The Limitations of Simple Two–Factor Interest Rate Models

Riccardo Rebonato

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ABSTRACT

Empirical correlations between forward interest rates and the relative prices of caps and swaptions suggest that changes in forward rates with adjacent maturities should not be too highly correlated. This article shows that this is very difficult to achieve for a general class of simple two–factor interest rate models, which severely limits the ability of such models to describe the market structure of cap and swaption prices simultaneously. The problem is shown to be related to the standard principal components analysis of interest rates.

This restriction is shown to hold for three important two–factor models: the Longstaff–Schwartz model, a version of the Brennan and Schwartz model, and the Chen and Scott model. Market prices of swaptions are shown to deviate systematically from prices based on the Brennan–Schwartz model parameterized to fit the cap curve. This difference is caused by the fact that the model is forced to imply too high a correlation between adjacent forward rates.

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I. INTRODUCTION

It is well known that one-factor interest rate models are intrinsically incapable of pricing simultaneously and consistently instruments which depend on the imperfect correlation between rates. All rates, if driven by a single factor, must display perfect instantaneous correlation.

Almost all interest rate options, however, depend in an important way on the correlation between rates. (Virtually the only exceptions are caps and some trivial cases of knock-outs.) Both academics and practitioners have therefore developed two-factor models as the panacea that will allow the consistent and simultaneous pricing of at least all the fundamental options (caps and European swaptions) observed in the market. Important examples include Brennan and Schwartz (1979), Heath, Jarrow, and Morton (1990), Longstaff and Schwartz (1992), and Chen and Scott (1992).

As the first two principal components of interest rate changes appear to explain a very large percentage of the dynamics of yield moves (Wilson 1994), one might expect two-factor models to be able to adequately price almost any interest rate options. It could be, however, that the part of yield behavior that is not adequately captured in two-factor models is particularly important in pricing some options. The purpose of this paper, therefore, is to explore to what extent simple two-factor models can capture an important feature of the covariance structure of rate changes. This feature is the way in which changes in adjacent forward rates are correlated.

<table>
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<th>2</th>
<th>3</th>
<th>5</th>
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</tbody>
</table>


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Table 1 shows the correlations between changes in forward interest rates in the period 1987–94. A key feature of this table is the rapid falloff of the correlation between forward rates as the difference between their delivery dates is increased. For instance, the correlation between changes in the six-month and one-year forward rates is .96, a decline of .04 caused by a time difference of six months. This falls by only a further .03 (to .93) when we look at the six-month and eighteen-month forward rate correlation, a further increase of six months in the maturity difference. Thus, the forward rates appear to "decorrelate" quite rapidly even for quite small differences between maturities. (This effect is even more pronounced if one looks at changes over periods shorter than a month.)

The same "decorrelation" appears in the pricing of swaptions. As an example, on February 8, 1995, the implied standard deviations of the 12-, 15-, 18-, and 21-month caplets on three month $US rates were all 18 percent. At the same time, the implied rate volatility of the one year option into a one year swap was 16.75 percent. The forward swap rate underlying this option is very close to an average of the four forward rates underlying the caplets. The low implied volatility of the swaption compared to the caplets implies that the average of the four rates underlying the caplets has a lower volatility than the rates themselves, implying that their correlations are significantly less than one.

II. A BASIC PROPERTY OF TWO–FACTOR MODELS

To investigate the ability of two–factor models to capture the effect described in the previous section, we consider models of the form:¹

\[ dF(t) = \mu_F dt + A(t)dY + B(t)dZ \]  

(1)

where:
- \( F(t) \) is the forward rate for delivery date \( t \)
- \( dY \) and \( dZ \) are orthogonal random variables with unit variances
- \( A(t) \) and \( B(t) \) are factor loading functions

We are interested in the structure of the correlations between changes in forward rates for different delivery dates. We define the function \( \rho(t,T) \) as the correlation between \( dF(t) \) and \( dF(T) \). Appendix 1 shows that, for any model of the form of Equation 1, this function has the property:

\[ \frac{\partial \rho(t,T)}{\partial T} \bigg|_{T=t} = 0 \]  

(2)

This condition for this class of models limits the rate at which changes in forward rates can "decorrelate" as we initially increase the difference between their maturities. It appears to be inconsistent with the empirical evidence.

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presented in Table 1, where the correlation between adjacent forward rates falls off rapidly as their difference is initially increased. Thus, simple two-factor models of the type described by Equation 1 appear to have an inherent limitation in capturing the empirical structure of forward rate changes.

