_{11} & \beta_{12} & \beta_{13} & \beta_{14} \\ \beta_{21} & \beta_{22} & \beta_{23} & \beta_{24} \end{bmatrix} \begin{bmatrix} x_{1,t} \\ x_{2,t} \\ x_{3,t} \\ x_{4,t} \end{bmatrix} = \begin{bmatrix} \beta_{11}x_{1,t} + \beta_{12}x_{2,t} + \beta_{13}x_{3,t} + \beta_{14}x_{4,t} \\ \beta_{21}x_{1,t} + \beta_{22}x_{2,t} + \beta_{23}x_{3,t} + \beta_{24}x_{4,t} \end{bmatrix}, \]

and each row of the matrix $\beta'$ represents a cointegrating relationship. Equation [3.15] shows that the cointegrating matrix converts the $4 \times 1$ vector of nonstationary variables into a $2 \times 1$ vector of stationary series.
B COMMON STOCHASTIC TRENDS

Cointegrating relationships are often interpreted as describing equilibrium configurations of economic variables toward which they converge over long periods. The error correction mechanism describes the dynamics of this convergence process. In every cointegrated system there are, in addition, exogenous shocks called common stochastic trends (or simply common trends), which drive the system away (or further away) from any initial equilibrium.

We began building our forecasting models by considering a set of $n$ time series $X_t$ that appear to be random walks. Economic theory may then suggest that $h$ cointegrating relationships prevail among these time series. We have also noted that the number of cointegrating relationships must be strictly smaller than the number of original random walks in $X_t$. This suggests that the random walk behavior of the $n$ time series is driven by a smaller number of random walks that cannot be reduced to stationarity by linear combination. If the vector $X_t$ behaves as a set of random walks, then there must be at least one irreducible random walk inducing the nonstationary behavior. The common stochastic trends are these irreducible random walks.$^9$

Thus there is more predictability in the set of time series $X_t$ than is suggested by their initial appearance as random walks. Rather than $n$ independent random walks, we are actually confronted with only $k$ random walks, while $h = n - k$ relationships are cointegrating relationships and hence are forecastable as more than simply their current values.

These exogenous shocks or $k$ random walks are called “common” because they affect each cointegrated set of time series in the system. They are called “trends” because they may or may not have a drift term, which drives the system in a predictable way each period. In the case of purchasing power parity, economic theory may suggest that the drift term is equal to zero. In the case of a set of spot and forward exchange rates, economic theory may suggest that the drift rate of the common trend is equal to the long-term expected inflation rate differential between the two currencies. Finally, the shocks are called stochastic because they possess a white noise component.

Given a set of nonstationary time series, researchers attempt to determine

- whether each series “is its own random walk” and, therefore, does not share any long-term movements with the other series, or

- whether the set of time series is tied-together by a smaller, common set of random walks, so-called common (stochastic) trends. The term “stochastic” is often used to describe the trend since the trend contains a random element.$^{10}$ In this document, however, we refer to such trends simply as ‘common trends.’

Economic theory may be applied to motivate cointegrating relationships among time series. In the many cases where economic theory cannot be relied on to suggest cointegrating relationships, researchers are left to rely on statistical theory and some intuition about how a certain set of time series may move together. The fundamental belief behind common stochastic trends is that financial and economic time series consist of long-term components (i.e., trend part) and short-term components (i.e., stationary part), each of which is separate and identifiable, and that different underlying economic forces govern the evolution of the components. For example, suppose we are studying the time series of prices in equity markets of the U.S., Japan, England, Germany, and Canada.$^{11}$ Instead of treating each market as completely separate from the others, one may ask, “If prices in each individual country’s stock market have a random component (with drift), are these random walk

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9 Stock and Watson (1988).

10 An example of a trend that is not random is a time trend. Such trends are often referred to as deterministic.

11 This example is taken from Kasa (1994).
components different from each other, or do they perhaps arise from the response of each country to a single, common, world growth factor? Or, more generally, how many common trends are there in the equity markets of the major industrialized countries?

The answers to these questions depend on the ability to decompose the equity price series into long- and short-term components. The long-term components originate from the elements driving earnings and dividends. Such components presumably depend on the underlying trends in national economies or industries. Conversely, the short-term components may represent transitory deviations from the earnings-dividend link arising from movements in discount rates and ‘national risk premia.’

We can use the time-series system presented in Equations [3.13a]–[3.13b] (which are reproduced in Equation [3.16]) to demonstrate the relationship between cointegration and common trends.

\[ \begin{align*}
  x_{1,t} &= \beta_{11} x_{2,t} + \epsilon_{1,t} \\
  x_{2,t} &= x_{2,t-1} + \epsilon_{2,t}
\end{align*} \]

In Equation [3.16] we have \( n = 2 \) nonstationary time series. Earlier, we established that there exists \( h = 1 \) cointegrating relationship, namely \( \beta_1 = (1, -\beta_{11}) \). Therefore, we should have \( k = 1 \) common trend. Upon inspection of Equation [3.16], we find that this is obviously the case because \( x_{2,t} \) is the common trend, which is shared by \( x_{1,t} \) and \( x_{2,t} \).

In general, we can write a model for a set of \( n \) nonstationary variables represented by \( X_t \), and driven by \( k = n - h \) common trends as

\[ \begin{align*}
  X_t &= AV_t + \epsilon_t \\
  V_t &= V_{t-1} + \epsilon_t,
\end{align*} \]

where \( V_t \) is a \( k \times 1 \) vector of random walks (common trends), \( \epsilon_t \) is a \( k \times 1 \) vector of stationary disturbances, and \( A \) is an \( n \times k \) matrix of parameters (factor loadings) that convert or ‘map’ the \( k \) random walks \( V_t \) into \( n \) nonstationary time series stored in \( X_t \). Also, \( \epsilon_t \) as defined previously, is an \( n \times 1 \) vector of stationary variables.

Recall that in cointegration, we define a matrix \( \beta \) that converts the \( n \) nonstationary variables into \( h \) stationary variables; that is, \( Z_t = \beta^\prime X_t \) is an \( h \times 1 \) vector of stationary variables. In terms of Equation [3.17], the cointegrating matrix must satisfy \( \beta^\prime A = 0 \) so that \( \beta^\prime X_t = \epsilon_t \). To see this relationship more explicitly, we write Equations [3.13a]–[3.13b] as follows:

\[ \begin{pmatrix}
  x_{1,t} \\
  x_{2,t}
\end{pmatrix} =
\begin{pmatrix}
  \beta_{11} \\
  1
\end{pmatrix}
\begin{pmatrix}
  x_{2,t} \\
  1
\end{pmatrix} + \begin{pmatrix}
  \epsilon_{1,t} \\
  0
\end{pmatrix}. \]

In terms of Equation [3.17], we have

\[ \begin{align*}
  X_t &= \begin{pmatrix}
    x_{1,t} \\
    x_{2,t}
  \end{pmatrix} \\
  Y_t &= x_{2,t} \\
  A &= \begin{pmatrix}
    \beta_{11} \\
    1
  \end{pmatrix}. \]
The cointegrating vector found previously was \( \beta_1 = (1, -\beta_{11})' \), and we see that \( \beta_1' A = 0 \), as expected.

C \ EXAMPLES AND LIMITATIONS

In many cases, cointegrating relationships are suggested by economic theory. For example, economic equilibrium states or no-arbitrage conditions can be represented as cointegrating relations. These conditions need not prevail at every instant, but must dominate the behavior of economic variables over long periods. There are a number of examples of how cointegration and error correction techniques can be applied to financial markets. We present two such examples.

\textit{Example 3.1}

\textbf{Purchasing power parity}

The theory of purchasing power parity (PPP) holds that, apart from transportation costs, goods should sell for the same effective price in two countries. Let \( P_t \) denote an index of the price level in the U.S. (in dollars per good), \( P_t^* \) a price index for, say, Italy (in lira per commodity unit), and \( S_t \) the spot exchange rate between the currencies (in USD per lira). Then PPP holds that the price of goods sold in the U.S. and Italy is the same, after accounting for the exchange rate between the U.S. dollar and the Italian lira. Mathematically, we can express this relationship as

\begin{equation}
P_t = S_t \times P_t^*
\end{equation}

or, taking logarithms,

\begin{equation}
p_t = s_t + p_t^* ,
\end{equation}

where \( p_t = \ln(P_t) \), \( s_t = \ln(S_t) \) and \( p_t^* = \ln(P_t^*) \). The notation \( \ln(X) \) represents the natural logarithm of \( X \).

In practice, errors in measuring prices, transportation costs, and differences in the quality of goods prevents Equation [3.21] from holding exactly. Moreover, a large body of research finds that \( p_t \), \( s_t \), and \( p_t^* \) are essentially random walks.

Therefore, one may reason that even if Equation [3.21] does not hold exactly, and the individual price and exchange rate series are unpredictable, any discrepancy or error from PPP should be stationary. That is to say, the variable \( z_{1,t} = s_t + p_t^* - p_t \), called the real exchange rate between U.S. and Italy in economic terms, is stationary although its individual components \( p_t \), \( s_t \), and \( p_t^* \) are not. That is, the price of a country’s goods in domestic currency units should not diverge “too much” from its price in foreign currency units. Otherwise it would be profitable to buy goods in one market and ship it to the other.

If we define \( X_t \) as

\begin{equation}
X_t = \begin{bmatrix}
s_t \\
p_t^* \\
p_t
\end{bmatrix},
\end{equation}

then the cointegrating vector is \( \beta_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \) such that \( z_{1,t} = \beta_1' X_t \), which represents the PPP condition, is a stationary process.

We construct a numerical illustration of PPP using the U.S. dollar and Italian lira. Chart 3.3a shows the Consumer Price Indexes (CPI) of the U.S. and Italy, and the USD per ITL exchange rate for
January 1970 to May 1998; each series is logarithmic and multiplied by 100. Also each series is normalized by its value on January 1970. Each series looks nonstationary, which means that the series moves far away from the starting point as time goes on. Italy experienced about twice the average inflation rate of the U.S. over the 1970–1998 period and the Italian lira dropped in value relative to the dollar by roughly this same proportion.

We also plot the real exchange rate between the U.S. and Italy ($z_{1,t}$) in Chart 3.3b. The real USD per ITL exchange rate is not always zero, but it does not move far away from zero, i.e., a stationary process. The stationarity implies that the two economies of the U.S. and Italy are not a perfect world of PPP as expressed by Equation [3.21] (or only one price across two countries), but neither do they deviate much from PPP.

**Chart 3.3**

**Example of Purchasing Power Parity**

- **a. Path of U.S. and Italian CPIs, with USD per ITL exchange rate**
- **b. Cointegration relationship: Real USD per ITL exchange rate**

---

**Example 3.2**

**Cash and forward rates**

Another important example of cointegrating relationships suggested by economic theory emerges from the discussion in Chapter 2 of the relationship between forward and spot rates. Contemporaneous spot and forward exchange rates tend to move much more closely together than do forward rates and the future spot rate. The same relationship holds for other asset classes. One consequence of this phenomenon is that spot and forward rates are cointegrated.

Specifically, if we assume that the spot exchange rate follows a random walk, as is empirically plausible, and if we also assume that the time series of differences between the forward rate and the expected future exchange rate are stationary, then the spot rate and all observable contemporaneous forward rates constitute a cointegrated system. The forward premium certainly appears to be stationary.

For example, a system of the spot and a single forward rate has a cointegrating vector $(1, -1)$, since the differences are stationary. This in turn implies that spot and forward rates can be written as an ECM, and thus, including information on lagged forward premiums can improve a forecast compared to one based on the random walk model.

While the preceding examples underscore the relatively simple idea that we plan to use to forecast financial and non-financial prices, numerous challenges arise in the implementation of this idea. Two examples of such challenges include:

1. **Stable relationships**: We need to determine the linear combination of rates/prices that help to predict the individual time series. In the preceding example, we simply assumed that
In general, however, $z_1$ could be any linear combination of $x_1$ and $x_2$ (e.g., $z_1 = 0.5 - 0.8x_2$), or $z_1$ may not even exist. Figuring out if $z_1$ exists (and if so, what the weights of $x_1$ and $x_2$ are) is an empirical matter and, therefore, must be determined by historical observations of $x_1$ and $x_2$. In addition, $z_1$ may be the function of another time series $x_3$ such that, for example, $z_1 = x_1 - x_2 - x_3$. Or, even more worrisome, we may find that $z_1$ changes so that sometimes $z_1 = x_1 - x_2$ and at other times $z_1 = x_1 - x_2 - x_3$. The result shows how important it is to know what variables constitute $z_1$, if it exists.

2. **Parameter stability**: What is the coefficient of $z_1$? In the preceding example, we simply posit that $0.5$ times the past value of $z_1$ equals the future changes in $x_1$. In practice, however, not only do we have to determine this coefficient, but also whether it changes frequently. If we find that the coefficient constantly changes, then we are most likely to conclude that it is very difficult to forecast changes in $x_1$ using $z_1$, because while we know $z_1$, we cannot establish a stable relationship between $z_1$ and $x_1$. Again, we must determine whether this relationship is stable by using historical data on $x_1$ and $z_1$.

### 3.4 VECM: The functional form of our chosen model

In this section we present the model that we use to forecast nonstationary time series. Chart 3.4 summarizes the candidates for our forecasting model. In order to make the best use of the macro fundamental variables and their long-term cointegration relationship, we construct our model under the VARM and ECM framework. Out of the wide variety of VARMs and ECMs that exist, we have chosen three specific types of models — Difference VARM (DVARM), Vector ECM (VECM), and Adaptive ECM (AECM)— as candidates for the functional form of our forecasting model.

**Chart 3.4**

**Candidates for the functional form of the forecasting model**

- **Framework**
  - VARM
  - ECM
  - LVAR
  - DVARM
  - VECM
  - AECM

- **Candidates for the functional form of model**

- **Chosen model**
  - VECM

In Section 3.4.1 we define three types of VARM and show how to connect ECM to the VARM. We show how to write a VARM in terms of an error correction representation in the case where cointegrating relationships exist among the time series. In Section 3.4.2 we define two types of ECM in detail. Section 3.4.3 then discusses the forecasting performance of models and finally chooses the VECM as the functional form of our model.
3.4.1 Three types of VARM

Our objective is to forecast the future value of a single series or a subset (e.g., relevant asset prices) of an $n$-dimensional vector $X_t$. As an example, we consider the future value of the U.S. dollar price of the deutsche mark, i.e., the USD per DEM exchange rate. We define

$$X_t = [s_t, p_t, y_t, i_t, m_t],$$

where $s_t$ is the logarithm of the spot USD per DEM exchange rate at time $t$, and $p_t, y_t, i_t,$ and $m_t$ are the logarithmic differences of the two countries’ prices, outputs, interest rates, and money supplies at time $t$, respectively.

A simple, yet often effective approach to modeling the vector $X_t$ is to employ a VARM with $p$ lags, which can be written as follows

$$X_t = \theta D + G_1 X_{t-1} + G_2 X_{t-2} + \ldots + G_p X_{t-p} + \varepsilon_t,$$

where

- $\theta$ represents the coefficients of the deterministic variables, $D$.
- $\varepsilon_t$ is a vector of independent (over time) and identically distributed normal random variables with zero mean and covariance matrix, $\Lambda_t$. In abbreviated form, $\varepsilon_t \sim MVN(0, \Lambda_t)$.

The VARM has been applied to stationary as well as nonstationary time series. Our focus is on modeling the latter since most of the time series that we deal with are nonstationary. We begin by assuming that the nonstationary time series $X_t$ needs to be differenced only once to become stationary (difference stationary), so that time series $X_t$ is integrated of order 1, or simply $I(1)$. We define the polynomial

$$G(L) = I + G_1 L + G_2 L^2 + \ldots + G_p L^p,$$

where $L$ denotes the lag operator and $G(1) = I + G_1 + G_2 + \ldots + G_p$ is referred to as the long-term multiplier. Let $h$ be the rank of $G(1)$. Since $h \leq n$, there are $k = n - h$ nonstationary components (random walks) among the $n$ time series in $X_t$.

Depending on the statistical properties of $X_t$, we have three potential models:

1. If $h = n$ (i.e., $k = 0$), where each element of $X_t$ is stationary, or $h = 0$ (i.e., $k = n$) so that $X_t$ is nonstationary but no cointegrated relationships among the variables exist, we may model $X_t$ as a level VARM (LVAR) model,

$$LVAR$$

$$X_t = \theta D + G_1 X_{t-1} + G_2 X_{t-2} + \ldots + G_p X_{t-p} + \varepsilon_t.$$
Since there are \( p \) lags in this model, it is denoted by LVAR\((p)\).

2. If \( h = 0 \) (i.e., \( k = n \)), then \( X_t \) is nonstationary with no cointegrating relations. In this case we model the first differences in \( X_t \), namely \( \Delta X_t = X_t - X_{t-1} \), as a function of past differences \( \{\Delta X_{t-1}, \Delta X_{t-2}, \ldots, \Delta X_{t-p+1}\} \). The result is a difference VARM (DVAR) model,

\[
\text{DVAR} \tag{3.27}
\]

\[
\Delta X_t = 0\theta D + \Gamma_1 \Delta X_{t-1} + \Gamma_2 \Delta X_{t-2} + \ldots + \Gamma_{p-1} \Delta X_{t-p+1} + \varepsilon_t,
\]

where \( \Gamma_i = -\sum_{j=i+1}^{p} G_j \) for \( i = 1, \ldots, p - 1 \).

3. If \( 0 < h < n \), then \( X_t \) contains \( h \) cointegrating relationships and the appropriate model for \( X_t \) is given by a Vector Error Correction Model (VECM),

\[
\text{VECM} \tag{3.28}
\]

\[
\Delta X_t = 0\theta D + \Pi X_{t-1} + \Gamma_1 \Delta X_{t-1} + \Gamma_2 \Delta X_{t-2} + \ldots + \Gamma_{p-1} \Delta X_{t-p+1} + \varepsilon_t,
\]

where \( \Pi = -G(1) \). The \( n \times n \) parameter matrix \( \Pi \) can be factored into the product of two \( n \times h \) matrices \( \alpha \) and \( \beta \) such that \( \Pi = \alpha \beta' \) where the rows of \( \beta' \) represent the system’s \( h \) cointegrating vectors, and the transformed series \( Z_t = \beta' X_t \) is stationary. The existence of the \( h \times 1 \) stationary series \( Z_t = \beta' X_t \) implies some long-term relationship among the variables of \( X_t \). The “error correction” parameters in the matrix \( \alpha \) determine the rate at which each of the elements of \( X_t \) adjust in response to lagged deviations from the \( h \) cointegrating relationships.\(^{12}\)

Of the three models we have so far introduced, we retain only DVAR and VECM to model the asset price in \( X_t \). We apply DVAR to asset prices that are nonstationary and do not have cointegration vectors. Although LVAR is also a viable model, we choose DVAR because asset price is usually difference stationary and the DVAR model accounts for its persistence.\(^{13}\) We apply VECM to asset prices that are nonstationary and do have a cointegration vector. We thus eliminate LVAR from consideration.

