

# Excess volatility of stock prices and Knightian uncertainty

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## 1. Introduction

This paper has two themes, both of which have preoccupied economists for many years. The first is the volatility of stock prices. It has long been suggested that the erratic movements of stock prices are incompatible with rational investor behaviour. More recently, formal empirical tests have been devised which suggest that stock prices are excessively volatile. These tests show apparently systematic violations of a variance bounds inequality.

The second theme is the question of how a rational person should behave under uncertainty, particularly when he or she has little information about the form of the uncertainty. Knight (1921) was a notable proponent of the view that this type of uncertainty is qualitatively different from risky situations where the parameters of the risk are well-known to the decision maker. However, the current standard model in economics, due to Savage (1954), predicts that agents should have subjective probability distributions which do not make this distinction.

In this paper, we show that models which formalize Knightian uncertainty can be used to explain high stock price volatility. We provide an example in which the variance bound is violated. Since the future profitability of companies depends heavily on many long-term factors, including political factors, which are extremely difficult to predict, it is natural to think that the

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stock market is characterized by a high degree of Knightian uncertainty. Therefore, we suggest this type of behaviour under uncertainty as a possible explanation of the high volatility of stock market prices.

## 2. Excess volatility

The variance bounds restrictions were developed by LeRoy and Porter (1981) and Shiller (1981). Here we give only the briefest outline of the restrictions and the evidence; the reader is referred to LeRoy's (1989) survey and Shiller's (1989) book for further details and references. The starting point is the principle that if agents are risk-neutral an asset's price should equal the expectation of the discounted value of the future dividends. Here we will assume for simplicity a situation where the asset pays a single liquidating dividend with present value  $V$ , and the current price is  $P$ .  $P$  depends on the agent's information set,  $I$ . The analysis applies to the general case with many dividend payments if  $V$  is interpreted to be the present value of the future dividends. Thus

$$P = E(V|I).$$

The variance bounds inequality is derived from the condition

$$\text{Var}(V) = \text{Var}[E(V|I)] + E[\text{Var}(V|I)]$$

which is in turn an immediate consequence of the relation  $E(V) = E[E(V|I)]$ . Since variances are non-negative and the price equals the expected value,

$$\text{Var}(V) \geq \text{Var}(P). \quad (1)$$

To test this requires some statistical assumptions. For example, if we had a large number of such assets whose liquidation values were iid, we could simply compare the sample variance of the set of prices and the set of realized values. The empirical studies [LeRoy and Porter (1981) and Shiller (1981)] used time series data on stock prices and dividend realizations, assuming a stationary dividend process (later studies, e.g., Campbell and Shiller (1988), made weaker assumptions on the dividend process).

How do the variance bounds apply to a situation of Knightian uncertainty? Then the agent does not know the actual probability distribution of the value. A risk neutral agent who satisfies the Savage axioms has a subjective probability distribution. This subjective distribution will in general differ from the actual distribution, and so the variance bound can be violated in a probabilistic sense (because the expectation, under the actual distribution, of the conditional expectation under the subjective distribution, will in

general differ from the actual unconditional expectation). However, it seems implausible to explain the large amount of evidence that has been collected by saying that agents systematically have subjective distributions with higher variance than actual distributions (although logically, this argument is dubious because it implies that the agent effectively knows the true probability distribution of the states but with a less precise information set). We therefore turn to another explanation: that the agents' decisions, in situations of Knightian uncertainty, are not represented by subjective probability distributions.