III. PRINCIPAL COMPONENTS ANALYSIS OF FORWARD RATE CHANGES

We can gain further insight into the restriction on the correlation structure placed by simple two-factor models by examining the effect of approximating a given correlation structure with its first two principal components. Table 1 shows that the historical correlation between changes in forward rates decreases with increasing difference in the maturity of the rates. The correlation falls away from unity quite quickly as the difference between the maturities of the rates is initially increased.

In order to focus on the essential features of the problem, we posit a simple exponentially decaying structure for the correlation function:

![Figure 1: Cap Volatility](image)

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\[ \rho(t,T) = \exp(-b(T - t)) \]  

(3)

In this case the correlation structure of forward rate changes is fully described by a single decay constant \(b\).

In order to give a concrete focus to the discussion, we address the particular issue of the correlation between a collection of 10 six-month US$ forward rates, with the first forward maturing in two years’ time. These are the forward rates that would enter the pricing of any \(2 \times 5\) swaption, or of any option to enter a five year cap in two years’ time, or of any five-year forward swap amortized in two years’ time by the realization of the six-month spot rate. We use the cap volatility curve displayed in Figure 1 and the correlation function given by Equation 3, with a parameter value for \(b\) equal to 0.8.²

Given this covariance matrix, a principal components (PC) analysis was undertaken. The resulting eigenvalues and eigenvectors are shown in Table 2. The usual interpretation of the first three principal components as level, slope, and curvature is borne out by the analysis. But, for this very “clean” example, another interesting interpretation is possible. Figure 2 shows the weights given to the original variables (the forward rates \(F_i\)) to obtain the principal components \(Y_i\). These figures suggest the interpretation of the eigenvectors as the basis functions in a Fourier series expansion, with each vector contributing higher and higher frequency components.

<table>
<thead>
<tr>
<th>Table 2</th>
<th>Principal Components Analysis of the Correlation Function (\exp[-b(T - t)])</th>
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</thead>
<tbody>
<tr>
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<td>Eigenvalues</td>
</tr>
<tr>
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<tr>
<td>Eigenvectors</td>
<td>.34</td>
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<tr>
<td></td>
<td>.36</td>
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<td>.25</td>
</tr>
<tr>
<td></td>
<td>.20</td>
</tr>
</tbody>
</table>

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The PC transformation can be seen as a rotation of the $n$–dimensional vector $F$ by means of a matrix $W$ into a new vector $Y$:

$$Y = WF$$

(4)

Alternatively, one can re–obtain the original quantities $F$ by using the inverse matrix $W'$:

$$F_i = \sum_{j=1}^{n} w'_{ij} Y_j$$

(5)

One can mimic the approach of an $m$–factor model by restricting the summation in Equation 5 to the first $m$ terms:

$$F_{m_i} = \sum_{j=1}^{m} w'_{ij} Y_j$$

(6)
The variance of the new variable $F_{m_i}$ is clearly smaller than the variance of the original variable $F_i$. To the extent, however, that the PC approach efficiently accounts for the observed variability, it is usually claimed that the loss of information should be small.

In order to obtain correct interest rate option pricing in an $m$–factor model, the variances of the included factors are implicitly re–scaled so as to compensate for the variance "lost" by carrying out the summation over $m$ rather than $n$ terms. When this is done, the correlation structure implied by using only $m$ PCs is given by:

$$\text{Covar}(F_{m_i}, F_{m_j}) \sum_{k=1}^{m} w_{ik} w_{jk} \text{var}(Y_k)$$  \hspace{1cm} (7)

where the orthogonality among the $Y$s has been used to eliminate their covariances.

**Figure 3**

*Correlations Between the Spot Rate and Forward Rates with Only the First Two Principal Components*

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Figure 3 shows the correlation between the first forward rate and the remaining forwards when only the first two principal components are used to generate rate changes. This has the interesting qualitative feature of a sigmoid–like correlation structure which does not exhibit the fast decorrelation found empirically between adjacent forward rates. It is actually rather surprising to find that, in order to recover an exponential–like correlation structure, one seems to have to employ a rather high number of factors (approximately four or five in this case).

Looking at Equation 7 one can appreciate the implications of the Fourier series interpretation of the eigenvectors outlined before. By limiting the set of basis functions to the first m terms, one constructs the model covariance matrix by using a linear combination of sine functions of increasing frequency, with positive and monotonically decreasing weights. Not surprisingly, the model correlation that can be obtained using a constant and a sine wave of frequency equal to half the full maturity span (this is what is implicitly used by a two–factor PC–based model) is constrained to look like a sigmoid.