### 3.4.2 Two types of ECM

We now present two models derived from the ECM, which incorporate cointegrating relationships and error corrections among time series. The first model is VECM, given by Equation [3.28]. VECM is multivariate in that it specifies a distribution for more than one variable at each forecast date. On the other hand, the second model—the Adaptive ECM (AECM)—specifies a distribution for only one variable, the variable we want to forecast at each forecast date. In fact, AECM can be viewed as a restricted form of VECM, i.e., a univariate version of the VECM.

#### A VECM

The VECM (Equation [3.28]) incorporates cointegrating relationships and error corrections among time series. Let us use our example again, i.e., forecasting the future value of the USD per DEM exchange rate. We define \( X_t = [s_t, p_t, y_t, l_t, m_t] \), where \( s_t \) is the logarithm of the spot USD per DEM exchange rate at time \( t \), and \( p_t, y_t, l_t, \) and \( m_t \) are the logarithmic differences between the two coun-

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\(^{12}\) Lin and Tsay (1996) explain why cointegration relations improve long-term forecast accuracy.

tries’ prices, outputs, interest rates, and money supplies at time $t$, respectively. Given this definition, we have

$$\Delta X_t = [\Delta s_t, \Delta p_t, \Delta y_t, \Delta i_t, \Delta m_t].$$

Next, we can define the $h \times 5$ ($0 < h < n$) matrix of cointegrating relations, $\beta$, so that $Z_t = \beta'X_t$ is simply an $h \times 1$ vector of long-term equilibrium (e.g., Purchasing Power Parity). Given these definitions, we can use the VECM (Equation [3.28]) to describe how the spot USD per DEM exchange rate and macro variable rates change over time. More specifically, the model for the change in spot exchange rate and macro variable rates is

$$\Delta X_{t+1} = \theta + \alpha Z_{t+1} + \Gamma_1 \Delta X_{t-1} + \epsilon_t, \quad \epsilon_t \sim MVN(0, \Lambda_t).$$

Equation [3.30] says that next period’s change in the spot exchange rates and macro variable rates is equal to a constant, plus the deviation from long-term equilibrium, plus last period’s change, plus some random disturbance. We estimate the parameters of the equation by using a set of time series, then generating forecasts of future spot rates by recursive substitution, i.e., by iteratively plugging the forecast values from the left-hand side of the equation into the right-hand side (one-period-ahead recursive feedback forecasting method).

**Parameter estimates**

To obtain estimates of the parameters $\theta$, $\alpha$, $\beta$, and $\Gamma_1$, we assume that we are justified in using historical prices and rates sampled at monthly intervals. We then obtain the estimates by using the Full Information Maximum Likelihood (FIML) as explained by Hamilton (1994, pp. 636–8) and reproduced in Appendix 3.A.

**Forecasts**

Having obtained estimates for $\hat{\theta}$, $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\Gamma}_1$, we can forecast the distribution of $X_{t+j}$ at each of the $t+j$ forecast dates as follows, where $j$ represents the $j$th month and $j = 1, 2, \ldots, J$. We first rewrite Equation [3.30] as

$$X_t = \theta + (I - \alpha\beta' + \Gamma_1)X_{t-1} + \Gamma_1\Delta X_{t-2} + \epsilon_t,$$

or, more compactly, as

$$h_t = A h_{t-1} + \tilde{\epsilon}_t,$$

where

$$h_t = \begin{bmatrix} X_t \\ X_{t-1} \\ \vdots \\ 1 \end{bmatrix}, \quad A = \begin{bmatrix} \Psi & \Gamma_1 & \theta \\ 0 & 0 & 1 \end{bmatrix}, \quad \tilde{\epsilon}_t = \begin{bmatrix} \epsilon_t \\ 0 \end{bmatrix}.$$

At time $t$, we can write the expression for forecast $h_{t+j}$ as

$$h_{t+j} = A^j h_t + \sum_{i=1}^{j} A^{j-i} \tilde{\epsilon}_{t+i}.$$
The expected value of $h_{t+j}$ is

$$[3.35] \hat{h}_{t+j} = A \hat{h}_t$$

and its covariance matrix is

$$[3.36] \hat{\Lambda}_{t+j} = \sum_{i=1}^{j} (A^i)^{j-i} \hat{\Lambda} A^{-i}.$$  

It follows from the distributional assumption on $e_{t+j}$ that the forecast $h_{t+j}$ follows a multivariate normal distribution with mean $\hat{h}_{t+j}$ and covariance matrix $\hat{\Lambda}_{t+j}$, i.e.,

$$[3.37] h_{t+j} \sim MVN(\hat{h}_{t+j} \mid \hat{\Lambda}_{t+j}).$$

The mean forecasts of $X_{t+j}$, denoted $\tilde{X}_{t+j}$, are given by the first $n$ elements of $\tilde{h}_{t+j}$.

**B AECM**

We begin by assuming that the logarithm of a financial variable at time $t$, denoted $s_t$, can be written as a deviation from the $m$th equilibrium value, $\tilde{s}_{m,t}$, $m = 1, \ldots, M$. Thus,

$$[3.38] s_t = \tilde{s}_{m,t} + z_{m,t},$$

where $z_{m,t}$ is a zero-mean stationary stochastic process and $\tilde{s}_{m,t}$ is a non-stationary unit-root process that represents some long-term equilibrium value of $s_t$. In other words, Equation [3.38] decomposes the financial variable $s_t$ into a permanent component $\tilde{s}_{m,t}$ and a transitory part $z_{m,t}$.

Equation [3.38] is based on the cointegration relationship among the time series. The equilibrium value or the permanent component $\tilde{s}_{m,t}$, can be any time series that is cointegrated with the financial variable $s_t$. For example, the equilibrium value $\tilde{s}_{m,t}$ is the path of the drunkard’s dog (Murray’s example of the drunk and her dog), or the difference between the U.S. and Italian inflation rate for the USD per ITL exchange rate (example of the PPP, Example 3.1), or one of the USD per JPY forward rates for the USD per JPY spot rate (example of the cash and forward, Example 3.2).

If we rewrite Equation [3.38] as $\tilde{z}_{m,t} = s_t - \tilde{s}_{m,t}$, we see that although the financial variable and its long-term value are nonstationary, the difference between them is stationary. In other words, $s_t$ and $\tilde{s}_{m,t}$ are cointegrated—they should not wander too far apart.\(^{14}\) We plan to use this cointegration relationship between $s_t$ and its equilibrium value $\tilde{s}_{m,t}$ when forecasting future changes in the spot exchange rate.\(^{15}\)

Let $s_t$ represent the logarithm of the spot USD per DEM exchange rate at time $t$ and assume that the logarithm of the exchange rate randomly fluctuates around the logarithm of its 12-month forward value, $f_{t,t+12}$. In this case we would set $\tilde{s}_{m,t} = f_{t,t+12}$ so that

$$[3.39] \tilde{z}_{m,t} = s_t - f_{t,t+12}$$

---

\(^{14}\) Recall the example in “A drunk and her dog” by Murray (1994).

\(^{15}\) In practice, it is not always possible to observe the long-term or equilibrium value and, therefore, we may have to estimate it from historical data. For an example, see Mark (1995).
We are interested in estimating the change in $s_t$ over the next $\tau_r$ months ($r = 1, \ldots, R$), making use of the long-term relationships described above. This leads to the regression

\[ s_{t+\tau_r} - s_t = \theta_r + \alpha_m, r z_{m,t} + e_{r, t+\tau_r}, \]

where $e_{r, t+\tau_r}$ is an independently and identically normally distributed disturbance with mean 0 and variance $\sigma_{r, t+\tau_r}$. Note that $e_{r, t+\tau_r}$ is univariate.

The intuition behind Equation [3.40] is that the convergence of $s_t$ towards $\tilde{s}_{m,t}$ implies some limit on the deviations between $s_t$ and $s_t$ for all forecast horizons $r = 1, \ldots, R$. For a given value of the equilibrium deviation at time $t$, the regression defined by Equation [3.40] provides an immediate forecast for the $\tau_r$-period-ahead price distribution.

Notice that since the regressor in Equation [3.40] depends on a long-term relationship in Equation [3.39], we should expect the model defined by Equation [3.40] to work relatively well for long-term forecasts (e.g., $\tau_r$ greater than 12 months). To improve the accuracy of our shorter-term forecasts (say, $\tau_r < 12$) we may add additional regressors to Equation [3.40] to account for short-term fluctuations.

In addition, we can include more than one measure of deviation from some equilibrium. For example, let $z_{1,t} = s_t - f_{1,t+1}$ so that we may define $z_{2,t} = s_t - f_{2,t+2}$, $z_{3,t} = s_t - f_{3,t+3}$ and $z_{4,t} = s_t - f_{4,t+4}$. The regression explaining the $\tau_r$-period change in the spot exchange rate is

\[ s_{t+\tau_r} - s_t = \theta_r + \alpha_1, z_{1,t} + \alpha_2, z_{2,t} + \alpha_3, z_{3,t} + \alpha_4, z_{4,t} + e_{t+\tau_r}. \]

An important feature of the AECM is that it allows us to estimate an entire path of forecast distributions all at once ($k$-period-ahead forecasting method). For example, suppose we are required to make quarterly forecasts of some financial variable over the next 2 years. In this case we need models for $\tau_r = 3, 6, 9, 12, 15, 18, 21, 24$ months ($R = 8$ quarters). We can form the regression system

\[
\begin{bmatrix}
    s_{t+3} - s_t \\
    s_{t+6} - s_t \\
    \vdots \\
    s_{t+24} - s_t
\end{bmatrix} =
\begin{bmatrix}
    \theta_1 \\
    \theta_2 \\
    \vdots \\
    \theta_8
\end{bmatrix} +
\begin{bmatrix}
    \alpha_{1,1} & \alpha_{2,1} & \cdots & \alpha_{M,1} \\
    \alpha_{1,2} & \alpha_{2,2} & \cdots & \alpha_{M,2} \\
    \vdots & \vdots & \ddots & \vdots \\
    \alpha_{1,8} & \alpha_{2,8} & \cdots & \alpha_{M,8}
\end{bmatrix}
\begin{bmatrix}
    Z_t \\
    e_{t+3} \\
    \vdots \\
    e_{t+24}
\end{bmatrix},
\]

where $Z_t$ is an $M \times 1$ collection of deviations from equilibrium, i.e., $Z_t = (z_{1,t}, z_{2,t}, \ldots, z_{M,t})$. More compactly, we can write Equation [3.42] as

\[ \Delta s_{t+\tau_r} = \theta + \alpha Z_t e_{t+\tau_r} \quad e_{t+\tau_r} \sim MVN(0, \Sigma_{t+\tau_r}). \]

where

\[
\Delta s_{t+\tau_r} = (s_{t+\tau_1} - s_t, s_{t+\tau_2} - s_t, \ldots, s_{t+\tau_R} - s_t)^T \quad \text{is an } R \times 1 \text{ vector of forecasted price changes.}
\]

$\theta$ is an $R \times 1$ vector of intercept parameters.

$\alpha$ is an $R \times M$ parameter matrix.

$Z_t$ is an $M \times 1$ vector of equilibrium deviations.
$e_{t+\tau}$ is an $R \times 1$ vector of multivariate normally distributed random variables with mean vector 0 and covariance matrix $\Sigma_{t+\tau}$. Alternatively expressed, $e_{t+\tau} \sim \text{MVN}(0, \Sigma_{t+\tau})$.

Note that unlike $\Lambda_t$ in Equation [3.30], the $R \times R$ covariance matrix $\Sigma_t$ captures the volatility at each of the $R$ forecast dates and measures covariation across the forecasted distributions.

The $k$-period-ahead forecasting method is simpler than the one-period recursive feedback forecasting method because it need not forecast all the series in the equation. However, it also yields unstable forecasts when the sample period is short. On the other hand, we cannot apply the one-period recursive feedback forecasting method to AECM, since AECM forecasts only the asset price in which we are interested. We can, however, apply both methods to VECM, but choose the method of one-period recursive feedback forecasting in order to increase forecasting stability.

### 3.4.3 Backtesting

Table 3.1 shows the main features of the three models we have investigated for the specific formulation.

<table>
<thead>
<tr>
<th>Model</th>
<th>Error correction</th>
<th>Forecasted variables</th>
<th>Estimation method</th>
<th>Forecasting method</th>
</tr>
</thead>
<tbody>
<tr>
<td>DVAR</td>
<td>No</td>
<td>all series</td>
<td>OLS</td>
<td>k-period-ahead</td>
</tr>
<tr>
<td>AECM</td>
<td>Yes</td>
<td>only relevant asset price</td>
<td>OLS</td>
<td>k-period-ahead</td>
</tr>
<tr>
<td>VECM</td>
<td>Yes</td>
<td>all series</td>
<td>FIML</td>
<td>one-period-ahead recursive</td>
</tr>
</tbody>
</table>

The two representative error correction models include error correction terms in their functional form. The AECM forecasts only the relevant asset price rather than all the prices of the series in the VARM system, which makes it impossible to use the one-period-ahead recursive feedback forecasting method. The parameters of DVAR and AECM are easily estimated by the Ordinary Least Square (OLS) regression method, but VECM needs to be estimated by the Full Information Maximum Likelihood (FIML) method.

In order to maximize our predictive power over a sample period, we can choose for each asset price a different model from among our three candidates. Generally, the more models one considers, the better the sample results will be. However, liberal consideration of various models will not necessarily help, and frequently hurts, when forecasting beyond the sample’s end. Econometricians call the problem ‘overfitting’ or ‘data-snooping.’ The primary source of the problem is having too many parameters (or explanatory variables) and applying too many specifications. A typical symptom is an excellent in-sample fit but poor out-of-sample performance, and the most effective means of reducing the impact of the problem is to impose some discipline on the search for a specification. To avoid the problem, we need to choose one best model and apply it, rather than all three, across asset prices.

**Which is the best forecasting model among the three?** One way to answer this question is to evaluate the quality of out-of-sample\(^\text{16}\) predictions produced by each forecasting model. We will

\(^{16}\)The difference between in-sample and out-of-sample forecasts can be explained as follows. First, define a data set for estimating the parameters of some forecasting model. In-sample forecasts are the forecasts generated from the data set that was used to estimate the forecasting model’s parameters. Out-of-sample forecasts, on the other hand, are forecasts generated from a data set that differs from the one used to estimate the forecasting model’s parameters.
explain the procedures for backtesting and the criteria for measuring accuracy in detail in the next chapter. Here we briefly show why we chose VECM from the three models, based on our backtesting results.

To assess forecasting performance, we invoke two criteria: the accuracy of the mean forecasts and the robustness of the confidence interval forecasts. We measure the accuracy of the mean forecasts according to two methods. One method is the mean absolute error (MAE), which shows how well the estimated model performs relative to a benchmark model in forecasting asset prices. The other method is the proportion of correct direction (PCD), which shows the percentage of time that a model predicts the correct direction in which an asset price moves.

We apply the three models in Table 3.1 to a total of 58 asset prices across four asset classes and calculate MAE and PCD. Table 3.2 shows the number of series that are forecasted the best by each model based on MAE and PCD.

Table 3.2
Backtesting the accuracy of mean forecasts
Number of best mean forecasts per asset class

<table>
<thead>
<tr>
<th>Model</th>
<th>Overall no. of time series</th>
<th>Foreign Exchange</th>
<th>Interest Rate</th>
<th>Equity Index</th>
<th>Commodity Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>DVAR</td>
<td>22</td>
<td>6</td>
<td>3</td>
<td>12</td>
<td>1</td>
</tr>
<tr>
<td>AECM</td>
<td>14</td>
<td>7</td>
<td>0</td>
<td>1</td>
<td>6</td>
</tr>
<tr>
<td>VECM</td>
<td>22</td>
<td>9</td>
<td>5</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>Total no. of time series</td>
<td>58</td>
<td>22</td>
<td>8</td>
<td>19</td>
<td>9</td>
</tr>
</tbody>
</table>

DVAR shows accurate mean forecasts for equity index, AECM for commodity price, and VECM for foreign exchange and interest rates. Overall, DVAR and VECM provide better mean forecasts than AECM.

The robustness of a confidence interval is measured by the percentage of outliers from the confidence interval (POC). If we construct 90% confidence intervals, then the percentage of samples outside the estimated confidence intervals should be 10%. If it is over 10% (or less than 10%), then the model underestimates (or overestimates) the true risk of asset price fluctuations.

We calculate the POC of a pool of four asset classes and plot them in Chart 3.5 at monthly forecast horizons, up to 12 months.
VECM provides better confidence interval forecasts than the other two models, since its POC is close to 10% across all forecast horizons. This means that the confidence interval calculated by VECM is not biased to overestimate or underestimate the true risk. However, DVAR is biased to overestimate the true risk (less than 10% outliers) across almost all the forecast horizons and AECM is biased to underestimate the true risk (more than 10% outliers) at the long forecast horizons.

In sum, we backtested three models proposed to assess the accuracy of mean forecasts and the robustness of confidence interval forecasts. Both DVAR and VECM showed almost the same accuracy of mean forecasts. While the confidence interval forecasts of DVAR were biased to overestimate the true risk level, VECM provided unbiased confidence interval forecasts. Hence, we will apply VECM across all the asset prices to avoid the overfitting, or data-snooping problem.

### 3.4.4 Summary

In this section we presented the models that LongRun can use to forecast asset prices. Chart 3.6 summarizes the path to the best forecasting model. In order to make the best use of the macro fundamental variables and their long-term cointegration relationship, we constructed our model under the VARM and ECM framework. We then chose the specific functional form of the model to be the VECM, i.e., ECM version of VARM (or the vector form of ECM), basing our decision on backtests of the accuracy of mean forecasts and the robustness of confidence interval forecasts. For the forecasting method, we chose the method of one-period recursive feedback forecasting rather than k-period ahead direct forecasting in order to increase forecasting stability. The next section discusses which time series across four asset classes should be included in the functional form of the model.
3.5 Specification of the VECM

To arrive at the specification of VECM, LongRun’s econometric model, we first discuss the general specification of the VARM, given by Equation [3.24] as,

\[ X_t = \theta D + G_1 X_{t-1} + G_2 X_{t-2} + \ldots + G_p X_{t-p} + \varepsilon_t, \]

where \( X_t \) is an \( n \)-dimensional vector and \( D \) represents deterministic variables. The parameters \( \theta \) and \( G_i \) are matrices of coefficients.

There are two approaches for specifying the VARM using economic theory (Holden, 1995).

- The VARM is viewed as a **restricted** reduced form of the structural model provided by economic theory, in which case, the economic relationships are imposed by restricting selected coefficients in the matrices \( \theta \) and \( G_i \) in Equation [3.24].

- The VARM is viewed as an **unrestricted** reduced form. The only assumptions required in this approach are that \( X_t \) have an accurate list of relevant variables and that we know the form in which they are to appear (i.e., level or difference).