### 3. Uncertainty aversion

Schmeidler (1982, 1989) and Gilboa (1987) have developed an axiomatic model of rational decision making in which agents' behaviour distinguishes between situations where agents know the probability distributions of random variables, and situations where they do not have this information. We refer to the former as *risk* and the latter as *uncertainty*, or Knightian uncertainty. Synonyms that are used in the literature include roulette lottery, for risk, and horse lottery and ambiguity, for uncertainty. We now give a very brief exposition of the main aspects of the model. The reader is referred to the papers by Schmeidler and Gilboa cited above for a complete description and for the underlying axioms, and to Dow and Werlang (1991) which contains an example and an application to portfolio choice. Simonsen and Werlang (1991) also describe the implications for portfolio choice. Bewley (1986) presents a similar model which is also designed to capture Knightian uncertainty. His model predicts that uncertainty leads to inertia, a tendency to favour the status quo, while in Schmeidler–Gilboa there is a tendency to choose acts where the agent does not end up bearing uncertainty.

The Schmeidler–Gilboa model predicts the agents' behaviour will be represented by a utility function and a (subjective) non-additive probability distribution. A non-additive probability  $p$  reflecting aversion to uncertainty satisfies the condition

$$p(A) + p(B) \leq p(A \cup B) + p(A \cap B), \quad (2)$$

rather than the stronger condition satisfied by (additive) probabilities:

$$p(A) + p(B) = p(A \cup B) + p(A \cap B). \quad (3)$$

In particular,  $p(A) + p(A^c)$  may be less than 1; the difference can be thought of as a measure of the uncertainty attached by the agent to the event  $A$ .

The agent maximizes expected utility under a non-additive distribution, where the expectation of a non-negative random variable  $X$  is defined as:

$$E(X) = \int_{\mathfrak{A}^+} p(X \geq x) dx. \quad (4)$$

Associated with a non-additive probability  $p$  is a set  $\Delta$  of additive probabilities called the core of  $p$ , which is defined (analogously to the core in cooperative game theory) as the set of additive probability measures  $\pi$  such that  $\pi(A) \geq p(A)$  for all events  $A$ . If the non-additive probability satisfies inequality (2) (reflecting aversion to uncertainty) the core is non-empty. (A closely related model of behaviour under uncertainty is for the agent to act to maximize the minimum value, over the elements of the core, of expected utility [Gilboa and Schmeidler (1989)]. It is tempting, but misleading, to interpret the core as the set of 'possible' probability distributions. For example, the model includes as a special case agents who satisfy the Savage axioms because they are not averse to uncertainty, but clearly such agents do not necessarily know the true probability distribution. However, for the analysis below we will need some relationship between the subjective non-additive distributions which represent agents' behaviour, and the frequencies observed by the econometrician. We will therefore assume that the econometrician observes realizations drawn from one element of the core.

When agents' preferences satisfy Savage's axioms, it is natural to assume that they update according to Bayes' rule. Although this is not a consequence of Savage's model, many considerations point to Bayes' rule (the standard argument is given in Kreps (1988); Brown (1976) goes further, showing that it is optimal for agents to use Bayes' rule). With non-additive probabilities, the situation is not so clear but the Dempster-Shafer rule is the natural generalization of Bayes' rule [Dempster (1968) and Shafer (1976)]. Dow, Madrigal and Werlang (1989) use the Dempster-Shafer rule. Schmeidler and Gilboa (1991) provide an axiomatic foundation for the Dempster-Shafer rule. The updating rule is:

$$p(A|B) = [p(A \cup B^c) - p(B^c)] / [1 - p(B^c)]. \quad (5)$$

We now apply this model to an example of equilibrium in a stock market with Knightian uncertainty.

#### 4. An example

There are three periods  $t=0, 1, 2$ . There is one risky asset in positive net supply ('the asset') and a safe asset (cash). The riskless rate of return is zero

or, equivalently, all values may be interpreted as present values discounted at the safe interest rate.

The asset will pay a liquidating dividend  $V$  at time  $t=2$ . There are three states of nature 1, 2 and 3. The value of  $V$  is different in each state and is, respectively, 1,  $\frac{1}{2}$  or 0.