**Figure 4a**

**Correlation Between the Spot Rate and Forward Rates for the Longstaff and Schwartz Model**

![Correlation Diagram](image-url)

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IV. IMPLICATIONS FOR TWO-FACTOR MODELS

The slow "decorrelation" of forward rates as their maturity difference is increased appears to be a general and significant feature in two-factor models of rate behavior based on principal components. It could be, however, that it is of limited importance in other types of two-factor models.

First, since principal components are constructed to explain with maximum efficiency the diagonal elements of the covariance matrix and hence the overall level of the observed variance, it is possible that different approaches (such as Factor Analysis), which explicitly focus attention on the off-diagonal elements, might give different results. Second, approaches such as Longstaff and Schwartz (1992), which have as the second factor a variable that is not a linear combination of rates (the variance of the short rate) could in principal describe the qualitative shape of the empirical correlation function in a more realistic way.

To investigate this, we examined the correlation structures of forward rates in three models: a version of the Brennan and Schwartz (1979) model (described in Appendix 2), the Longstaff and Schwartz (1992) model, and the Chen and

Figure 4b
Correlation Between the Spot Rate and the Forward Rates for the Modified Brennan and Schwartz Model
Scott (1992) model. All produced exactly the sigmoid correlation function predicted by this study. The results for the Longstaff and Schwartz, and Brennan and Schwartz models are shown in Figures 4a and 4b.

V. PRICING CAPS AND SWAPTIONS

The implication of these results is that it is impossible, within the framework of principal component-based (and probably all simple) two-factor models, to price all caps and swaptions consistently with the market. A single-factor model can account for all the cap prices, but no swaption could be simultaneously and consistently priced. A simple two-factor model can price at the same time all the caps and one arbitrary swaption. It may also price the particular family of swaptions which happen to have a combination of expiry time and final maturity implied by a line generated by the intersection of the model covariance matrix and the market covariance matrix, but not all swaptions.

Figure 5
Modified Brennan and Schwartz (BS Swapt), Market (Market) and Perfect Correlation (BlackCorr1) Swaption Prices

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This can be seen in Figure 5, which shows the market and model swaption prices for a variety of swaptions for the modified Brennan and Schwartz model (denoted BS Swapt). It also shows the model prices for a version of the Black (1976) model computed using perfect correlation between forward rates (a single-factor model denoted BlackCorr1). Both models are calibrated to fit precisely the market values of all caps. The data underlying Figure 5 are mid-market prices for February 2, 1995, given in Table 3.

The additional degrees of freedom of the two-factor model have been used to ensure that, for each swaption series, there is always one intersection point between the market and BS model curves. Thus, the two-factor model is clearly superior to the one-factor model in explaining the general relative prices of swaptions and caps. Any decorrelation of rates brings about a significant improvement with respect to any one-factor model.

The two-factor model cannot, however, match the shape of the swaption value curve.\(^3\) It overprices swaptions that are exercisable into relatively short-term rates for a given option maturity. This reflects the inability of the two-factor model to generate sufficient “decorrelation” of forward rates that are close together. Forward swap rates of short maturity are averages of forward rates that are close together, so the variance of these swap rates is exaggerated in the model relative to their actual behavior.

This relative pricing bias of the two-factor model is a permanent feature, and not just the effect of trading pressure in part of the market on a particular day. The same effect was observed on all days checked by the authors and whichever side of the market (bid, ask, or mid-price) was analyzed. The bias is also large economically. The omitted structure amounts, for some of the swaptions considered, to 15–20 basis points out of a total price of 250. It is also large relative to the bid/ask spread faced by some traders. The bid/ask spread on swaption trades depends on the expiry date, swap maturity, and counterparty but, as an order of magnitude, the spreads for the trades analyzed here would have been a few basis points.

VI. MORE COMPLEX MODELS

Both the cap rates and the swaption rates can be fitted by increasing the complexity of the models. If one were to introduce a third factor, one probably could obtain the desired correlation shape, but at the expense of giving a spuriously large weight to the third component. This would run against the econometric interpretation of the underlying factors and create a forced fitting to a misspecified model. This is, to a small extent, what is done by all one-factor models. By constructing each forward rate by using a truncated summation of factors, one is creating new variables which display a smaller variance than the original components. By forcing exact pricing of caps, one is implicitly re-scaling
Table 3
Data Underlying Figure 5
Mid-market dealer quotes for at-the-money caps and
at-the-money options into par forward swaps

3A: Zero Coupon Interest Rates

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<th>Maturity in Years</th>
<th>Rate (%)</th>
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<td>2</td>
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<td>7</td>
<td>8.20</td>
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<tr>
<td>10</td>
<td>8.22</td>
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</table>