If the restrictions in the first approach are invalid, it is better to ignore them and let the data determine the specification of the model. However, should those restrictions be valid, ignoring them would cause a loss of efficiency. Since it is impossible to validate the restrictions in the first ap-
We choose specific explanatory variables based on economic theory and let the data determine the specific relationships between them. Now we explain economic theories and choose explanatory variables by four asset classes.

### 3.5.1 Foreign exchange

There exist two popular economic theories that explain how to determine foreign exchange rates according to macroeconomic fundamentals. They are the Casselian PPP and the Monetary theory.

- **Casselian PPP**

  One of most popular ways of modeling long-term exchange rates is the Casselian view of Purchasing Power Parity. The long-term equilibrium relationship based on Casselian PPP is expressed as,

  \[ s_t = p_t - p_t^* - \gamma (i_t - i_t^* - \Delta s_{t+k}^e) + \xi_t, \]

  where

  - \( s_t \) is the nominal spot exchange rate defined as domestic currency per unit of foreign currency (e.g., USD per DEM and USD per FRF).
  - \( p_t \) is the domestic price level. (The asterisk in Equation [3.44] denotes a foreign variable.)
  - \( i_t \) is the yield on bonds with maturity \( k \).
  - \( \Delta s_{t+k}^e \) represents the expected change in the exchange rate over the subsequent \( k \) periods.
  - \( \xi_t \) is a random error term.

  We use the end-of-month spot exchange rate, the CPI, and the long-term government bond yield rate for \( s_t, p_t \), and \( i_t \), respectively. As a proxy variable for \( \Delta s_{t+k}^e \), we use net exports because the trade balance is considered to be a leading indicator of foreign exchange rate.

- **Monetary theory**

  The long-term equilibrium relationship based on monetary theory is expressed as,

  \[ s_t = m_t - m_t^* - \gamma (y_t - y_t^*) + \xi_t, \]

  where \( m_t \) indicates the money supply and \( y_t \) is real income.

  We proxy M1 for \( m_t \) and industrial production (IP) for \( y_t \). Since monetary theories based on expectation model argue that the equilibrium relationship in Equation [3.45] needs to be ad-

---

17 See MacDonald and Marsh (1997).
18 See Mark and Choi (1997).
justed by the expectation and the adjustment speed of exchange rate, we also include net exports as a proxy variable.

Based on the preceding two approaches, we can construct a broad specification (the so called extended monetary or general equilibrium approach), which includes the differences in the CPI, IP, M1, government bond yield rate, and net exports of the two countries as explanatory variables.

Table 3.3 shows which time series are used in the specifications for the exchange rate model in previous empirical studies. We do not consider the forward premium specification because many emerging markets do not have reliable forward data. Also, it is difficult to apply the cross-country cointegration specification, since the number of country-sets is large.

<table>
<thead>
<tr>
<th>Time series</th>
<th>Theory</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>Spot and 4-, 13-, 26-, and 52-week forward FX rates</td>
<td>Forward premium</td>
<td>Clarida and Taylor (1997)</td>
</tr>
<tr>
<td>Spot FX rates of CAD, DEM, FRF, GBP, and JPY</td>
<td>Cross-country cointegration</td>
<td>Lin and Tsay (1996)</td>
</tr>
<tr>
<td>Spot FX rate, money supply (M1 or M3), and real GDP</td>
<td>Monetary</td>
<td>Mark (1995)</td>
</tr>
<tr>
<td>Spot FX rate, CPI, M1, IP, manufacturing production index and employment index, short-term interest rate, government expenditure</td>
<td>General equilibrium model</td>
<td>Mark and Choi (1997)</td>
</tr>
<tr>
<td>Spot FX, CPI and long-term interest rate</td>
<td>Casselian PPP</td>
<td>MacDonald and Marsh (1997)</td>
</tr>
<tr>
<td>Spot FX, M1, IP, short-, and long-term interest rate</td>
<td>Monetary</td>
<td>MacDonald and Taylor (1994)</td>
</tr>
</tbody>
</table>

### 3.5.2 Interest rate

The behavior of the nominal interest rates of a country critically depends on the country’s openness to the rest of the world (degree of free capital movement). One extreme scenario assumes that the country is completely closed to the rest of the world. In this scenario, the standard Fisher approach, we can specify the nominal interest rate as

\[ i_t = r_t + \pi_t^e, \]

where \( i \) is the nominal interest rate, \( r \) is the real (ex ante) rate of interest, and \( \pi^e \) is the expected rate of inflation.

The real interest rate in turn can be specified as

\[ r_t = \rho - \lambda EMS_t + \omega_t, \]

where \( \rho \) is a constant and represents the long-term equilibrium real interest rate, \( EMS \) represents the excess money supply, and \( \omega \) is a random error term.

The excess money supply is calculated by subtracting the money demand (i.e., the desired equilibrium money stock), from the money supply or the actual money stock. The money demand is modeled as a function of the transaction volume or income level, and two opportunity costs of money holdings, the expected rate of inflation, and the interest rate. The empirical form for interest rate for the closed economy, assuming the above money demand function, then becomes

\[ \frac{dM}{M} = \alpha \frac{dY}{Y} + \beta \pi^e - \gamma r, \]

where \( \alpha, \beta, \gamma \) are parameters. For countries with free capital movement, the excess money supply can be expressed as a function of the foreign interest rate and the expected rate of inflation.

---

\[ \psi_t = \psi_0 + \psi_1 \log y_t + \psi_2 \log m_{t-1} + \psi_3 \pi_t + \psi_4 i_{t-1} + \epsilon_t, \]

where \( y_t \) is the income level and \( \epsilon_t \) is a random error term.

At the other extreme we assume that the economy is completely open to the rest of the world and capital flow is unimpeded. Then, domestic and foreign interest rates will be closely linked. In particular, in a world with no transaction costs and risk neutral agents, the following uncovered interest arbitrage relation will hold:

\[ i_t = i_t^* + \hat{\epsilon}_t, \]

where \( i_t^* \) is the world interest rate for a financial asset of the same characteristics (e.g., maturity) as the domestic instrument, and \( \hat{\epsilon}_t \) is the expected rate of change of the exchange rate during the maturity of the financial asset.

The preceding interest rate determinations are the two polar cases related to the degree of openness of the economy. However, in the general case, the economy allows foreign capital flows with some restrictions. Then, both open and closed economy factors will affect the behavior of domestic interest rates. The appropriate form for the general case becomes

\[ i_t = \delta_0 + \delta_1 \log y_t + \delta_2 \log m_{t-1} + \delta_3 \pi_t + \delta_4 i_{t-1} + \delta_5 (i_t^* + \hat{\epsilon}_t) + \epsilon_t. \]

As a proxy variable of the world interest rate, we use the U.S interest rates (for non-European countries) and Deutsche interest rates (for European countries). Also, the actual inflation rate and the spread of long-term and short-term interest rates are included as proxy variables of the expected inflation rate. Thus, the general case specification includes domestic short-, mid- and long-term interest rates, money supply, CPI, industrial production, change of foreign exchange rate, and U.S. or Deutsche interest rate.

Table 3.4 shows which time series were used in the specifications for the interest rate model in previous empirical studies. We have already incorporated the cross-country cointegration specification as including interest rates of core countries like the U.S. and Germany. However, we do not apply the yield curve cointegration specification because many emerging markets do not have reliable yield curve data.

<table>
<thead>
<tr>
<th>Table 3.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Specifications for the interest rate model in various empirical studies</td>
</tr>
<tr>
<td>Time series</td>
</tr>
<tr>
<td>Interest rates on DEM, FRF, GBP, JPY, and USD in the eurocurrency market</td>
</tr>
<tr>
<td>Domestic short-term interest rate, GDP, government deficit, inflation and foreign real interest rate (DEM and USD)</td>
</tr>
<tr>
<td>Domestic and foreign interest rates, price index, money supply, real income, and forward FX rate,</td>
</tr>
<tr>
<td>U.S. 1- to 12-month treasury bill yields</td>
</tr>
<tr>
<td>Domestic interest rate of time deposit, government expenditure, exports, M1, GDP deflator, price of imports, energy price, exchange rate, and eurodollar interest rate</td>
</tr>
<tr>
<td>Domestic short- and long-term interest rates and foreign short-term interest rates (DEM and USD)</td>
</tr>
</tbody>
</table>
3.5.3 **Equity index**

The present value model of stock valuation equates stock price \( P_t \) to the present discounted value of future expected dividends \( D_{1+j} \) as follows:

\[
[3.51] \quad P_t = E_t \sum_{j=0}^{\infty} \beta^j D_{t+j},
\]

where \( \beta \) is a discount factor used to discount future dividends, \( 0 < \beta < 1 \), and \( E_t \) is the conditional expectations operator based on information available to investors at time \( t \). Using log-linear approximations, we can convert Equation [3.51] to relate the dividend-price ratio \( d_t - p_t \) to the present value of the first difference in the log dividend \( \Delta d_t \):

\[
[3.52] \quad p_t = d_t + E_t \sum_{j=0}^{\infty} \phi^j \Delta d_{t+j},
\]

where \( d_t - p_t = \log(D_t) - \log(P_t) \) is the dividend-price ratio, \( \phi \) is the average ratio of the stock price to the sum of the stock price and the dividend, and \( \Delta d_t = \log(D_t) - \log(D_{t-1}) \) is the growth rate of an expected dividend. The equation states that the log price-dividend ratio is an expected discounted value of all future real dividend growth rates.

This simple present value model may not be sufficient to describe stock price behavior due to a time-varying discount factor, speculative bubbles, and the omission of other relevant variables. However, many empirical studies (e.g., Lee, 1996) show that the long-term cointegration relation based on the simple present value model holds. As a proxy variable of the current and future dividends, we use industrial production, price level, net export and foreign exchange rates. We also include the long-term interest rates as proxy variables of the discount factor.

3.5.4 **Commodity price**

A commodity price can be viewed in two ways:

1. As the price of a good that is consumed or used to make other goods and services. The price of a good is determined by the market equilibrium of demand and supply. The demand of a good is a function of income, the good’s price, and the prices of other goods. The supply of a good is a function of the good’s price and the cost of production.

2. As the price of an *asset that is traded in the financial market*. The present value model is the most basic description of rational asset pricing (Pindyck, 1993):

\[
[3.53] \quad P_t = \sum_{j=0}^{\infty} \beta^j E_t(q_{t+j}),
\]
where $P_t$ is the commodity spot price, $E_t(\varphi_{t+j})$ is the expected future payoff, and $\beta$ is the discount rate.

For a storable commodity, the payoff stream $\varphi_t$ is the convenience yield that accrues from holding inventories, i.e., the value of any benefits that inventories provide, including the ability to smooth production, avoid stockouts, and facilitate the scheduling of production and sales.

The arbitrage condition implies that the net marginal convenience yield is measured by forward price,

$$[3.54] \quad \varphi_{t,t+T} = (1 + r_{t+T})P_t - f_{t,t+T}$$

where $\varphi_{t,t+T}$ is the flow of marginal convenience yield net of storage costs over the period $t$ to $t + T$, $f_{t,t+T}$ is the forward price for delivery at $t + T$, and $r_{t+T}$ is the risk-free $T$-period interest rate. Using Equation [3.54], the present value model can be converted to specify a relationship between the spot price and the forward price of a commodity.

Considering both the demand and supply model and the present value model, we can construct a general regression form for commodity price:

$$[3.55] \quad \Delta P_t = \alpha_0 + \beta_1 P_{t-1} + \beta_2 f_{t-1} - \beta_3 + \beta_4 b_i Z_{t-1} \epsilon_t + \epsilon_t,$$

where the $Z_i$’s are any variables that might affect commodity price, including all the variables in the demand and supply functions.

We include U.S. industrial production as a proxy for income, the U.S. general price level as a proxy for the prices of other goods and the cost of production based on the demand and supply model. In addition, we include the forward rates of the commodity and the U.S. short-term interest rate according to the present value model. Since there are close comovements in a group of asset prices (e.g., energy, industrial metal, and precious metal, see Pindyck and Rotemberg, 1990), the prices of other commodities within the same group are included. We also include the trade-weighted USD index (International Financial Statistics [IFS] definition) because the commodity is priced in USD.

**Where can we collect all time series data used in our specification?** Appendix 3.B reports on the source of the time series data that LongRun uses to generate forecasts.

### 3.6 Economic regimes in forecasts

In the last two sections we presented time series models to forecast price distributions over long horizons. Now we are ready to discuss economic regimes, which establish the relationship between the variable we want to forecast, the model, and the historical time series used for the explanatory variables. A reasonable conclusion from this discussion is that the forecast quality of the model depends heavily on the time series. In other words, the forecasts are in some sense a mirror of the historical information that was used to estimate the model’s parameters. The inability to forecast well is not necessarily a weakness of the model since poor forecasts may reflect that either (1) the period of historical data used to estimate the models parameters is too short to represent the future regime, or (2) the historical data set used is so long that too many regimes are represented in the data; in this case, different economic regimes have different relationships between the variables that we want to forecast and the explanatory variables.

This section has three parts. In Section 3.6.1, we discuss general issues about structural regimes based on long-horizon forecasting and introduce some statistical tools that allow us to identify structural breaks, which divide a sample period into structural regimes. In Section 3.6.2 and
Section 3.6.3, we use actual data to explain how to identify structural breaks and how to estimate a model’s parameters so as to reflect a particular regime.

### 3.6.1 Accounting for economic regimes in forecasts

The idea of a regime is reasonable for financial prices because they are subject to monetary intervention from central banks and to other exogenous shocks to the financial markets. For any financial time series, we define a regime as an extended period of time over which a price or rate series remains mostly within a certain range of values (bands) and either trends upwards, downwards or remains constant. If we believe that a certain economic regime such as high inflation will exist over the next year, it follows that we should select a historical time series with the same or similar features. As a result, the parameter estimates based on our chosen series will incorporate our notion of what we believe will hold in the future.

**Using regimes in model estimation**

It is well documented that failure to capture changes in structural regimes leads to poor forecasts (Clements and Hendry [1996]). For example, if the only history available is a high inflation period, then we should not be surprised at our inability to forecast in a low inflation period. The issue of capturing economic regimes is relevant to all models whose predictions are based on historical information.

Here we present a simple example that shows how to incorporate economic regimes into forecasts in the context of the VECM model. There are effectively two ways to do this:

1. Adjust only the model’s constant term to reflect regimes.
2. Estimate all of the model’s parameters using only time series data associated with a particular regime.

In either case, users specify a regime and estimate either one parameter—the constant—or all the parameters, using data that is consistent with the regime. Note that we are not trying to forecast economic regimes, but only to identify them and then use them to estimate the model’s parameters.

We assume that we know when a particular regime begins and ends. While this seems to be a rather strong assumption, it actually is not. For example, business cycle dates are available for many countries, e.g., NBER business cycle dates for the United States. We introduce the following dummy variable $\delta_t$ to denote business cycle regimes. Let

$$\delta_t = \begin{cases} 1, & \text{economic expansion} \\ 0, & \text{economic recession} \end{cases}$$

Given these definitions, we can modify the VECM model, Equation [3.30], so that it becomes

$$\Delta X_t = \zeta \delta_t + \alpha Z_{t-1} + \Gamma_1 \Delta X_{t-1} + \epsilon_t,$$

In the practice of forecasting, rather than aggressively forecast the next regime, we passively assume the next regime to be identical or similar to the current regime. Our concern is to maintain consistency: either ensure that the time series used in parameter estimation belongs to the current regime, or implement a regime-change technique before applying time series from an unrelated regime to the current regime.
where $\zeta$ is a parameter. The first term of the right-hand side represents the effect of the regime on the asset price.

Rather than specifying the dummy variable in the intercept term, we apply the dummy variable to the regressors, $Z_t$. In other words, we estimate Equation [3.57] only over the time series associated with a particular historical period.

**B IDENTIFYING STRUCTURAL BREAKS**

Rather than guessing what the future may hold and then selecting a data set that is consistent with this view, we can use standard statistical tests to determine the most recent time period in which price history is stable. We present two structural break tests and apply them to our models as appropriate. The Recursive Johansen trace test focuses on the stability of cointegration vectors in the VECM and the Rolling Chow test focuses on the stability of coefficients.

- **Recursive Johansen trace test**

  Johansen proposes a trace test method for estimating the number of cointegration vectors.\(^2\) The trace test is a likelihood ratio test. Its null hypothesis states that there exists a maximum of $h$ cointegration vectors ($0 < h < n$, where $n$ is the dimension of VECM) against the alternative hypothesis of $n$ cointegration vectors. Appendix 3.A describes how to calculate Johansen trace test statistics.

  As we calculate the test statistics of the Johansen trace test recursively using rolling sample periods, we can identify structural breaks if they exist. If two test statistics that are calculated over two sample periods differ significantly, it implies that the number of cointegration vectors has changed. This would indicate the occurrence of an event that affected the stable cointegration relationship among the variables in the VECM.

- **Rolling Chow test**

  One of the most common tests of structural change is the Chow test. This application of the F-test tests the null hypothesis that one specified regression fits all the sample periods (no structural break) against the alternative hypothesis that some or all of the regression coefficients differ from each other in subperiods and thus introduce structural breaks.

  The test statistic is given by,

  \[
  [3.58] \quad \lambda_t = \frac{[SSR_T - (SSR_1 + SSR_2)]/k}{(SSR_1 + SSR_2)/T}.
  \]

  where

  $SSR_T$ denotes the sum of the squared residuals with restriction of null hypothesis, i.e., residuals are calculated from the regression on the entire $T$ observations.

  $SSR_1$ and $SSR_2$ denote the sum of the squared residuals without restriction of null hypothesis, i.e., residuals calculated from the regression on the first $t$ observations and remaining $T-t$ observations, respectively.

  $k$ is the number of parameters in the regression.

---

The test statistic $\lambda_t$ has an F-distribution with $(k, T-2k)$ degrees of freedom. As we calculate the test statistics of the Chow test repeatedly using rolling sample periods, we can identify structural breaks. If the test statistic exceeds the critical value and we can reject the null hypothesis, then we have found a structural change between two subperiods, which affects the parameter stability across the whole sample period.

### 3.6.2 Applying structural break tests

To illustrate structural break detection, we apply the Johansen and Chow tests to the USD per DEM exchange rate. We set the sampling window backward from November 1997 to January 1983. We then initialize the test procedure by estimating the test statistic on the set of observations from November 1997 to December 1993. Next, we widen the estimation period by recursively cumulating observations to the initial observation set, i.e., we estimate the second test statistic on the period from November 1997 to November 1993, the third test statistic on the period from November 1997 to October 1993, and so on. Finally, we estimate the last test statistic on the complete data set, i.e., from November 1997 to January 1983.

We plot the test statistics of the recursive Johansen trace test and the rolling Chow test. Chart 3.7 shows that the test statistics of the recursive Johansen trace test exceed the 5% critical value at five points, with June 1992 showing a structural break with the highest probability. This means that we can reject the null hypothesis of no structural break.