Agents are all identical, and are risk neutral. In period  $t=1$ , agents receive (public) information about the value of the asset, represented by a partition,  $I$ , of the set of states. The partition is  $I = \{\{1\}, \{2, 3\}\}$ ; in other words agents are informed whether or not state 1 has occurred. (The initial period  $t=0$ , before agents receive any information, plays no role in the variance bounds violation but we include it here for completeness' sake.) Agents' beliefs and attitudes towards uncertainty are represented by the following non-additive probability measure:

number 1

$$\begin{aligned} p_1 = p_2 = p_3 = \frac{1}{4}, \\ p_{12} = p_{23} = p_{13} = \frac{1}{2}. \end{aligned} \tag{6}$$

Here  $p_{ij}$  is the (non-additive) probability of state  $i$  or  $j$ : because eq. (3) does not hold it is necessary to specify these in addition to the (non-additive) probability of each state. The core,  $A$ , of this measure is the convex hull of the following three (additive) probability measures:

$$\pi_1 = \frac{1}{2}, \pi_2 = \frac{1}{4}, \pi_3 = \frac{1}{4}, \tag{7}$$

$$\pi_1 = \frac{1}{4}, \pi_2 = \frac{1}{2}, \pi_3 = \frac{1}{4}, \tag{8}$$

$$\pi_1 = \frac{1}{4}, \pi_2 = \frac{1}{4}, \pi_3 = \frac{1}{2}. \tag{9}$$

Let  $P_t$  denote the price at time  $t$ . Then since the asset is in positive net supply, it follows [Dow and Werlang (1991)] that  $P_0 = E(V)$ ,  $P_1 = E(V|I)$ , and naturally  $P_2 = V$ . Thus in the initial period,

$$P_0 = E(V) = 0 + (\frac{1}{2} - 0)p_{12} + (1 - \frac{1}{2})p_1 = 3/8.$$

If the agents receive good news (state 1) then  $P_1 = 1$ . (It is straightforward to show that this is implied by the Dempster-Shafer rule.) If they receive bad news (state 2 or 3) we first need to calculate the conditional probability of state 2. By the Dempster-Shafer rule [eq. (4)] this is given by

$$(p_{12} - p_1)/(1 - p_1) = (\frac{1}{2} - \frac{1}{4})/(1 - \frac{1}{4}) = 1/3,$$

so the price is  $P_1 = 1/6$ .

To summarize:

State	1	2	3
	3/8		
	1		
	1		

12a

7

Let the actual distribution of prices be  $\pi$ . Then (as may be easily verified) the variances are

$$\text{Var}(P_1) = \pi_1(1 - \pi_1) + \pi_{23}(1 - \pi_{23})/36 - \pi_1\pi_{23}/3,$$

$$\text{Var}(V) = \pi_1(1 - \pi_1) + \pi_2(1 - \pi_2)/4 - \pi_1\pi_2,$$

where  $\pi_{23} = \pi_2 + \pi_3$  since  $\pi$  is an additive probability distribution. As previously discussed, it is not clear what relationship  $\pi$  should bear to the subjective non-additive distribution  $p$ . For example, if the agent were not uncertainty averse  $p$  would be an additive distribution, but since this is consistent with the agent not knowing the distribution, it would be quite natural to have  $p \neq \pi$ . Nevertheless, for the reasons outlined above, we will assume that  $\pi \in \Delta$ , where  $\Delta$  is the core of  $p$ .

For  $\pi \in \Delta$ , the variance of  $P_1$  ranges over the interval

$$[25/192, 100/576] = [0.1302, 0.1736],$$

while the variance of  $V$  ranges over the interval

$$[8/64, 11/64] = [0.1250, 0.1719].$$

Notice that both endpoints of the interval of possible variances are larger for the price than for the value. This is a stronger property than having an element of  $\Delta$  for which the value has lower variance than the price.

For the three probability distributions listed above, we have:

In the case of (7),  $\pi_1 = \frac{1}{2}, \pi_2 = \frac{1}{4}, \pi_3 = \frac{1}{4}$ ,

$\text{Var}(P_1) = 0.1736, \text{Var}(V) = 0.1719$ , violating the variance bound.