3B: Cap Rates (%ISDs From Black Model)

<table>
<thead>
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<th>Maturity Date</th>
<th>ISD(%)</th>
<th>Maturity Date</th>
<th>ISD(%)</th>
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<tbody>
<tr>
<td>Mar–95</td>
<td>12.75</td>
<td>Dec–97</td>
<td>22.00</td>
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<tr>
<td>Jun–95</td>
<td>14.75</td>
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<td>Sep–95</td>
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<td>22.25</td>
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3C: Swaption Rates (% ISDs from Black Model)

<table>
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<th>Exercise Maturity (years)</th>
<th>Underlying Swap Maturity in Years</th>
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<tbody>
<tr>
<td></td>
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<td>1</td>
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<td>5</td>
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the variance of the first component to match the market data. Due to the lack of structure in the first principal component, which is roughly a constant, this has no major effect. The same would no longer be true, however, if one allocated in an arbitrary fashion the variances of the further principal components. Indeed, an exponential–looking correlation structure can be obtained with two factors using implausibly fluctuating (and time dependent) variances. As usual, blind, brute-force fitting to market quantities does not generate realistic time series properties of the variables.

VII. SUMMARY AND CONCLUSIONS

We have shown that simple two–factor models share a problem in being unable to simultaneously explain the market prices of all caps and all swaptions. This stems from the inability of such models to generate low enough correlation between changes in adjacent forward rates. This restricts the covariance structure of rates, as can be clearly seen in models that use the first two principal components of rate changes as the factors.

This restriction on the covariance structure does not, however, appear to be a result of the particular assumptions of specific models, but a general consequence of the low dimensionality of simple two–factor models. Such details of the correlation structure are increasingly important in pricing. More and more assets (such as range chooser notes, index principal swaps, trigger notes, and captions) depend on the correlation between rates rather than individual rate moves or volatilities.
APPENDIX I

Consider any two-factor model of the form:

\[ dF(t) = A(t)dY + B(t)dZ \]

where:
- \( F(t) \) is the instantaneous forward rate for date \( t \)
- \( dY \) and \( dZ \) are orthogonal random variables with unit variances
- \( A(t) \) and \( B(t) \) are factor loading functions

Then:

\[ \rho(t,T)^2 = \left[ A(t)A(T) + B(t)B(T) \right]^2 / \left[ (A(t)^2 + B(t)^2)(A(T)^2 + B(T)^2) \right] \]

where:
- \( \rho(t,T) \) is the correlation between \( dF(t) \) and \( dF(T) \)

So:

\[
\frac{\partial \rho(t,T)^2}{\partial T} = \frac{2[A(t)A'(T) + B(t)B'(T)]/[\left( A(t)^2 + B(t)^2 \right)(A(T)^2 + B(T)^2)] - 2[A(t)A(T) \cdot B(t)B(T)][A(T)A'(T) + B(T)B'(T)]}{\left( A(t)^2 + B(t)^2 \right)^2\left( A(T)^2 + B(T)^2 \right)^2}
\]

where prime denotes a derivative.

\[
\frac{\partial \rho(t,T)^2}{\partial T}_{\tau=t} = 0
\]

\[
\frac{\partial \rho(t,T)}{\partial T}_{\tau=t} = 0
\]
APPENDIX II

The Brennan and Schwartz (1979) model was modified to avoid the criticism of Hogan (1993). The version used is:

\[ \frac{dC}{C} = \mu_C dt + \sigma_C dZ_C \]

\[ dsp = \mu_{sp} dt + \sigma_{sp} dZ_{sp} \]

where:

- \( C \) is the price of a consol bond
- \( sp \) is the spread between \( L \) and the short rate
- \( dZ_{sp} \) and \( dZ_L \) are increments to Wiener processes with \( E[dZ_{sp} dZ_L] = \rho dt \)

and \( \mu_C \) and \( \mu_{sp} \) are determined to preclude arbitrage between bonds of different maturities.

NOTES

1. The drift term \( \mu_p \) will, in general, ensure that no-arbitrage conditions are met. See, for example Heath, Jarrow, and Morton (1990). If the two factors used are taken to be the first two principal components, then Equation 1 describes a popular implementation of the Heath, Jarrow, and Morton model.

2. Changing \( b \) does not change the shapes of the eigenvectors, so the qualitative results are insensitive to the value of \( b \).

3. Brace and Musiela (1994) simultaneously price caps and swaptions. They are not able, however, to compare their model swaption prices with those in a liquid market.

REFERENCES