#### Chart 3.7

**Recursive Johansen Trace Test, USD per DEM exchange rate**

![Chart 3.7](image)

Chart 3.8 shows that the test statistics of the rolling Chow test exceed the 5% critical value in May 1991.
3.6.3 Incorporating structural breaks

Having identified structural breaks in the USD per DEM exchange rate, we can incorporate them into the forecasts of asset prices by setting a structural break dummy and including it as one of the deterministic variables \((D\) in Equation \([3.24]\)) in the VECM forecasting models.

The structural dummy is expressed as

\[ \delta_t = \begin{cases} 1, & \text{period } t \text{ is before structural break} \\ 0, & \text{period } t \text{ is after structural break.} \end{cases} \]

[3.59]

Chart 3.9 plots the USD per DEM exchange rate from January 1985 to December 1993.
Sec. 3.6 Economic regimes in forecasts 113

Chart 3.9
Structural break of USD per DEM exchange rate

The DEM steadily appreciated against the USD until early 1991. After that, the trend disappeared and moved up and down with almost the same probability after the break. The structural break identified by our methods matches well the change in the USD per DEM trend. The structural break dummy equals 1 for the samples before May 1991 and zero for the samples after that date.

Chart 3.10 plots two paths of the USD per DEM exchange rate forecasted by the VECM model. We assume that today is June 30, 1996 and forecast the USD per DEM exchange rate up to 24 months at monthly intervals. One path uses the structural break dummy, the other does not. We can see that the forecasted path with the structural break dummy slopes downward (DEM depreciation) in the same direction as the true foreign exchange rates, but the forecasted path without the structural break dummy slopes upward (DEM appreciation), in the opposite direction.

Because we use the sample of July 1987 to June 1996 (10 years) to calculate parameters for the model, the parameters that are calculated without the structural break dummy are still affected by the strong trend of DEM appreciation during July 1987 to May 1991 and provide the DEM appreciation path as a set of mean forecasts. However, the parameters with the structural break dummy can be free from the out-of-date appreciation trend. The structural break dummy therefore improves the accuracy of the mean forecasts for the USD per DEM exchange rate.
Before closing our discussion on the use of economic regimes in forecasting, we point out a limitation of the current econometric techniques in handling economic regimes—that is, their subjectivity. The prudent reader may already see from our example just how difficult it is to incorporate structural breaks into forecasts. First, even if statistical inference (e.g., recursive Johansen trace test and rolling Chow test) provides candidate structural breaks, the forecaster must have a general knowledge of the economy and must know when to select certain structural break (and which ones) and when to ignore all of them. In many cases, it is hard to validate the structural breaks suggested by statistical inference based on economic events. Second, even if a forecaster knows which structural break to use for given asset prices, she must decide, on the basis of intuition about future economic regimes, whether to incorporate the structural break into the forecasting model. As an example, the Korean won (USD per KRW) has a clear structural break in October 1997 of the Asian Crisis, but some forecasters may decide not to incorporate the structural break in forecasts for the next 2 years because they believe that in the near future the Korean economy will be free from the effects of the Asian Crisis. Because of such subjectivity, it is difficult for us to include our technique of accounting for economic regimes in a standard forecasting model. However, individual users can still adapt the technique to create forecasts based on their views of the future regime or to implement their views in stress testing.

### 3.7 Estimating the VECM

In this section we describe our methods for resolving the issues that we encountered in estimating the parameters of the VECM, and provide two examples of estimation. For a given time series $X_t$, we first determine the number of lags in the VARM by Akaike’s Information Criterion (AIC), then directly select the number of cointegration vectors in the VECM based on its forecasting performance. As an example, we apply VECM to USD per ZAR and USD per JPY and compare forecasts of the VECM to those of the random walk model.

#### 3.7.1 Selecting order of lags and cointegration vectors

To estimate the VECM, we need to select two parameters: the number of lags $p$ (in Equation [3.24]) that are included in the regressor, and the number of cointegration vectors $h$ (in Equation [3.28]) for VECM. To select the number of lags, we apply AIC to the estimation of VARM for a set of asset prices. First, we estimate the VARM on the level of variables as
where $X_t$ is an $n \times 1$ vector of $I(1)$ variables, which are selected according to the specifications of Section 3.5. We determine the number of lags $p$ based on the vector form of AIC (Harvey [1994], p. 251) as follows:

\[\text{minimize } AIC = \left| \hat{\Sigma} \right| \exp\left[\frac{(2u)}{T}\right].\]

where $\hat{\Sigma}$ is the estimated variance-covariance matrix of the error term ($\hat{\epsilon}_t$) and $u$ is the number of parameters in Equation [3.60].

If Equation [3.61] is minimized by using $p$ lags, then the number of lags in the VECM model (Equation [3.28]) is $(p - 1)$.

\[\Delta X_t = \theta + \Pi X_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta X_{t-j} + \epsilon_t.\]

To estimate the number of cointegration vectors $h$, it is common to use the Johansen trace test (described in Appendix 3.A). However, the test is not powerful enough and too sensitive to the model specifications. Lin and Tsay (1996) show that even for the cases in which the error correction term improves the forecasts, the best forecasts do not necessarily result from the model that uses the Johansen trace test for determining the number of cointegration vectors. Since our goal is to arrive at the best forecast, we select the number of cointegration vectors not from the Johansen trace test, but directly from forecasting performance. The method is a standard backtesting procedure and will be explained in Chapter 4.

### 3.7.2 USD per ZAR exchange rate example

Let us assume that today is June 30, 1996 and we need to forecast the USD per ZAR exchange rate up to 24 months at monthly intervals.

1. First, we estimate all the parameters of our VECM model using the last 8 years of foreign exchange data. The series in the data set are determined by the extended monetary specification.

2. Next, we input recent data into the estimated model and calculate monthly forecasts up to 24 months. Usually we use less than 1 year of data for forecasting, depending on the lag length included in the chosen model.

3. Last, we compare our results to that of the benchmark, the random walk with zero expected return model.

Chart 3.11 plots our forecasts: the mean forecasts, the 90% confidence interval of the forecasts, the 90% confidence interval of the random walk with zero expected return model, and the true values (we can know the true value only ex-post). Chart 3.11 shows how well VECM performs relative to the random walk with the zero expected return model; i.e., the mean forecasts track the direction of the steadily depreciating true values.
3.7.3 USD per JPY exchange rate example

Using the method of Section 3.7.2, we forecast the USD per JPY exchange rate. The results are plotted in Chart 3.12. The mean forecasts of VECM correctly predict the direction of the USD per JPY movement.

3.8 Summary

In this chapter we illustrated how to generate long-horizon scenarios that are derived from economic structure. Specifically, we presented time series forecasting models based on vector autoregression and error correction concepts. These models attempt to explicitly model relationships among time series.
Although a great deal of research supports the success of the econometric models, a growing body of literature refutes such claims. The next chapter addresses such criticism by assessing the forecasting performance of the various forecasting methods proposed in the last two chapters, i.e., random walk with zero expected return model, random walk with forward premium expected return model, and VECM.
Appendix 3.A  Estimating and testing the VECM

3.A.1 Full information maximum likelihood (FIML) estimation of VECM

Johansen suggests full information maximum likelihood (FIML) estimation of a system characterized by exactly $h$ cointegration relations.\(^{22}\) From Equation [3.28] of the VECM,

$$\Delta X_t = \theta + \Pi X_{t-1} + \sum_{j=1}^{p-1} \Gamma_j \Delta X_{t-j} + \epsilon_t \quad \epsilon_t \sim MVN(0, \Lambda_t),$$  

where $X_t$ is an $n \times 1$ vector of $I(1)$ variables. For simplicity, we include only a constant in the deterministic variables $D$ in Equation [3.28].

Equation [3.30] is a specific case of Equation [3.1]. For the estimation of Equation [3.30], we just need to replace $\Pi X_{t-1}$ with $\alpha Z_{t-1}$ and set $p = 1$.

To estimate VECM defined by Equation [3.1] using maximum likelihood, we calculate two auxiliary regressions:

$$\Delta X_t = \hat{\phi}_0 + \sum_{j=1}^{p-1} \hat{\Phi}_j \Delta X_{t-j} + \hat{\epsilon}_t$$  

and

$$X_{t-1} = \hat{\zeta}_0 + \sum_{j=1}^{p-1} \hat{G}_j \Delta X_{t-j} + \hat{\nu}_t.$$  

Next, we calculate the sample variance-covariance matrices of the OLS residuals $\hat{\epsilon}_t$ and $\hat{\nu}_t$;

$$\hat{\Sigma}_{\hat{\epsilon}\hat{\epsilon}} = \frac{1}{T} \sum_{t=1}^{T} \hat{\epsilon}_t \hat{\epsilon}_t',$$

$$\hat{\Sigma}_{\hat{\epsilon}\hat{\nu}} = \frac{1}{T} \sum_{t=1}^{T} \hat{\epsilon}_t \hat{\nu}_t',$$

and

$$\hat{\Sigma}_{\hat{\nu}\hat{\nu}} = \hat{\Sigma}_{\hat{\epsilon}\hat{\epsilon}}.$$  

\(^{22}\) This appendix draws heavily from Hamilton (1994), pp. 635–45.
From these, we find the eigenvalues of the matrix

\[ \Sigma_{VV}^{-1} \Sigma_{uv} \Sigma_{uu} \Sigma_{uv} \]

with the eigenvalues ordered \( \hat{\lambda}_1 > \hat{\lambda}_2 > ... > \hat{\lambda}_n \).

The maximized value of the log likelihood function subject to the constraint that there are \( h \) (0 < \( h \) < \( n \), where \( n \) is dimension of VECM) cointegration vectors is given by

\[ L_0^* = -\left( \frac{TN}{2} \right) \log(2\pi) - \left( \frac{TN}{2} \right) \log|\hat{\Sigma}_{uu}| - \left( \frac{T}{2} \right) \sum_{i=1}^{h} \log(1 - \hat{\lambda}_i) \].

Let \( \hat{a}_1, \hat{a}_2, \ldots, \hat{a}_h \) denote the \( n \times 1 \) eigenvectors of Equation [3.A.8] associated with the \( h \) largest eigenvalues. These provide a basis for the space of cointegrating relations; i.e., the maximum likelihood estimate is that any cointegrating vector can be written in the form

\[ a = b_1 \hat{a}_1 + b_2 \hat{a}_2 + ... + b_h \hat{a}_h \]

for some choice of scalars \( (b_1, b_2, \ldots, b_h) \). Johansen suggested normalizing these vectors \( \hat{a}_i \) so that \( \hat{a}_i ' \Sigma_{vv} \hat{a}_i = 1 \). Collect the first \( h \) normalized vectors in an \( n \times h \) matrix \( \hat{A} \):

\[ \hat{A} = [\hat{a}_1 \quad \hat{a}_2 \quad ... \quad \hat{a}_h] \]

Then the Maximum Likelihood Estimator (MLE) of \( \Pi \) is given by \( \hat{\Pi} = \hat{\Sigma}_{uv} \hat{A} \hat{A}' \). The MLE of \( \Gamma_j \) for \( j = 1, 2, \ldots, p-1 \) is \( \hat{\Gamma}_j = \Phi_j - \Pi \hat{\Gamma}_j \), the MLE of \( \theta \) is \( \hat{\theta} = \Phi_0 - \Pi \hat{\theta}_0 \), and the MLE of covariance matrix \( \Lambda \) is

\[ \hat{\Lambda} = (1/T) \sum_{t=1}^{T} (\hat{u}_t - \hat{\Pi} \hat{v}_t)(\hat{u}_t - \hat{\Pi} \hat{v}_t)' \].

### 3.A.2 Johansen trace test

The maximized value of the log likelihood restricted by the null hypothesis of a maximum of \( h \) cointegration vectors (0 < \( h \) < \( n \), where \( n \) is the dimension of VECM) against the alternative equal to \( n \) is calculated from Equation [3.A.9].

The unrestricted log likelihood is then,

\[ L_A^* = -\left( \frac{TN}{2} \right) \log(2\pi) - \left( \frac{TN}{2} \right) \log|\hat{\Sigma}_{uu}| - \left( \frac{T}{2} \right) \sum_{i=1}^{n} \log(1 - \hat{\lambda}_i) \].

A test statistic of likelihood ratio test would then be \( 2(L_A^* - L_0^*) \).
Chapter 3. Forecasts based on economic structure

Appendix 3.B Historical time series data

Historical time series of prices and rates are the most widely available type of data. This data falls into two general categories: financial and economic. Financial time series data includes spot and forward rates and prices of financial assets, whereas economic time series data includes macroeconomic variables such as measures of output (e.g., industrial production) and money supply (e.g., M1). Table 3.B.1 reports on the historical time series that LongRun uses to generate forecasts. Chapter 9 of the RiskMetrics Technical Document details the definitions and sources of the time series.

Table 3.B.1
Historical time series
Frequency: monthly. Source: Reuters

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<th>Country</th>
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<th>Exchange Rate Data</th>
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</table>

† FX: foreign exchange rates, Spot: spot exchange rates; FWD1m: 1-month-ahead forward exchange rates; FWD2m: 2-months-ahead forward exchange rates, and so on.
‡ Trade weighted exchange rate index (IFS definition).
Table B.1 (continued)

Historical time series

Frequency: monthly. Source: Reuters

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<th>Country</th>
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<td>• 1986:01</td>
<td>• • • • 1995:01</td>
</tr>
</tbody>
</table>

§ Generally, rates are treasury bill and treasury bond yield rates. For the countries that do not have treasury bill and bond yield rates, rates are money market (M) and swap (S) rates.

** Spot: spot prices; FUT1m: 1-month-ahead future prices or first nearby future prices; FUT3m: 3-months-ahead future prices or third nearby future prices, and so on.
Chapter 4. Backtesting LongRun’s forecasting models

How good are LongRun’s forecasting models? One way to answer this question is to evaluate the quality of out-of-sample predictions produced by the various forecasting models. In this chapter we demonstrate the ability of our models to forecast foreign exchange rates, interest rates, and equity and commodity prices over long horizons. We calculate numerous statistics to gauge the effectiveness of our models. Specifically, we compare mean forecasts and confidence interval forecasts across the four models we proposed, i.e., the random walk model with zero expected return, the random walk model with forward premium expected return using historical volatility, the random walk model with forward premium expected return using implied volatility, and the VECM. Our results show that for many price series, VECM and the two random walk models with forward premium expected returns are able to consistently outperform the random walk model with zero expected return.

The evaluation procedure consists of the following steps:

1. Dividing the sample period into in-sample and out-of-sample.
2. Estimating the parameters of each model using the in-sample data.
3. Forecasting the asset prices using estimated parameters.
4. Comparing forecasts with the out-of-sample data.

4.1 Performance assessment framework

4.1.1 Sample period

We divide the sample period into in-sample and out-of-sample. The in-sample is used to estimate the parameters of the model, whereas the out-of-sample is used as a data set of true asset prices to which the forecasts will be compared. We set the size of the in-sample to 96 monthly observations (an 8-year window), and the size of the out-of-sample to 12 monthly observations. In this chapter, we reduce the maximum forecast horizon from 24 months to 12 months to increase the backtesting sample number.

Generally we set the analysis date to January 1995, when the first forecast is made. We then use an 8-year period of data from January 1987 through December 1994 (in-sample) to estimate the parameters of the model, and finally, we forecast the asset prices up to 12 months ahead, for the period January 1995 through December 1995. We then compare our forecasts to the out-of-sample data in the same period, January 1995 through December 1995.

As the analysis date moves forward by 1 month, both the in-sample window and the out-of-sample window are rolling updated by 1 month until the analysis date reaches September 1997. The rolling updated in-sample allows us to use the latest 8 years of data to calculate parameters of the models (see Chapter 3). Since we update the analysis date from January 1995 to September 1997, we have a total of 33 backtesting samples for each asset price.

---

1 For some asset prices, we reduce the in-sample and out-of-sample period, since their historical data is limited.
4.1.2 Benchmarks

We use the RiskMetrics random walk model for the short horizon as a benchmark; that is, we take the forecasts of the VECM and both random walk models with forward premium expected return and compare them to the forecasts of the random walk with zero expected return model.

4.1.3 Accuracy measures

Ideally, the forecasted distribution is the best criterion for comparing models; however, there is no simple way to compare forecasted distributions. We suggest three alternative accuracy measures derived from the forecasted distribution. Two of the measures, Mean Absolute Error (MAE) and Proportion of Correct Direction (PCD), indicate the accuracy of mean forecasts, while the third measure, Percentage of Outliers from Confidence Interval (POC) indicates the robustness of confidence interval forecasts.

- Mean Absolute Error
  The ratio of the mean absolute error of an estimated model to the mean absolute error of a benchmark model shows how well the estimated model performs relative to the benchmark model in forecasting asset prices.\(^2\) The MAE ratio is expressed as

\[
[4.1] \quad \text{MAE} = \frac{\frac{1}{T} \sum_{t=1}^{T} |EM_t - TR_t|}{\frac{1}{T} \sum_{t=1}^{T} |BM_t - TR_t|}
\]

where \(EM_t\) is the asset price forecasted by the estimated model, \(BM_t\) is the asset price forecasted by the benchmark model (i.e., random walk with zero expected return), and \(TR_t\) is the true asset price.

- Proportion of Correct Direction
  If an analyst is interested only in the direction of future asset prices, she can evaluate the different models by comparing their PCD’s, i.e., the percentage of time that each model predicts the correct direction in which an asset price moves.

\[
[4.2] \quad \text{PCD} = (1/T) \times (\text{number of correct-direction forecasts of the estimated model}).
\]

The PCD of a random walk without drift is 0.5 in a large sample if the time series does not have a time trend.

- Percentage of Outliers from Confidence Interval
  The percentage of outliers from the confidence interval compares the robustness of the confidence interval forecasts predicted by each model as follows:

\[
[4.3] \quad \text{POC} = (1/T) \times (\text{number of samples outside of estimated confidence interval}).
\]

\(^2\) Mean Squared Error (MSE) also shows how near the forecasts are to true values. However, one problem of MSE is its extreme sensitivity to outliers.
If we construct 90% confidence intervals, then the fraction of samples outside the estimated confidence intervals should be 10%. If it is over 10% (or under 10%), then the model underestimates (or overestimates) the true risk of asset price fluctuations.

### 4.1.4 Non-overlapping backtesting samples

Which sample do we use to calculate accuracy measures — overlapping or non-overlapping? This is one of the problems that makes the backtesting of long-horizon forecasting difficult. Since we are able to calculate samples large enough to backtest, we use only non-overlapping backtesting samples for the short-horizon forecasting. However, there exists a trade-off to using overlapping and non-overlapping backtesting samples for long-horizon forecasting.

In the case of overlapping backtesting samples, we can obtain a large number of backtesting samples regardless of how far the forecast horizon is projected. Generally, larger numbers of samples yield more accurate conclusions. However, overlapping backtesting samples make the consecutive forecasts autocorrelated.

Consider the following example: First, we calculate a 6-month forecast for the USD per FRF exchange rate using the most recent data set in each month. Chart 4.1 plots the 6-month forecast at each month starting from July 1995 (i.e., the first analysis date is set at January 1995) and ending March 1998 (i.e., the last analysis date is set at September 1997). As a result, we have a total of 33 overlapping samples of 6-month forecasts.