In the case of (8),  $\pi_1 = \frac{1}{4}, \pi_2 = \frac{1}{2}, \pi_3 = \frac{1}{4}$ ,

$\text{Var}(P_1) = 0.1302, \text{Var}(V) = 0.1250$ , violating the variance bound.

In the case of (9),  $\pi_1 = \frac{1}{4}, \pi_2 = \frac{1}{4}, \pi_3 = \frac{1}{2}$ ,

$\text{Var}(P_1) = 0.1302, \text{Var}(V) = 0.1719$ , which does not violate the variance bound.

This example therefore shows that agents who are averse to uncertainty in the sense of Schmeidler and Gilboa may violate the variance bound inequality.

## 5. Absence of arbitrage opportunity

We conclude by showing that the prices in our example do indeed constitute an equilibrium in the sense that there is no arbitrage opportunity.

In defining arbitrage opportunity, we must clearly not introduce additional agents into the model who have different preferences and beliefs, e.g. agents who are not uncertainty-averse. In equilibrium, there will be no trade since agents are identical. Thus to verify absence of arbitrage opportunity we consider whether an agent has any incentive *ex ante* to carry out intertemporal trades, e.g. trades which involve buying the asset in one period and selling it back in another period.

The following discussion will make use of several properties of expectation under non-additive measures, for details of which the reader should refer as necessary to the works cited above. For example,  $E(\lambda X) = \lambda E(X)$  for  $\lambda \geq 0$  while  $-E(-X) \geq E(X)$ , also  $E(X + Y) \geq E(X) + E(Y)$  but with equality in case  $X$  and  $Y$  satisfy a condition called comonotonicity.

An example of an arbitrage strategy is for an agent to buy an extra unit in period 0 and sell it again in period 1. Assuming without loss of generality that the initial holding is one unit, the expected utility from this strategy is  $E[V(s) - P_0 + P_1(s)]$  and from continuing to hold the asset it is  $E[V(s)]$ . Here random variables have been explicitly expressed as functions of the state of nature,  $s$ , for clarity. We therefore require

$$E[V(s) - P_0 + P_1(s)] \leq E[V(s)],$$

or, by comonotonicity of  $V(s)$  and  $P_1(s)$ ,

$$E[V(s)] - P_0 + E[P_1(s)] \leq E[V(s)],$$

i.e.,  $P_0 \geq E[P_1(s)]$ . In the example, we have

$$E[P_1(s)] = 1/6 + (1 - 1/6)(1/4) = 3/8 = P_0,$$

so the arbitrage is not profitable.

Equally, the agent must not have an incentive to sell the asset and buy it back one period later:

$$E[V(s) + P_0 - P_1(s)] \leq E[V(s)].$$

In state 1, the wealth if the arbitrage is carried out has value  $1 + P_0 - 1 = P_0$ ; in state 2,  $1/2 + P_0 - 1/6 = P_0 + 1/3$ ; and in state 3,  $0 + P_0 - 1/6 = P_0 - 1/6$ . Hence the arbitrage has expected value

$$(P_0 - 1/6) + (1/6 \cdot 1/2) + (1/3 \cdot 1/4) = P_0 = E[V(s)].$$

Again the arbitrage is not profitable.

The complete set of conditions to rule out arbitrage is

$$E[V(s) + \alpha P_0 - \alpha V(s) + \beta V(s) - \beta P_1(s)] \leq E[V(s)],$$

in other words the agent does not want to sell  $\alpha$  units at  $t=0$  and buy  $\beta$  at  $t=1$ . It may be verified (by considering the various cases, depending on the signs and relative magnitudes of  $\alpha$  and  $\beta$ ) that this inequality is always satisfied. Hence, the equilibrium described in this paper is indeed free of arbitrage opportunities.

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