Using the forecasted means and standard deviations we construct 90% confidence intervals. The bars in Chart 4.1 represent the 90% confidence intervals of the VECM. We observe six outliers: the rates of December 1995, February 1996, February 1997, June 1997, July 1997, and August 1997. Among the six outliers, three are consecutive. We can easily find some degree of autocorrelation between outliers in the overlapping sample. If the realized USD per FRF exchange rate becomes an outlier in July 1997, then the probability of an outlier in August 1997 is even higher. Next we choose only non-overlapping samples of 6-month forecasts (see Chart 4.2). The sample size shrinks from 33 to 6. Among six backtesting samples, we observe one outlier: the rates of July 1997.

As the forecast horizon becomes longer, the autocorrelation between consecutive forecasts produced by overlapping samples becomes stronger because the longer period is shared by consecutive forecasts. Also, the autocorrelation is significant for the large shocks that occurred during the forecasting horizon. In this chapter, to avoid autocorrelation produced by overlapping samples, we will use only non-overlapping samples for backtesting. The price of using non-overlapping backtesting samples, is that the backtesting sample size is reduced for the longer forecast horizons, as shown in Table 4.1.

| Table 4.1 |
| Sample size of non-overlapping backtesting as a function of horizon |
| Horizon, months | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| Sample size | 33 | 17 | 11 | 9 | 7 | 6 | 5 | 5 | 4 | 4 | 3 | 3 |
Chapter 4. Backtesting LongRun’s forecasting models

Chart 4.1
USD per FRF exchange rate, overlapping samples at 6-month forecast horizons

Chart 4.2
USD per FRF exchange rate, non-overlapping samples at 6-month forecast horizons
4.2 Assessing the accuracy of mean forecasts

4.2.1 An example

Using the performance assessment framework in Section 4.1, we present an example that shows how the accuracy of mean forecasts can be evaluated. To illustrate our point, we use forecasts of the USD per FRF exchange rates. Table 4.2 reports the MAE and the PCD by forecast horizon for both models, the VECM and the random walk with forward premium expected return. Note that the random walk with forward premium expected return model using historical volatility and the same model using implied volatility share the forward premium as mean forecasts. Therefore, we do not distinguish between the two models in this section. As we have already discussed in Section 4.1, we use the random walk with zero expected return model as a benchmark.

Table 4.2
Accuracy of mean forecasts, USD per FRF

Benchmark: Random walk with zero expected return model

<table>
<thead>
<tr>
<th>Horizon (months)</th>
<th>MAE</th>
<th>PCD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>VECM</td>
<td>RWF</td>
</tr>
<tr>
<td>1</td>
<td>1.2164</td>
<td>0.9969</td>
</tr>
<tr>
<td>2</td>
<td>1.1646</td>
<td>1.0029</td>
</tr>
<tr>
<td>3</td>
<td>1.2789</td>
<td>0.9784</td>
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<tr>
<td>4</td>
<td>0.8978</td>
<td>1.0088</td>
</tr>
<tr>
<td>5</td>
<td>1.5479</td>
<td>0.9391</td>
</tr>
<tr>
<td>6</td>
<td>1.0511</td>
<td>0.9969</td>
</tr>
<tr>
<td>7</td>
<td>1.0600</td>
<td>1.0348</td>
</tr>
<tr>
<td>8</td>
<td>0.9434</td>
<td>1.0080</td>
</tr>
<tr>
<td>9</td>
<td>1.3510</td>
<td>0.9812</td>
</tr>
<tr>
<td>10</td>
<td>1.3622</td>
<td>0.9481</td>
</tr>
<tr>
<td>11</td>
<td>0.4976</td>
<td>0.9788</td>
</tr>
<tr>
<td>12</td>
<td>0.7066</td>
<td>1.0049</td>
</tr>
</tbody>
</table>

The MAE and PCD values are interpreted as follows:

- MAE shows how closely the forecasted value approaches the true value at each forecast horizon. In our example, at the 12-month horizon, the MAE of the VECM is 0.7066 and the MAE of the random walk with forward premium expected return model is 1.0049. That is, the estimated VECM forecasts exchange rates with a mean absolute error that is 70.66% of the error produced by the benchmark, while the estimated random walk with forward premium expected return model forecasts with an MAE that is 100.49% of the error produced by the benchmark. If an analyst now using the benchmark to model USD per FRF exchange rates switches to the VECM, he can expect to improve the accuracy of the mean forecasts by 29.34% (= 100.00% − 70.66%).

- PCD shows the proportion of correct-direction forecasts. In the FRF example, at the 12-month horizon, the PCD of the VECM is 0.6667 and the PCD of the random walk with forward premium expected return model is 0.3333, which means that the estimated VECM provides correct-direction mean forecasts with a probability of 66.67%, while the estimated random walk with forward premium expected return model forecasts correct direction with a probability of 33.33%. The PCD arising from the benchmark averages to 0.5000. Switching from the benchmark to the estimated VECM to forecast USD per FRF exchange rates can increase the probability of correct-direction forecasts by 16.67% (= 66.67% − 50.00%).
4.2.2 Results by asset class

We carried out estimation and backtesting for the accuracy of mean forecasts across 65 time series of four asset classes, following the procedure in our example of the USD per FRF exchange rate (Section 4.2.1). We needed to decide which forecast horizon to use, given that in some cases, the performance of a model is sensitive to the forecast horizon. Generally, the random walk based models perform better in short-term forecasting, whereas VECM performs better in long-term forecasting, since it contains a long-term error correction term that the random walk model lacks. In this section, we set the forecast horizon to 1, 3, 6, and 12 months, and report the model that showed the best mean forecasting performance.

A FOREIGN EXCHANGE

We applied the backtesting procedure described in the previous section to the mean forecasts of the following currencies:3

- Canadian dollar (CAD)
- Swiss franc (CHF)
- Dutch guilder (NLG)
- French franc (FRF)
- Danish kroner (DKK)
- Norwegian kroner (NOK)
- German deutsche mark (DEM)
- Spanish peseta (ESP)
- New Zealand dollar (NZD)
- British sterling (GBP)
- Finnish mark (FIM)
- Philippine peso (PHP)
- Italian lira (ITL)
- Indonesian rupiah (IDR)
- Portuguese escudo (PTE)
- Japanese yen (JPY)
- Irish pound (IEP)
- Swedish krona (SEK)
- Austrian shilling (ATS)
- Korean won (KRW)
- Singapore dollar (SGD)
- Australian dollar (AUD)
- Mexican peso (MXN)
- Thailand baht (THB)
- Belgian franc (BEF)
- Malaysian ringgit (MYR)
- South African rand (ZAR)

Since monthly industrial production (IP) figures are not available for Australia, Indonesia, Sweden, Singapore, Switzerland, and Thailand, we use the share price index (SP) as a proxy variable to represent the real income for these countries. Table 4.3 summarizes our estimated results. The “Best model” column shows the best mean forecasting model based on the MAE and PCD at the 1-, 3-, 6-, and 12-month forecast horizons. In the total of 27 exchange rate series, the models provide the best mean forecasts for the time series as follows: The random walk with zero expected return provides the best mean forecasts for 10 series; the random walk with forward premium expected return, 6 series, and the VECM, 11 series.

---

3 All foreign exchange rates are measured against the U.S. dollar (e.g., USD per DEM, USD per FRF).
### Table 4.3
Accuracy of mean forecasts, foreign exchange rate

<table>
<thead>
<tr>
<th>Country</th>
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<th>Horizon</th>
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<td>RWF</td>
<td>VECM</td>
<td>RWF</td>
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<td>0.4546</td>
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<tr>
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B INTEREST RATE

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Accuracy of mean forecasts, interest rate

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* 9-month horizon
In Table 4.5, we apply the models to the equity indices of the U.S., Canada, France, Germany, U.K., Italy, Japan, Austria, Australia, Denmark, Spain, Finland, Ireland, Korea, Netherlands, Norway, Philippines, Sweden, and South Africa. Since monthly industrial production (IP) is not available for Australia and Sweden, it is not included in the model estimation. For the equity index forecasts, we applied only the random walk with zero expected return model and the VECM. In the total of 19 equity indices, the random walk with zero expected return provides the best mean forecasts for 8 series, and the VECM for 11 series.

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<td></td>
<td>6</td>
<td>1.2315</td>
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<td>0.9202</td>
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<tr>
<td>GBP</td>
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<td>0.9745</td>
<td>0.6452</td>
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<td>VECM</td>
<td>1</td>
<td>0.9641</td>
<td>0.6774</td>
</tr>
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<td>0.4546</td>
<td></td>
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<td>3</td>
<td>1.1439</td>
<td>0.6364</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>0.8557</td>
<td>0.6667</td>
<td></td>
<td></td>
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<td>1.0648</td>
<td>0.6667</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>0.8723</td>
<td>0.6667</td>
<td></td>
<td></td>
<td>12</td>
<td>0.9161</td>
<td>0.6667</td>
</tr>
<tr>
<td>ITL</td>
<td>VECM</td>
<td>1</td>
<td>1.0285</td>
<td>0.5807</td>
<td>NOK</td>
<td>VECM</td>
<td>1</td>
<td>0.8714</td>
<td>0.7419</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>0.7819</td>
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<td></td>
<td></td>
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<td>0.6592</td>
<td>0.8333</td>
<td></td>
<td></td>
<td>6</td>
<td>0.8011</td>
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<td></td>
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<td>12</td>
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<td></td>
<td>12</td>
<td>0.7625</td>
<td>1.0000</td>
</tr>
<tr>
<td>JPY</td>
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<td>1</td>
<td>1.1005</td>
<td>0.5161</td>
<td>PHP</td>
<td>RWZ</td>
<td>1</td>
<td>1.5262</td>
<td>0.5484</td>
</tr>
<tr>
<td></td>
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<td>3</td>
<td>1.0853</td>
<td>0.5455</td>
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<td>1.7706</td>
<td>0.3636</td>
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<td>1.0233</td>
<td>0.6667</td>
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<td>1.7462</td>
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<td></td>
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<td></td>
<td></td>
<td>12</td>
<td>3.2859</td>
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<tr>
<td>ATS</td>
<td>RWZ</td>
<td>1</td>
<td>1.3854</td>
<td>0.5484</td>
<td>SEK</td>
<td>RWZ</td>
<td>1</td>
<td>1.0155</td>
<td>0.6129</td>
</tr>
<tr>
<td></td>
<td></td>
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<td>1.2710</td>
<td>0.6364</td>
<td></td>
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<td>3</td>
<td>1.0084</td>
<td>0.5455</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>1.2130</td>
<td>0.6667</td>
<td></td>
<td></td>
<td>6</td>
<td>1.3272</td>
<td>0.3333</td>
</tr>
<tr>
<td></td>
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<td>0.6023</td>
<td>1.0000</td>
<td></td>
<td></td>
<td>12</td>
<td>0.9131</td>
<td>0.6667</td>
</tr>
<tr>
<td>AUD</td>
<td>RWZ</td>
<td>1</td>
<td>1.1203</td>
<td>0.5484</td>
<td>ZAR</td>
<td>VECM</td>
<td>1</td>
<td>1.0297</td>
<td>0.7097</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>0.9397</td>
<td>0.6364</td>
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<td>3</td>
<td>0.8808</td>
<td>0.9091</td>
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<tr>
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<td>6</td>
<td>1.1981</td>
<td>0.3333</td>
<td></td>
<td></td>
<td>6</td>
<td>0.9283</td>
<td>0.8333</td>
</tr>
<tr>
<td></td>
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<td>12</td>
<td>1.3314</td>
<td>0.3333</td>
<td></td>
<td></td>
<td>12</td>
<td>1.2065</td>
<td>0.6667</td>
</tr>
</tbody>
</table>
In Table 4.6, we backtested the commodity prices\(^4\) of NY Harbor #2 unleaded gas (UNL), WTI Light Sweet Crude (WTI), Heating Oil (HTO), Aluminum (ALU), Nickel (NIC), Copper (COP), Gold (GLD), Silver (SLV), and Platinum (PLA). In the total of 9 commodity price series, the models provide the best mean forecasts for the series as follows: The random walk with zero expected return provides the best mean forecasts for 4 series; the random walk with forward premium expected return, 3 series, and VECM, 2 series.

**Table 4.6**

<table>
<thead>
<tr>
<th>Commodity</th>
<th>Best model</th>
<th>Horizon</th>
<th>MAE</th>
<th>PCD</th>
</tr>
</thead>
<tbody>
<tr>
<td>UNL</td>
<td>VECM</td>
<td>1</td>
<td>0.9552</td>
<td>0.9552</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>0.8583</td>
<td>0.8583</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>0.9934</td>
<td>0.9934</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>1.0797</td>
<td>1.0797</td>
</tr>
<tr>
<td>WTI</td>
<td>RWZ</td>
<td>1</td>
<td>1.2726</td>
<td>1.2726</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>1.2432</td>
<td>1.2432</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>1.3538</td>
<td>1.3538</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>1.3734</td>
<td>1.3734</td>
</tr>
<tr>
<td>HTO</td>
<td>RWZ</td>
<td>1</td>
<td>1.0474</td>
<td>1.0474</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>1.0261</td>
<td>1.0261</td>
</tr>
<tr>
<td></td>
<td></td>
<td>6</td>
<td>1.0881</td>
<td>1.0881</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>1.3590</td>
<td>1.3590</td>
</tr>
<tr>
<td>ALU</td>
<td>RWZ</td>
<td>1</td>
<td>1.2330</td>
<td>1.2330</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>1.3023</td>
<td>1.3023</td>
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<td></td>
<td>6</td>
<td>1.6762</td>
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<td></td>
<td></td>
<td>12</td>
<td>2.5322</td>
<td>2.5322</td>
</tr>
<tr>
<td>NIC</td>
<td>RWF</td>
<td>1</td>
<td>1.0577</td>
<td>1.0577</td>
</tr>
<tr>
<td></td>
<td></td>
<td>3</td>
<td>1.0384</td>
<td>1.0384</td>
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<td></td>
<td></td>
<td>6</td>
<td>1.2554</td>
<td>1.2554</td>
</tr>
<tr>
<td></td>
<td></td>
<td>12</td>
<td>1.5590</td>
<td>1.5590</td>
</tr>
</tbody>
</table>

\(^4\) First nearby future price is used for UNL, WTI, and HTO, and spot price is used for others.

**4.2.3 Summary of results**

Table 4.7 shows the number of series that are forecasted the best by each model based on MAE and PCD. None of the models dominates the others across four asset classes in the accuracy of mean forecasts.

**Table 4.7**

<table>
<thead>
<tr>
<th>Asset class</th>
<th>Overall no. of time series</th>
<th>Foreign Exchange</th>
<th>Interest Rate</th>
<th>Equity Index</th>
<th>Commodity Price</th>
</tr>
</thead>
<tbody>
<tr>
<td>RWZ</td>
<td>25</td>
<td>10</td>
<td>3</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>RWF</td>
<td>10</td>
<td>6</td>
<td>1</td>
<td>–</td>
<td>3</td>
</tr>
<tr>
<td>VECM</td>
<td>30</td>
<td>11</td>
<td>6</td>
<td>11</td>
<td>2</td>
</tr>
<tr>
<td>Total no. of time series</td>
<td>65</td>
<td>27</td>
<td>10</td>
<td>19</td>
<td>9</td>
</tr>
</tbody>
</table>
4.3 Assessing the robustness of confidence interval forecasts

4.3.1 An example

Consider the previous example of the 6-month forecast for the USD per FRF exchange rate in Chart 4.2. We have a total of six samples of 6-month forecasts. Using the forecasted means and standard deviations we construct 90% confidence intervals. We then determine how many of the ex-post true values of the USD per FRF exchange rate fall out of the forecasted 90% confidence intervals. We observe one outlier: the rates of July 1997.

Next we calculate the POC to backtest how robust are the confidence intervals. In this example, the POC of the VECM is calculated to be 16.67% (≈1/6 × 100). The calculated POC is higher than the expected POC at our risk level at the 90% confidence interval.

We can apply this method to the random walk with zero expected return, the random walk with forward premium expected return using historical volatility, and the VECM in the 1- to 12-month forecast horizon. Table 4.8 shows the number of outliers and the calculated POC across the models and horizons in the USD per FRF example.

Table 4.8
USD per FRF exchange rate, robustness of confidence interval forecasts

<table>
<thead>
<tr>
<th>Outliers</th>
<th>RWZ</th>
<th>RWF</th>
<th>VECM</th>
</tr>
</thead>
<tbody>
<tr>
<td>12345678</td>
<td>3</td>
<td>4</td>
<td>9</td>
</tr>
<tr>
<td>Total Samples</td>
<td>33</td>
<td>17</td>
<td>11</td>
</tr>
<tr>
<td>POC</td>
<td>0.0909</td>
<td>0.1212</td>
<td>0.2727</td>
</tr>
<tr>
<td>RWZ</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>POC</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>RWF</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>POC</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>VECM</td>
<td>0.1176</td>
<td>0.1667</td>
<td>0.2500</td>
</tr>
<tr>
<td>POC</td>
<td>0.1176</td>
<td>0.2000</td>
<td>0.0000</td>
</tr>
</tbody>
</table>

4.3.2 Results for historical volatility

It is difficult to backtest individual asset prices using POC. The sample size is too small to represent both tails of the distribution. In our estimates, we have only three samples for the 12-month forecast horizon. As a result, the weight of a single outlier is 33.33% (≈1/3 × 100), a condition that cannot provide a smooth tail in the distribution. If we set 90% confidence intervals and all the three samples are inside the confidence intervals, the POC is calculated to be 0.00% and is far less than the expected POC of 10% at our risk level, 10%. On the contrary, if one of the three samples is outside of the confidence intervals, the POC is calculated to be 33.33% and is far larger than the expected POC at our risk level, 10%. Thus, it is impossible to test the robustness of confidence intervals.

To smooth out the tails of the distribution, we create a large pool of asset prices when we calculate the POC of each forecast horizon. We use 65 asset prices, where each asset price is made up of three samples for the 12-month forecast horizon, giving us a total of 195 samples (≈65 × 3). The weight of one outlier is therefore 0.51% (≈1/195 × 100), which is small enough to yield a

---

Footnote: 5 One example of a panel data approach for evaluation methods is given in Lopez and Saidenberg (1998). Given the long horizons typical of credit risk models, only a few forecasts are available with which to analyze a model’s performance. They overcome the problem of credit risk model evaluation by adding cross sectional simulation.
smooth tail. In this case, if the number of outliers is around 20, the POC is calculated to be near 10% and is consistent to the expected POC of 10% at our risk level, 10%. Then, we can know that the model provides robust confidence intervals.

Chart 4.3 shows the backtesting results of a pool of 65 prices across four asset classes.

*Chart 4.3*
**Robustness of confidence interval forecasts**
*Pool of 65 prices across four asset classes*

If a model provides a robust confidence interval, the POC should be 10%. In Table 4.9, we compare the robustness of confidence intervals between the models.

*Table 4.9*
**POC at 90%**

<table>
<thead>
<tr>
<th>Model</th>
<th>POC</th>
</tr>
</thead>
<tbody>
<tr>
<td>RWZ</td>
<td>3.23–8.60%</td>
</tr>
<tr>
<td>RWF</td>
<td>5.40–14.63%</td>
</tr>
<tr>
<td>VECM</td>
<td>6.64–10.48%</td>
</tr>
</tbody>
</table>

The POC of VECM is closest to 10%. However, the random walk with zero expected return is biased to overestimate (less than 10% outliers) the true risk across almost all the forecast horizons.

To formally test our hypothesis, we construct the band of critical values based on the binomial distribution method suggested by Kupiec (1995). If the calculated POC is within the band of critical values, we cannot reject the null hypothesis that the probability of outliers from the confidence interval constructed by the model equals the specified percent $\alpha$. Chart 4.3 shows the band of critical values of 10%. As the forecast horizon becomes longer, the backtesting sample size is reduced (see Section 4.1.4, “Non-overlapping backtesting samples”) and the band of critical values widens.

The random walk with zero expected return, the random walk with forward premium expected return using historical volatility, and the VECM generate a POC for each of the 12 horizons, of which we reject 10, 4, and 2 POCs, respectively. The hypothesis test shows that the VECM and the random walk with forward premium expected return using historical volatility provide robust confidence intervals.
4.3.3 Results for implied volatility

It is hard to backtest the robustness of confidence interval forecasts produced by implied volatility because of the availability of data. We construct only the random walk model with forward premium expected return using implied volatility for six foreign exchange series: AUD, CAD, CHF, DEM, GBP, and DEM. Chart 4.4 plots the POCs of four forecasting models for the six FX rates. All the calculated POCs of the random walk model with forward premium expected return using implied volatility are within the band of critical values. It means that implied volatility also provides robust confidence intervals.

Chart 4.4
Robustness of confidence interval forecasts, pool of six FX rates

4.4 Conclusion

Although a good deal of research asserts the success of long-term forecasting models for producing accurate forecasts, there also exists a growing body of literature that refutes such claims. Many of these papers criticize the statistical techniques that are used to evaluate forecast accuracy. To address such criticism, we clearly defined backtesting procedures and predetermined performance assessment criteria. Our estimation and backtesting results across 65 time series of four asset classes point toward the relative success of our forecasting models, which extract information from the current market or economic structure. In backtesting the accuracy of mean forecasts and the robustness of confidence interval forecasts, the random walk with forward premium expected return model using historical volatility and implied volatility and VECM provide better forecasts than the random walk with zero expected return model.

---

6 As an example, see Bekaert and Hodrick (1992).
Chapter 5. Scenario simulation

Up to this point we have described various ways to forecast distributions over long horizons. Our attention now turns to producing scenarios consisting of paths of monthly and daily prices that are consistent with the forecast distributions and reflect the correlation between these prices. In this chapter we introduce a routine to simulate prices from the current period through the last forecast date. Briefly, this routine consists of the following four steps:

1. Using the results presented in Chapters 2 and 3, estimate the distributions for the prices of each asset of interest (i.e., foreign exchange, interest rate, equity, and commodity) at each forecast date.

2. Using the forecasted distributions from Step 1 and the historical correlation among monthly prices, simulate a monthly path of prices for each asset. (Level I simulation).

3. Generate a correlated set of random variables in a way that reflects their historical daily correlation. Then, using the generated random variables and each simulated price path from Step 2, construct a corresponding path of daily prices in a way that the forecasted monthly price distributions are maintained. (Level II simulation).

4. Repeat Steps 2 and 3 until you have a reasonable estimate of the monthly and daily evolution of prices from the analysis date through the final forecast date. (If only monthly paths of prices are needed, Level II simulation may be skipped.)

We now explain the second and third steps in detail.

5.1 Level I simulation: Monthly prices

We assume that the vector of asset returns follows a multivariate normal distribution with mean $\hat{X}$ and covariance matrix $\Sigma_X$, i.e.,

$$[5.1] \quad X \sim MVN(\hat{X}, \Sigma_X).$$

It is a straightforward exercise to simulate returns from a multivariate normal distribution and to generate $M$ price paths for each asset, since given a covariance matrix, Equation [5.1] completely specifies the distribution of asset returns at all forecast dates.$^1$ Unfortunately we do not have the covariance matrix $\Sigma_X$. The models from earlier chapters provide forecasts of the mean and variance at multiple forecast horizons, but they use separate procedures from which it is impossible to infer correlation information between prices at different horizons.

For instance, our models provide forecasts of the mean and variance of the DEM per USD exchange rate in 1 month and in 2 months, but cannot provide information about the correlation between the exchange rates at these two horizons. The goal of this section is to explain how one can obtain the correlation structure necessary to perform Monte Carlo simulations for prices at Level I forecast dates, with as little additional estimation as possible and without corrupting the consistency of the mean and variance forecasts we have already obtained. In addition, the resulting correlation matrix must be positive definite (or at least positive semi-definite).

The covariance matrix we want to estimate can be decomposed in the following way:

---

$^1$ There are well-established methods for generating correlated random variates from a covariance matrix, which include Cholesky and Singular Value decompositions. See the RiskMetrics Technical Document (1996) for details.
In Equation [5.2], each diagonal element $\Sigma_{ii}$ is an $R \times R$ matrix that represents the covariance of returns within one asset for different time horizons (autocovariance), and each off-diagonal element $\Sigma_{ij}$ is an $R \times R$ matrix representing the covariance of returns between asset $i$ and asset $j$ (cross covariance). $R$ denotes the number of forecast dates and $n$ is the number of assets.

At this point it is important to note that if we are interested in calculating a risk measure depending only on the distribution of a given set of prices at a fixed horizon, we do not need the full covariance matrix depicted in Equation [5.2]. An example of this type of application would be a 1-year VaR calculation for a portfolio consisting of $n$ assets for which we need only an $n \times n$ covariance matrix of 1-year returns. Such a matrix can be estimated from historical data using LongRun’s forecasted means and volatilities.

Lacking information about the correlations between prices at different horizons, we will introduce some assumptions in order to construct the covariance matrix. Since the positive definiteness of a high-dimensional matrix is extremely sensitive to changes in the matrix elements, it is very difficult to guarantee that any realistic set of assumptions will always yield a positive definite matrix. With this in mind, we introduce in Appendix 5.A a technique to slightly modify a non-positive definite matrix and turn it into a positive definite one. In the meantime, the procedure in the next two sections aims at constructing a positive definite covariance matrix that is consistent with LongRun’s forecasts and closely approximates the true correlation between prices at different horizons.

In the following section we give a procedure to obtain the diagonal matrices $\Sigma_{ii}$ in Equation [5.2].

### 5.1.1 Single price series: Autocorrelation

Let us now consider a single log price series for which we have obtained variance forecasts $(\sigma_1^2, \ldots, \sigma_R^2)$, for Level I dates $t + \tau_1, t + \tau_2, \ldots, t + \tau_R$. Let $r_{m,n}$ indicate this price series return from time $t + \tau_m$ to $t + \tau_n$. Then

$$\text{Var}[r_{0,m}] = \sigma_m^2$$

for $m = 1, 2, \ldots, R$.

In order to generate Level I simulations, we need information about the correlations between the returns over distinct horizons. In particular, we need $\text{Cov}[r_{0,m}, r_{0,n}]$, for $m \leq n \leq R$.

To infer this covariance, we first observe that returns are additive; that is

$$r_{m,n} = r_{m,k} + r_{k,n}$$

for $m < k < n$.

This allows us to express the covariance between the 1-period and 2-period returns as
Although we have not estimated the covariance between the first and second period returns \( r_{0,1} \) and \( r_{0,2} \), we can infer something about this term from our estimate of \( \sigma_2^2 \) through the expression

\[
\sigma_2^2 = \text{Var}[r_{0,2}]
\]

Solving Equation [5.6] for \( \text{Cov}[r_{0,1}, r_{1,2}] \) and substituting this into Equation [5.5] yields

\[
\text{Cov}[r_{0,1}, r_{0,2}] = \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 - \text{Var}[r_{1,2}] \right).
\]

To finalize our covariance expression, we need an estimate of \( \text{Var}[r_{1,2}] \). At this point, rather than rely on historical or other estimations, we make the assumption that at our analysis date, future 1-period returns, regardless of how far in the future they occur, all have variance \( \sigma_1^2 \). This assumption implies that \( \text{Var}[r_{1,2}] = \text{Var}[r_{0,1}] = \sigma_1^2 \), and that our covariance is simply

\[
\text{Cov}[r_{0,1}, r_{0,2}] = \frac{1}{2} \sigma_2^2.
\]

To generalize the autocovariance expression in Equation [5.8] to arbitrary horizons, we assume that future \( k \)-period returns, regardless of how far in the future they occur, all have variance \( \sigma_k^2 \), implying that

\[
\text{Var}[r_{m,n}] = \text{Var}[r_{0,n-m}] = \sigma_{n-m}^2.
\]

for \( m < n \).

Applying Equation [5.9], and proceeding as in Equations [5.3] through [5.8], we can derive the general covariance expression:

\[
\text{Cov}[r_{0,m}, r_{0,n}] = \frac{1}{2} \left( \sigma_m^2 + \sigma_n^2 - \sigma_{n-m}^2 \right).
\]

We remark that Equation [5.9] is intuitively appealing and does not contradict any of the previous forecasting techniques; however, it is consistent with the covariance structure only if

\[
(\sigma_m - \sigma_{n-m})^2 \leq \sigma_n^2 \leq (\sigma_m + \sigma_{n-m})^2
\]

for all \( m < n \).

To see why this condition must hold, observe that the covariance term in Equation [5.10] is bounded above by the product of the standard deviations of \( r_{0,m} \) and \( r_{0,n} \), and below by the negative of this product.

The upper bound imposed by Equation [5.11] allows volatility to increase as a linear function of time. We do not believe this bound to be particularly restrictive since growth in volatility is often
assumed to be much slower (e.g., square root of time). The lower bound in Equation [5.11] establishes the extent to which volatility can move up or down at any time, given its previous values. The restriction imposed by this lower bound is usually not binding in practice.

Equation [5.10] yields the covariance matrix $\Sigma_{ij}$ (for $R = 3$):

$$\Sigma_{i} = \text{Cov} \left[ \begin{array}{c} r_{0,1} \\ r_{0,2} \\ r_{0,3} \end{array} \right] = \begin{bmatrix} \sigma_1^2 & \frac{1}{2} \sigma_2^2 & \frac{1}{2} \left( \sigma_1^2 + \sigma_2^2 - \sigma_3^2 \right) \\ \frac{1}{2} \sigma_2^2 & \sigma_2^2 & \frac{1}{2} \left( \sigma_2^2 + \sigma_3^2 - \sigma_1^2 \right) \\ \frac{1}{2} \left( \sigma_1^2 + \sigma_3^2 - \sigma_2^2 \right) & \frac{1}{2} \left( \sigma_2^2 + \sigma_3^2 - \sigma_1^2 \right) & \sigma_3^2 \end{bmatrix}.$$ 

The generalization to arbitrary $R$ is straightforward.

Observe that we have constructed a covariance structure where there was none before. We have avoided further model estimation and ensured that our construction conforms with our Level I mean and volatility forecasts. The assumption in Equation [5.9] is a rather weak one, and so it is reasonable to assert that the covariance matrix in Equation [5.12] is the most natural matrix that is consistent with our mean and variance forecasts.

### 5.1.2 Multiple price series: Cross correlation

Now that we have a correlation structure across time for each asset, we need to consider correlations across assets (cross correlations) in our portfolio. For example, consider a portfolio consisting of DEM per USD and JPY per USD exchange rate exposures at Level I dates $t + \tau_1$, $t + \tau_2$, ..., $t + \tau_R$. We follow the notation above and denote returns for each asset with a superscript and volatilities with a subscript, that is, the returns and volatilities for the DEM per USD are $r_{Dm,n}$ for $m < n \leq R$ and $(\sigma_{D1}, \ldots, \sigma_{DR})$, respectively.

In our example, the covariance matrix can be decomposed as

$$\Sigma_X = \begin{bmatrix} \Sigma_{DD} & \Sigma_{DJ} \\ \Sigma_{DJ} & \Sigma_{JJ} \end{bmatrix}.$$ 

The $R \times R$ matrices $\Sigma_{DD}$ and $\Sigma_{JJ}$ can be calculated according to the procedure outlined in Section 5.1.1. In this section we explain the calculation of $\Sigma_{DJ}$.

In order to reduce the number of parameters to estimate, we assume that the cross correlation between contemporaneous returns over the same period is constant. To be precise, we write this assumption as

$$\text{Cov}[r_{m,n}^{D}, r_{m,n}^{J}] = \rho \sigma_{Dn-m} \sigma_{Jn-m}$$

for $m < n \leq R$.

The correlation coefficient $\rho$ can be estimated by using historical returns $r$ and our forecasted means $\hat{r}$ for those returns:
Since returns are additive we can write

\[ \text{Cov}[r^D_{m \times n}, r^D_{m \times n}] = \text{Cov}[r_{0 \times n}^D, r_{0 \times n}^D] + \text{Cov}[r_{0 \times m}^D, r_{0 \times m}^D] - \text{Cov}[r_{0 \times m}^D, r_{0 \times n}^D] \]

If we assume that \( \text{Cov}[r_{0 \times m}^D, r_{0 \times n}^D] = \text{Cov}[r_{0 \times m}^D, r_{0 \times m}^D] \), then Equation [5.16] can be rewritten as

\[ \text{Cov}[r^D_{m \times n}, r^D_{m \times n}] = \text{Cov}[r_{0 \times m}^D, r_{0 \times n}^D] + \text{Cov}[r_{0 \times m}^D, r_{0 \times m}^D] - 2\text{Cov}[r_{0 \times m}^D, r_{0 \times n}^D] \]

Using Equation [5.17], we find that the covariance between the \( m \)-period JPY per USD return and the \( n \)-period DEM per USD returns \((m < n)\) is

\[ \text{Cov}[r_{0 \times m}^J, r_{0 \times n}^D] = \frac{\rho}{2}(\sigma_{Dn}\sigma_{Jn} + \sigma_{Dm}\sigma_{Jm} - \sigma_{Dn-m}\sigma_{Jn-m}) \]

Note that the cross correlation \( \rho \) between returns of the same asset over different time horizons is equal to one. Therefore, the covariance structure given by Equation [5.18] is consistent with the autocovariance structure in Equation [5.10].

This method yields the cross covariance matrix \( \Sigma_{DJ} \) (for \( R = 3 \)):

\[
\Sigma_{DJ} = \begin{bmatrix}
\rho \sigma_{D1} \sigma_{J1} & \frac{\rho}{2} \sigma_{D2} \sigma_{J2} & \frac{\rho}{2} (\sigma_{D3} \sigma_{J3} + \sigma_{D1} \sigma_{J1} - \sigma_{D2} \sigma_{J2}) \\
\frac{\rho}{2} \sigma_{D2} \sigma_{J2} & \rho \sigma_{D2} \sigma_{J2} & \frac{\rho}{2} (\sigma_{D3} \sigma_{J3} + \sigma_{D2} \sigma_{J2} - \sigma_{D1} \sigma_{J1}) \\
\frac{\rho}{2} (\sigma_{D3} \sigma_{J3} + \sigma_{D1} \sigma_{J1} - \sigma_{D2} \sigma_{J2}) & \frac{\rho}{2} (\sigma_{D3} \sigma_{J3} + \sigma_{D2} \sigma_{J2} - \sigma_{D1} \sigma_{J1}) & \rho \sigma_{D3} \sigma_{J3}
\end{bmatrix}
\]

It is important to keep in mind that we are imposing some structure on the covariance matrix, and as the number of assets \((n)\) or the number of horizons \((R)\) gets larger, it becomes increasingly difficult to maintain the positive definiteness of our matrix. In Appendix 5.A, we derive a method to slightly modify the correlation structure and obtain a positive definite (or at least positive semi-definite) matrix without changing our volatility forecasts. Matrices perturbed by this technique are in practice very similar to the original ones.

It is important to emphasize that the covariance matrices obtained by following this procedure are not unique in the sense that a different covariance structure would result from a different set of assumptions. However, our final objective is to produce not an exact covariance matrix, but one that allows for an estimation of the market risk implied by the joint fluctuations of prices at different time horizons. We constructed our covariance matrix to serve this purpose.
5.2 Level II simulation: Daily prices

5.2.1 Overview

The Level I simulation discussed in Section 5.1 gave us a framework to generate scenarios at a small number of longer horizons (referred to as the Level I dates). A typical application of this might be to generate scenarios for the DEM per USD exchange rate 1, 2, 3, and 4 months into the future. To complete our simulation, we generate scenarios for arbitrary horizons in between the Level I dates. In our example, this means generating DEM per USD scenarios on all days between the monthly horizons. This is the goal of Level II simulation, and is the focus of this section.

The methods presented below are intended to illustrate the generation of price scenarios for every day, but they are just as applicable to the case where forecasts are desired for only a small number of days between the Level I dates. One way to think of the two levels of simulation is that Level I gives a coarse view—scenarios at a small number of critical horizons—and Level II “fills in” scenarios at horizons in between. Chart 5.1 illustrates Level I and Level II simulation for one possible path of an asset price. The points marked by “◊” indicate the Level I scenarios simulated at the three horizons \(t + \tau_i\) \((i = 1, 2, 3)\). The solid line indicates a possible daily path derived from the Level II scenarios, i.e., the “filled in” points on all the days in between the Level I horizons.

**Chart 5.1**
Simulated daily prices for one forecast path

\(s_t\) and \(d_t\) denote the logarithm of Level I prices and Level II prices, respectively

![Chart 5.1](image)

The next scenario would be generated by repeating the same two steps: first Level I simulation to produce a new set of three Level I prices, then Level II simulation to produce a path of daily prices that pass through the Level I prices. Repeating the process produces multiple paths, as shown in Chart 5.2.
An important concept arises when we consider the two levels of simulation—that of conditional and unconditional volatility. This concept arises from the fact that Level I prices are fixed within a given scenario (conditional volatility), but not from one scenario to another (unconditional volatility).

For example, once we generate a Level I scenario, the prices on the Level I dates are fixed, since we restrict our daily path to go through these points. The prices on days in between Level I dates are still random, but the prices on Level I dates are not. It appears from Chart 5.1 that the log price (recall that this is the first Level I date), has zero volatility. Chart 5.2 reveals a different perspective, as the prices on all dates from one simulation to the next, including the Level I dates, appear to be random. Here, it is clear that the volatility of is not zero.

To clarify this apparent contradiction, we observe that we are discussing two distinct volatilities: The first is the conditional volatility of , given that we observe, through a single realization of a Level I simulation, a single set of Level I prices. The second is the unconditional volatility of , that is, the price volatility we would obtain after several realizations of Level I simulation. The same distinction can also be drawn for the prices between Level I dates: conditional on the Level I prices, for example at 1 and 2 months, the 1-month plus 1-day price has relatively little volatility, while unconditionally, this price is much more uncertain.

The methods in this section deal with generating scenarios from the conditional distribution of daily prices, given specific monthly (Level I) prices. To specify this distribution, we stipulate that given the Level I prices, an asset’s daily price process is a random walk, with volatility equal to the asset’s estimated daily volatility, and constrained to pass through each of the predetermined monthly prices. If we assume that the random walk is composed of normally distributed steps, then this process is referred to as a “Brownian bridge.”

### 5.2.2 Brownian bridges and the Level II algorithm

We begin by introducing some notation. As in the previous section, let , , , ..., be the logarithms of Level I prices in scenario for the asset in question. We adopt the convention that , so that is equal to the current price in all scenarios. Let be the asset’s daily volatility. For , we generate daily log prices with the following properties:
1. On Level I dates, that is, at \( b = \tau_1, \ldots, \tau_R \), the Level II scenarios “match” the Level I scenarios, such that \( d^q_{t+\tau} = s^q_{t+\tau} \).

2. Between Level I dates, for \( \tau_{r-1} < b < \tau_r \), the Level II scenarios are distributed according to our assumptions on the daily dynamics of \( d \), conditional on \( d \) “passing through” \( s^q_{t+\tau_{r-1}} \) and \( s^q_{t+\tau_r} \).

If the daily price process for \( d \) is a random walk with normally distributed returns and volatility \( \tilde{\sigma} \), then we may utilize the Brownian bridge process to generate our Level II scenarios. In short, a Brownian bridge process has the same distribution as a random walk in between two time points, where the value at each time point is known. Supposing that between Level I dates, \( d \) is distributed as a Brownian bridge, we can conclude\(^2\) that for each \( q \), \( \{d^q_{t+b}; b = \tau_{r-1}, \ldots, \tau_r\} \) are jointly normally distributed\(^3\). For \( b = \tau_{r-1}, \tau_{r-1} + 1, \ldots, \tau_r - 1 \), the conditional moments of \( d^q_{t+b} \) are given by:

\[
E_q[d^q_{t+b}] = \frac{\tau_r - b}{\tau_r - \tau_{r-1}} s^q_{t+\tau_{r-1}} + \frac{b - \tau_{r-1}}{\tau_r - \tau_{r-1}} s^q_{t+\tau_r},
\]

\[
Var_q[d^q_{t+b}] = \frac{\tilde{\sigma}^2}{\tau_r - \tau_{r-1}} (b - \tau_{r-1}) \frac{\tau_r - b}{\tau_r - \tau_{r-1}},
\]

and

\[
Cov_q[d^q_{t+b}, d^q_{t+b'}, d^q_{t+b' + b}] = \frac{\tilde{\sigma}^2}{\tau_r - \tau_{r-1}} (b - \tau_{r-1}) \frac{\tau_r - b}{\tau_r - \tau_{r-1}},
\]

for \( b < b^* \).

The subscript \( q \) in the equations above indicates that these moments are conditional on the Level I prices in scenario \( q \) (\( s^q_{t+\tau_{r-1}} \) and \( s^q_{t+\tau_r} \)). The conditional mean of \( d \) is just a linear interpolation between \( s^q_{t+\tau_{r-1}} \) and \( s^q_{t+\tau_r} \). Note that the variance and covariance terms, if the random walk were only anchored at \( \tau_{r-1} \), would be simply \( \tilde{\sigma}^2 (b - \tau_{r-1}) \); the extra term ensures that the conditional variance approaches zero as \( b \) becomes close to \( \tau_r \).

A sample realization of a Brownian bridge process is illustrated in Chart 5.3. The confidence bands (one standard deviation) around the linear path are widest exactly halfway between \( t + \tau_{r-1} \) and \( t + \tau_r \), and approach zero at both endpoints.

---

\(^2\) See Karatzas and Shreve (1991), pp. 358–360, for details.

\(^3\) In fact, \( d^q \) is a Gaussian process.
In order to simulate the Brownian bridge process for \( d \), we employ the following algorithm:

1. Generate independent, normally distributed random variables \( u_{l}^{q} \) for \( l = 1, \ldots, \tau_{R} \), with mean 0 and variance \( \sigma_{b}^{2} \).

2. On the Level I dates (that is, for \( b = \tau_{r}, \ r = 1, \ldots, R \), set \( d_{i+\tau_{r}}^{q} = s_{i+\tau_{r}}^{q} \).

3. For all other dates (that is \( b \in (\tau_{r-1}, \tau_{r}) \) for \( r = 1, \ldots, R \), set

\[
\tau_{r} - b \frac{s_{i+\tau_{r-1}}^{q}}{\tau_{r} - \tau_{r-1}} - \sum_{l = \tau_{r-1} + 1}^{\tau_{r}} \sum_{l = \tau_{r-1} + 1}^{b} u_{l}^{q} \frac{s_{i+\tau_{r-1}}^{q}}{\tau_{r} - \tau_{r-1}}
\]

It is clear that the daily price scenarios generated in this way are normally distributed, and verifying that the conditional moments of \( d \) are the same as in Equations [5.20] through [5.22] is a straightforward exercise.

The algorithm above is also applicable if we do not assume a random walk for the daily process. In general, we may generate the \( u_{l}^{q} \) according to an arbitrary distribution. This allows us to account for any special characteristics of the daily moves while maintaining the covariance structure of the Brownian bridge.

The extension of the algorithm to the multivariate case (let \( N \) be the number of price series in question) is straightforward. For each scenario \( q \), we first generate Level I prices, accounting for longer horizon correlations as in Section 5.2. In Step 1 of the algorithm, we generate \( \tau_{R} \) independent random vectors \( (u_{1}^{q}, u_{2}^{q}, \ldots, u_{N_{q}}^{q}) \), \( l = 1, 2, \ldots, \tau_{R} \), according to the joint distribution of daily returns on the \( N \) price series. We then apply Steps 2 and 3 of the algorithm for each of the \( N \) price series, using the sequence \( u_{i+1}^{q}, u_{i+2}^{q}, \ldots, u_{i+\tau_{R}}^{q} \) for the \( i \)th price.
5.2.3 Putting the steps together: Unconditional moments

At this point, we have fully specified our scenario simulation process. While the results are sufficient for us to perform the Monte Carlo simulation and allow us to produce results such as those in Chart 5.4, there is an air of incompleteness, as we have never explicitly stated the unconditional distribution of prices between the Level I dates; we have only specified their conditional distribution given the Level I scenarios. An exact description of the unconditional distribution is awkward, but the computation of its moments is tractable.

Chart 5.4
Simulation of multiple daily paths

Suppose that at times $t + \tau_1, \ldots, t + \tau_R$, the log prices of a given asset follow a multivariate normal distribution, with means $(\bar{s}_1, \ldots, \bar{s}_R)$, and variances $(\sigma_1^2, \ldots, \sigma_R^2)$. From Equation [5.10], the covariance between prices at times $t + \tau_{r-1}$ and $t + \tau_r$ is given by

$$\text{Cov}[s_{t+\tau_{r-1}}, s_{t+\tau_r}] = \frac{1}{2}(\sigma_{r-1}^2 + \sigma_r^2 - \sigma_1^2).$$

Moving now to the daily price series $(d_1, d_2, \ldots, d_{\tau_r})$, we know that at Level I dates (for $b = \tau_r$), the Level II step introduces no volatility, so that the means and variances at these dates are just

$$E[s_{t+\tau_r}] = \bar{s}_r$$

and

$$\text{Var}[s_{t+\tau_r}] = \sigma_r^2$$

for $r = 1, \ldots, R$.

Between Level I dates (for $b \in (\tau_{r-1}, \tau_r)$), we must consider both simulation steps. The mean daily price is just the expectation of the conditional mean in Equation [5.20], giving
Thus the mean daily prices are simply linear interpolations of the mean Level I prices.

The variance of \( d_{t+b} \) comes from two sources: the variance of the conditional mean (which we can think of as Level I variance), and the expectation of the conditional variance (Level II variance). The Level I variance is due to the volatility of prices at the Level I dates, and is the variance of the linear interpolation expression in Equation [5.20]. By our assumptions on the Level I prices, this yields

\[
E \left[ d_{t+b} \right] = \frac{\tau_r - b}{\tau_r - \tau_{r-1}} \tau_{r-1} + \frac{b - \tau_{r-1}}{\tau_r - \tau_{r-1}} \tau_r.
\]

[5.27]

Note that we used Equation [5.24] for the covariance term. The Level II variance is the variance due to the Brownian bridge simulations and is given by Equation [5.21]. Adding these two components gives

\[
Var\left[ d_{t+b} \right] = \frac{\tau_r - b}{\tau_r - \tau_{r-1}} \sigma_{\tau_{r-1}}^2 + \frac{b - \tau_{r-1}}{\tau_r - \tau_{r-1}} \sigma_r^2 + 2 \frac{(\tau_r - b)(b - \tau_{r-1})/2}{(\tau_r - \tau_{r-1})^2} \left( \sigma_{\tau_{r-1}}^2 + \sigma_r^2 - \sigma_1^2 \right).
\]

[5.28]

Two immediate observations are that \( Var\left[ d_{t+b} \right] \) goes to \( \sigma_{\tau_{r-1}}^2 \) as \( b \) approaches \( \tau_{r-1} \) and to \( \sigma_r^2 \) as \( b \) approaches \( \tau_r \), which is crucial for consistency with the unconditional volatility of Level I prices.

### 5.3 Simulation example

In this section we describe with an example how to simulate monthly prices using Level I simulation and then “fill in the gaps” with daily prices generated using Level II simulation. In our example we consider DEM per USD and JPY per USD exchange rates at three Level I dates 1, 2 and 3 months into the future.

In Level I simulation we want to simulate the prices of DEM per USD and JPY per USD at each of the Level I dates. At this stage, we need the forecasted mean and volatility for each asset at each of those dates. This information is summarized in Table 5.1.

**Table 5.1**

<table>
<thead>
<tr>
<th>Horizon</th>
<th>DEM per USD</th>
<th>JPY per USD</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>Volatility</td>
</tr>
<tr>
<td>Today</td>
<td>1.5274</td>
<td>—</td>
</tr>
<tr>
<td>1 month</td>
<td>1.5342</td>
<td>2.43%</td>
</tr>
<tr>
<td>2 months</td>
<td>1.5275</td>
<td>3.62%</td>
</tr>
<tr>
<td>3 months</td>
<td>1.5253</td>
<td>4.53%</td>
</tr>
</tbody>
</table>

To construct the covariance structure we also need the correlation \( \rho \) between monthly non-overlapping returns of the DEM per USD and JPY per USD exchange rates. This correlation can be calculated using Equation [5.15]. In our example \( \rho \) is equal to 45.91%.
Following the procedure outlined in Section 5.1, we construct the correlation matrix for Level I simulation, as shown in Chart 5.5.

**Chart 5.5**  
**Correlation matrix for DEM per USD and JPY per USD rates at Level I dates**

We can now simulate a set of six correlated normal variables (i.e., two exchange rates at each of the three Level I dates) by using Equation [5.1] with the correlation matrix given in Chart 5.5 and volatility given in Table 5.1, thus obtaining a Level I scenario. If we generate $N$ sets of six correlated normal variables in the same way, we obtain $N$ Level I scenarios. Chart 5.6 shows three Level I scenarios for DEM per USD and JPY per USD.

**Chart 5.6**  
**a. DEM per USD Level I simulation**  
**b. JPY per USD Level I simulation**

For Level II simulation we want to keep the Level I scenarios unchanged and “fill in the gaps” between Level I dates with daily prices for each of the three scenarios generated in Level I. In this stage we need the daily volatility of each rate as well as the correlation between daily returns. This information is given in Table 5.2 below.

**Table 5.2**  
**Volatility and correlation between DEM per USD and JPY per USD daily returns**

<table>
<thead>
<tr>
<th>JPY Daily Volatility</th>
<th>DEM Daily Volatility</th>
<th>Correlation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.67%</td>
<td>0.53%</td>
<td>69.44%</td>
</tr>
</tbody>
</table>
We now use the information in Table 5.2 to construct a $2 \times 2$ daily covariance matrix:

**Chart 5.7**

**Covariance matrix of DEM per USD and JPY per USD daily returns**

<table>
<thead>
<tr>
<th></th>
<th>DEM per USD</th>
<th>JPY per USD</th>
</tr>
</thead>
<tbody>
<tr>
<td>DEM per USD</td>
<td>0.000028</td>
<td>0.000024</td>
</tr>
<tr>
<td>JPY per USD</td>
<td>0.000024</td>
<td>0.000045</td>
</tr>
</tbody>
</table>

The next step is to generate, for each pair of successive Level I dates, 30 sets (one for each day) of two correlated normal variables (one for each asset) with the covariance matrix shown in Chart 5.7. In our example we have three pairs of Level I dates: 0 to 1 month, 1 to 2 months, and 2 to 3 months, giving a total of 90 sets of bivariate normal variables. Using these random numbers we can apply the algorithm (Equation [5.23]) to generate a Brownian Bridge process described in Section 5.2.2, and obtain a daily price path for each asset. We repeat this procedure three times using Equation [5.23] with each of the monthly prices generated in Level I simulation to obtain three daily paths. Note that this procedure is computationally intensive since we need to generate $90 \times 2 \times N$ random variables, where $N$ is the number of paths we want to generate (in our example $N = 3$).

Chart 5.8 shows three daily paths for DEM per USD and JPY per USD, respectively. Note that the daily paths cross the Level I points at each of the Level I dates.

**Chart 5.8**

a. DEM per USD Level II simulation  
b. JPY per USD Level II simulation

### 5.4 Summary

In this chapter we described the procedure for constructing a feasible covariance matrix and discussed its use in the simulation of Level I prices. Once we explained how to generate a path of Level I prices we discussed how to “fill in the gaps” between Level I dates with simulated daily returns following a Brownian Bridge process. Finally, we gave a complete example of the simulation procedure (including both Level I and Level II simulation) resulting in a set of daily paths over three forecast horizons (1, 2 and 3 months) for each of two assets (DEM per USD and JPY per USD exchange rates). The resulting set of daily paths can be used to obtain the distribution of a portfolio’s potential changes in value from which a risk measure can be derived. The discussion on LongRun’s scenario generation procedure is now complete.
The conclusion of this chapter showed how to use the forecasts obtained in Chapter 2 and Chapter 3 to simulate price scenarios. In the next chapter, we outline some risk management practical applications in which LongRun’s scenarios can be used.
Appendix 5.A  Matrix perturbations

In this appendix we discuss a method for transforming a non-positive definite matrix into a positive definite one.

A matrix $A$ is said to be symmetric if $A = A'$, where $A'$ is the transpose of $A$. Therefore, covariance and correlation matrices are real and symmetric.\(^4\)

A real symmetric matrix $A$ is said to be positive definite if $x'Ax > 0$ for every vector $x \neq 0$. Given this definition, one can prove that a real symmetric matrix $A$ is positive definite if and only if all its eigenvalues are greater than zero.\(^5\)

Let us assume that we know the eigenvalues of a real symmetric matrix $A$ and that the smallest of those eigenvalues is smaller than or equal to zero. In this case, the matrix is not positive definite and we would like to turn it into a positive definite one. The question we want to answer is: "How do the eigenvalues of a matrix $A$ change if it is subject to a perturbation $A \to A + E$?" Because the eigenvalues are continuous functions of the entries of $A$, it is natural to think that if the perturbation matrix $E$ is small enough, then the eigenvalues should not change too much. It would be undesirable for our purposes to drastically change a matrix in order to turn it into a proper covariance matrix.

We propose the following perturbation:

\[ [5.A.1] \quad A \to A + \epsilon(I - A) = B, \]

where $\epsilon$ is a scalar and $I$ is the identity matrix.

If $\epsilon = 0$, then $A \to A$; and if $\epsilon = 1$, then $A \to I$, which is clearly a positive definite matrix. Therefore, since the eigenvalues of a matrix are continuous functions of its entries, there are an infinite number of scalars $\epsilon$ in the interval $[0, 1]$ such that $B$ given by Equation [5.A.1] is a positive definite matrix.

How should we choose $\epsilon$? Clearly, we do not want $\epsilon$ to be too large because that would imply a radical transformation of our original matrix $A$; but, if $\epsilon$ is too small we might not be able to guarantee that the smallest eigenvalue of $B$ is a positive number. The following result, known as the Rayleigh-Ritz theorem (Equation [5.A.2]), will guide us in our election of $\epsilon$.\(^6\)

Let us denote the smallest eigenvalue of a matrix $M$ by $\lambda_M$.

If $A$ is a real symmetric matrix, then

\[ [5.A.2] \quad \lambda_A = \min_{x'x = 1} x'Ax. \]

Using Equation [5.A.1] and Equation [5.A.2] we can write

\[ [5.A.3] \quad \lambda_B = \min_{x'x = 1} x'Bx = \min_{x'x = 1} \left[ (1 - \epsilon)x'Ax + \epsilon x'x \right] = (1 - \epsilon)\lambda_A + \epsilon. \]

\(^4\) The results in this appendix hold not only for real symmetric matrices, but more generally for Hermitian matrices where $A = A^*$ and $A^*$ denotes the conjugate transpose of $A$. Note that a real symmetric matrix is Hermitian.

\(^5\) All the eigenvalues of a Hermitian matrix are real numbers.

\(^6\) See Horn and Johnson (1985), pp. 176–177, for a proof of the Rayleigh-Ritz theorem.
If $\lambda_A \leq 0$, Equation [5.A.3] implies that $\lambda_B \geq 0$ if $\varepsilon \geq \frac{-\lambda_A}{1 - \lambda_A}$.

We now have a clear choice for $\varepsilon$ depending on the method selected to generate random variables: If one wants to use Cholesky decomposition to generate correlated random variables, then $\varepsilon$ must be slightly greater than $-\lambda_A/(1 - \lambda_A)$ since the Cholesky algorithm can be applied to only strictly positive definite matrices. If one uses Eigenvalue or Singular Value decompositions, $\varepsilon$ can be chosen exactly equal to $-\lambda_A/(1 - \lambda_A)$ since these methods allow for positive semi-definite matrices.

The reader should note that if one applies the perturbation in Equation [5.A.1] to the correlation matrix, the elements in the main diagonal of the covariance matrix, corresponding to LongRun’s volatility forecasts, remain unchanged. This provides a mechanism to obtain a positive definite (or positive semi-definite) covariance matrix by altering the correlation structure without corrupting the consistency of the forecasts we have already obtained.

Once we have our positive definite matrix, we can use the following metric to measure how different the matrix is from the original one:

$$[5.A.4] \quad \eta = \frac{\|B - A\|_2}{\|A\|_2},$$

where $\|A\|_2 = \left( \sum_{i,j=1}^{R \times n} |a_{ij}|^2 \right)^{1/2}$.

The variable $\eta$ given by Equation [5.A.4] provides a measure of the relative difference between the original matrix $A$ and the perturbed matrix $B$. 
Chapter 6. Applications of LongRun

LongRun can be applied to any risk management situation that requires the measurement of risk arising from long-term fluctuations in market prices and rates. As we demonstrated in Chapter 4, the out-of-sample predictions produced by LongRun’s forecasting procedures are able to consistently outperform (at horizons longer than three months) the random walk model with zero expected return used in the RiskMetrics methodology. With this in mind, we can view LongRun as an extension of the RiskMetrics methodology to the cases where the relevant risk horizon is long.

In this chapter we explore some of the risk management applications of LongRun, showing how the relevant variables and definitions of appropriate risk measures vary across applications. The examples we analyze are

- long-term VaR for a portfolio manager,
- Cash-Flow-at-Risk (CFaR) and Earnings-at-Risk (EaR) for corporations, and
- VaR for pension plans and mutual funds.

Although the problems faced in each of these three situations are quite different, they share a common feature: the solution to all of them involves the use of long-term forecasts to assess risk at long horizons.

The problem solving techniques are based on the first five chapters in which we described forecasting methodologies that allow us to obtain the individual (marginal) distributions of market prices and rates at various horizons (Chapter 2 and Chapter 3), and to integrate those individual distributions through the introduction of a covariance structure (Chapter 5). As a result of the integration of the individual distributions, we obtain the joint distribution for all the market prices and rates at a set of horizons. Once we have the joint distribution of market prices and rates, we can obtain the distribution of changes in the value of a portfolio of assets. This can be done analytically if the relationship between market prices and the assets in the portfolio is simple enough (e.g., linear), or by Monte Carlo simulation if the assets in our portfolio are more complicated functions of market prices and rates.

The distribution of the positions’ potential changes in value is a common starting point in most risk management situations, being an important input to any risk management framework. The differences between applications arise in the selection of the key market prices and rates driving the value of the assets in the portfolio, the actual valuation of the individual assets needed in a particular situation, the relevant risk horizon (or horizons), and the selection of an appropriate risk measure (e.g., VaR, CFaR, or EaR) for each application.

The cases presented here are not intended to be exhaustive, but rather illustrative of the various ways in which LongRun can be applied to solving risk management problems where the relevant horizon is long.

6.1 Long-term VaR for a portfolio manager

The RiskMetrics methodology provides a robust framework to calculate VaR for risk horizons up to approximately three months. In this example, we explain how LongRun can be used to extend the VaR measurement introduced in the RiskMetrics Technical Document to longer horizons.

To extend the RiskMetrics VaR measurement to longer horizons, we need a set of mean and volatility forecasts that are adequate for those horizons. LongRun introduces two different methodologies that provide an alternative to RiskMetrics for computing long-term mean and volatility forecasts for each component in a portfolio.
Once we have calculated long-term means and volatilities using LongRun, the only information that we need to obtain the distribution of changes in the value of a portfolio is the correlation structure between each pair of components in the portfolio. It is important to emphasize that in the long-term VaR calculation we need to make forecasts at only a single long horizon. Hence, the introduction of an autocorrelation structure between forecast horizons is not necessary. The covariance matrix needed in this situation can be obtained as a particular case (only one forecast horizon) of the matrix introduced in Chapter 5. Once we have constructed a covariance matrix using LongRun’s forecasts, we can proceed as in RiskMetrics and calculate VaR as a percentile of the distribution of changes in value of the portfolio. Note that VaR can be calculated from the procedure described in the RiskMetrics Technical Document by using the covariance matrix constructed with LongRun, instead of the RiskMetrics covariance matrix.

For example, let us suppose that we have a portfolio consisting of a long position of US$100 in DEM, and a short position of US$100 in JPY. Let us further assume that we want to calculate a three-month VaR for our portfolio. Using the information in Table 5.1 and Chart 5.5, we can calculate the covariance matrix of three-month returns for the USD per DEM and USD per JPY exchange rates:

{\[ \Sigma = \begin{bmatrix} 0.0453 & 0 \\ 0 & 0.0469 \end{bmatrix} \begin{bmatrix} 1 & 0.4591 \\ 0.4591 & 1 \end{bmatrix} \begin{bmatrix} 0.0453 \\ 0.0469 \end{bmatrix} = \begin{bmatrix} 0.00205 & 0.00097 \\ 0.00097 & 0.00219 \end{bmatrix}. \]}

Then the three-month 95% VaR can be calculated as\(^1\)

{\[ \text{VaR} = -1.64 \sqrt{\Sigma \delta} = -1.64 \sqrt{\begin{bmatrix} 0.00205 & 0.00097 \\ 0.00097 & 0.00219 \end{bmatrix} \begin{bmatrix} 100 \\ -100 \end{bmatrix}} \approx -7.86. \]}

### 6.2 CFaR and EaR for corporations

In this example we discuss the application of our long-term scenario generation methodology to corporate risk management. A brief description of the problem is given below, but we refer readers interested in the subject to the CorporateMetrics Technical Document, which provides a complete discussion of corporate risk management describing in detail the market risk management needs of corporations.\(^2\)

The application of risk management principles in the corporate environment has been growing as companies seek to improve decision-making processes that affect their risk profiles and strive to enhance communication with shareholders and potential investors.

Additionally, companies are being required by regulators to provide more quantitative reporting of their market risk exposures, thus making the definition of appropriate risk measures a critical factor in regulatory reporting. For example, the SEC requires that companies provide quantitative information about their market risk sensitive instruments\(^3\) and has approved VaR-type measures that express potential loss of earnings, cash flow, or fair value.

Cash flows and earnings are two performance variables that are commonly used to gauge the value of a company. If we project these variables as functions of future market prices and rates and obtain

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\(^1\) See the RiskMetrics Technical Document (1996) for details.

\(^2\) The CorporateMetrics Technical Document (1999) is a RiskMetrics Group publication.

\(^3\) See Securities and Exchange Commission Item 305 of Regulation S-K and Item 9A of Form 20-F.
their distribution, we can define sensible corporate risk measures in a VaR-based framework. In this way, we can define CFaR as the worst case potential shortfall in cash flow (arising from market risk) with respect to targeted cash flow with a given probability over a specified reporting period, where cash flow is defined as the net change in cash balances. Similarly, EaR is defined as the worst case potential shortfall in forecasted earnings relative to targeted earnings with a given probability over a specified reporting period.

The risk horizon in corporate risk management is generally longer than three months, and the key market prices and rates must be forecasted at various horizons corresponding to the timing of the cash flows or earnings’ recognition (which is usually in line with accounting principles). The methods provided in LongRun allow us to jointly forecast key market prices and rates at a set of given long horizons, making possible the calculation of the CFaR and EaR measures.

For example, let us consider a fictitious U.S.-based multinational corporation called ABC. ABC uses oil in its production process, sells its products in Mexico, and funds its operations by raising debt indexed to 3-month T-Bill rates. Suppose ABC wants to calculate its CFaR.

Let us assume that ABC knows with a high degree of certainty the amounts of oil it will buy in the next four quarters. These amounts are shown in Table 6.1.

<table>
<thead>
<tr>
<th>Table 6.1</th>
<th>ABC’s risk exposure to oil price</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Millions of barrels purchased:</td>
</tr>
<tr>
<td></td>
<td>1st Quarter</td>
</tr>
<tr>
<td></td>
<td>0.4</td>
</tr>
</tbody>
</table>

Suppose ABC’s projected sales in Mexico (with a high degree of certainty) over the next four quarters are as shown in Table 6.2.

<table>
<thead>
<tr>
<th>Table 6.2</th>
<th>ABC’s risk exposure to the USD per MXP exchange rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Sales in MXP, mm:</td>
</tr>
<tr>
<td></td>
<td>1st Quarter</td>
</tr>
<tr>
<td></td>
<td>80</td>
</tr>
</tbody>
</table>

To support its operation, ABC needs to raise US$20mm and anticipates paying a 200 basis point spread over the 3-month T-Bill rate. This means that ABC has an exposure of US$5mm (i.e., 20/4) to the prevailing (annualized) 3-month T-Bill rate at the beginning of each of the following four quarters as shown in Table 6.3.

<table>
<thead>
<tr>
<th>Table 6.3</th>
<th>ABC’s risk exposure to the prevailing 3-month T-Bill rate</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Exposure in USD, mm:</td>
</tr>
<tr>
<td></td>
<td>1st Quarter</td>
</tr>
<tr>
<td></td>
<td>5</td>
</tr>
</tbody>
</table>

Tables 6.1–6.3 give us a map of ABC’s exposures to each price or rate at each horizon. For example, ABC’s exposure to oil price changes in the third quarter is 0.5 million barrels (Table 6.1), its expo-

---

4 In this example, the risk horizon is equal to the reporting period.
sure to the USD per MXP exchange rate in the second quarter is 85 mm pesos (Table 6.2), and its exposure to the first quarter 3-month rate is 5mm U.S. dollars (Table 6.3).

In other words, we can express ABC’s cash flow (in millions of dollars) at the end of the second quarter as:

\[
\text{Cash flow} = -0.45 \times \text{OilPrice} + 85 \times \text{MXP} - (0.1 + 5R)
\]

where OilPrice is the price of oil at the end of the second quarter, MXP is the prevailing USD per MXP exchange rate at the end of the second quarter and R is the 3-month T-Bill rate at the beginning of the second quarter.\(^5\)

In order to calculate CFaR in this example, we need forecasts for the mean and volatility of three types of data: the oil price, the USD per MXP exchange rate, and the 3-month U.S. T-bill rate at the end of each quarter. The methods presented in LongRun can be used to obtain these forecasts.

The next step in obtaining CFaR is to construct the 12 × 12 covariance matrix (3 assets at 4 horizons) as explained in Section 5.1.\(^6\) Since our positions are linear in the risk factors, we could use a parametric approach to calculate the CFaR,\(^7\) or, alternatively, we could jointly simulate the price of oil, the USD per MXP exchange rate, and the 3-month U.S. T-bill rate at the end of each quarter by taking into account the correlation structure.

We can finally calculate the 95% CFaR as the 5th percentile of the distribution of the difference between the forecasted and the targeted cash flows.

### 6.3 VaR for pension plans and mutual funds

A risk management system can help pension plan and mutual fund managers to control their risk/return profile—that is, to obtain acceptable returns while keeping only the risks to which they want to be exposed. An important step in the implementation of a risk management system is defining and calculating an appropriate risk measure. The definition that we establish depends on the type of fund we are taking under consideration. The main difference between pension and mutual funds resides in that pension plans have liabilities in the form of retirement benefits. For pension plans, the definition that we establish depends on whether we are talking about defined contribution or defined benefit plans, but regardless of the particular risk measure used in each case, the main source of risk for pension and mutual funds arises from long-term fluctuations in asset prices and interest rates used for discounting purposes. Therefore, LongRun can be used as a tool to quantify the risk introduced by those fluctuations via the calculation of a risk measure.

In this example, we analyze the risk management needs of pension plans and mutual funds, discuss how to adapt the traditional VaR measure to those needs, and briefly explain how LongRun can be used to calculate the VaR measure that we have thus defined.

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\(^5\) The 0.1 in the third term of Equation [6.1] comes from the 200 basis point spread (20 × .02/4).

\(^6\) Note that in this example we have exposures to various risk factors across a set of different time horizons. Therefore, we need to take into account the autocorrelation between returns at each pair of horizons as explained in Chapter 5.

6.3.1 Defined contribution pension plans and mutual funds

A defined contribution plan is one in which retirement benefits are completely determined by returns on fixed contributions. Since there are no significant conceptual differences in market risk measurement for mutual funds and for defined contribution pension plans, we will refer only to defined contribution pension plans in the discussion below, but all the concepts are applicable to mutual funds as well.

Managers of defined contribution plans must make investment decisions, taking some risk in order to earn an adequate return for the beneficiaries. The decision making process includes asset allocation into broad classes (e.g., equity and fixed income), specific security selection within each asset class, and timing of each particular trade. Empirical evidence suggests that asset allocation is the most important determinant of investment performance. Thus, for risk management purposes, portfolios held by fund managers can be accurately mapped to a broad set of indices characterizing the main asset classes. Examples of possible indices are the MSCI World index for equities, the J.P. Morgan WGBI index for fixed income, and the 3-month Libor rate for cash equivalent positions.

The risk faced by a defined contribution plan is the risk of a negative change in the present value of the plan’s assets when market prices and rates change. In addition, since the immediate goal is to invest the assets competitively, defined contribution plan managers are generally concerned with the risk of underperforming relative to alternative investments or benchmarks at a long horizon.

The preceding discussion suggests that a relative risk measure could be appropriate for defined contribution plans. In other words, we want to measure the risk that the difference between the value of the fund’s assets and the benchmark at a future date will be, with a given probability, smaller than a certain amount.

We can use LongRun to calculate the corresponding risk measure in the following way: once we have selected a set of indices, and mapped our portfolio and the benchmark to those indices, we can use any of the models explained in Chapter 2 and Chapter 3 to obtain the long-term mean and volatility for each of our indices. Then, we can calculate the covariance matrix at the relevant risk horizon for our representative set of indices. Once we have the mean and covariance matrix, we have fully characterized the distribution of the difference between the plan’s portfolio and the benchmark portfolio. Following the RiskMetrics methodology, we can now define our 5% VaR measure as the 5th percentile of the distribution. This risk measure indicates that 5% of the time the fund’s portfolio underperforms relative to the benchmark by an amount at least as large as the VaR.

6.3.2 Defined benefit plan

A defined benefit plan is one where pensioners’ benefits are pre-defined, often as a formula involving final salary or inflation, among other factors. Defined benefit plans face funding risk in addition to market risk; that is, the risk that the plan’s assets will be insufficient to fund the plan’s liabilities at a long horizon. Hence, one must take into account the plan’s liabilities in order to capture both the market and funding risks.

The risk associated with the plan’s liabilities is determined by several factors. For example, demographics, such as mortality rates, play an important role on the liability side, but are beyond the control of the plan and are often stable for annuity pricing purposes. An important and volatile factor...
affecting future liability payments is interest rates, which are correlated with most of the factors
driving the assets’ performance. This suggests that if we keep the demographic assumptions fixed
(revising them periodically, for example, at the end of each year), the liabilities can be approximat-
ed as a function of a set of interest rates. In fact, one can roughly approximate the liabilities as a
portfolio consisting of bonds with different maturities, some of which may be indexed to
inflation.\textsuperscript{11}

Following the discussion in the previous paragraph, we can use \textit{LongRun}'s methods to forecast the
interest rates driving the liabilities, construct the correlation structure between those interest rates
and the plan’s assets, and integrate the liability side with the VaR measure described above for de-
defined contribution plans. The difference is that with defined benefit plans we need to quantify the
risk of underperforming relative to the cost of the liabilities.

6.4 Summary

In this chapter, we provided three brief examples to illustrate the use of \textit{LongRun} in areas such as
long-term VaR calculation, corporate risk management, and pension and mutual fund risk
management.

The first example in this chapter extends the VaR measure introduced in the \textit{RiskMetrics Technical
Document} to longer horizons. It is important to emphasize that once we obtain a covariance matrix
using \textit{LongRun}'s long-term forecasting procedures, the extension of VaR to longer horizons is
straightforward.

The second example describes the application of \textit{LongRun}'s scenarios in the calculation of risk
measures in the corporate environment (i.e., CFaR and EaR). In our effort to provide a complete
corporate risk measurement framework, the RiskMetrics Group is simultaneously introducing the
\textit{CorporateMetrics Technical Document}, which addresses the market risk management needs of cor-
porations.

The last example deals with the use of \textit{LongRun}'s scenarios to calculate risk measures for defined
contribution and defined benefit pension plans, where the relevant risk horizon is long. Pension risk
management is also an active area of research at the RiskMetrics Group.

\textit{LongRun}'s applications are broader than those shown above. However, our choices for the exam-
iples in this chapter were driven by the increasing demand for long-term risk management solutions
for portfolio managers, corporations, and pension funds.

In summary, \textit{LongRun} provides a variety of tools that are applicable to solve many risk management
problems, and can be used in any situation for which long-term forecasts or scenarios are needed.

\textsuperscript{11} However, factors such as future salary growth, inflation, employment growth, and retirement patterns have an
influence on liability streams and are difficult to predict, making liabilities less “bond-like.” For a discussion on
this subject and the application of VaR to defined benefit plans see Gupta and Stubbs (1998).
### List of acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>AECM</td>
<td>Adaptive Error Correction Model</td>
</tr>
<tr>
<td>AIC</td>
<td>Akaike’s Information Criterion</td>
</tr>
<tr>
<td>AR</td>
<td>Auto-Regressive model</td>
</tr>
<tr>
<td>ARCH</td>
<td>Auto-Regressive Conditional Heteroskedasticity model</td>
</tr>
<tr>
<td>ARIMA</td>
<td>Auto-Regressive-Moving Average model</td>
</tr>
<tr>
<td>CPI</td>
<td>Consumer Price Index</td>
</tr>
<tr>
<td>DVAR</td>
<td>Difference Vector Auto-Regressive model</td>
</tr>
<tr>
<td>ECM</td>
<td>Error Correction Model</td>
</tr>
<tr>
<td>FIML</td>
<td>Full Information Maximum Likelihood</td>
</tr>
<tr>
<td>IFS</td>
<td>International Financial Statistics</td>
</tr>
<tr>
<td>IP</td>
<td>Industrial Production</td>
</tr>
<tr>
<td>LVAR</td>
<td>Level Vector Auto-Regressive model</td>
</tr>
<tr>
<td>MAE</td>
<td>Mean Absolute Error</td>
</tr>
<tr>
<td>MLE</td>
<td>Maximum Likelihood Estimator</td>
</tr>
<tr>
<td>MSE</td>
<td>Mean Squared Error</td>
</tr>
<tr>
<td>MVN</td>
<td>Multi-Variate Normal distribution</td>
</tr>
<tr>
<td>OLS</td>
<td>Ordinary Least Squares</td>
</tr>
<tr>
<td>PCD</td>
<td>Proportion of Correct Direction</td>
</tr>
<tr>
<td>POC</td>
<td>Percentage of Outliers from Confidence interval</td>
</tr>
<tr>
<td>PPP</td>
<td>Purchasing Power Parity</td>
</tr>
<tr>
<td>RWF</td>
<td>Random Walk model with Forward premium expected return</td>
</tr>
<tr>
<td>RWFHis</td>
<td>RWF using historical volatility</td>
</tr>
<tr>
<td>RWFImp</td>
<td>RWF using implied volatility</td>
</tr>
<tr>
<td>RWZ</td>
<td>Random Walk model with Zero expected return</td>
</tr>
<tr>
<td>SBD</td>
<td>Structural Break Dummy</td>
</tr>
<tr>
<td>SP</td>
<td>Share Price index</td>
</tr>
<tr>
<td>SSR</td>
<td>Sum of the Squared Residuals</td>
</tr>
<tr>
<td>VaR</td>
<td>Value-at-Risk</td>
</tr>
<tr>
<td>VARM</td>
<td>Vector Auto-Regression Model</td>
</tr>
<tr>
<td>VECM</td>
<td>Vector Error Correction Model</td>
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</tbody>
</table>


Jondeau, Eric and Michael Rockinger (1999). Reading the smile: the message conveyed by methods which infer risk neutral densities, mimeo.


