THE RELATIONSHIP BETWEEN TWO METHODS OF VALUING CONVERTIBLE BONDS

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1. Introduction

The traditional method of valuing convertible debt or setting the terms on new convertible issues is the so-called 'cross-over' method. This involves forecasting the first date at which the cash flow from dividends on the equity into which the bond is convertible will exceed the coupon flow from the debt. The excess of the coupon flow over the dividend flow up to that date is then present valued. This value is added to the current value of the equity underlying the bond to give the 'value' of the convertible.

A more modern method of valuing convertibles is to treat the instrument as a package of a bond and an option. The bond is the straight bond component included in the convertible. The option is an option to switch out of this bond into the equity underlying the convertible. This option is equivalent to a call option on the underlying equity, with an exercise price determined by the characteristics of the bond. The value of the convertible is the sum of the bond value and the value of this call option.

The purpose of this note is to explain the relationship between these two valuation approaches.
2. Assumptions

This section lists the assumptions that are used throughout this paper. They are:

A1. The convertible pays a continuous coupon at a rate \( y \). It has maturity \( M \) and face value of unity.

A2. The equity into which one unit of the convertible can be converted has current price \( S \). It pays a continuous dividend yield \( d \). The dividend yield (and therefore also the stock price) is expected to grow at a constant rate \( g \).

A3. The bond flows are riskless. The yield curve is flat, the continuous interest rate is \( r \), which is constant.

A4. The risk-adjusted discount rate for the equity is \( k \).

These assumptions leave out the impact of periodic interest and dividend flows, variation in interest rates, sloped yield curves, the default-risk of the debt, and variations in the growth rate and yield of the equity. They are, however, close to the assumptions used in the 'cross-over' method of valuation and capture most of the first order effects in convertible valuation.
3. The Cross-over method

3.1 The Cross-over date

Given the assumptions, the divided flow from the equity will equal the coupon flow from the bond at some date T. T is the solution to:

\[ y = S_0 e^{gT} \]  \hspace{1cm} (1)

The left-hand side of (1) is the coupon flow from the bond. The right-hand side is the initial dividend, S0, compounded at the continuous growth rate, g, to the date T. When these are equal, the dividend flow equals the coupon. From that point onwards, the dividend will exceed the coupon. The cross-over method assumes that the holder will convert the bond at the date T.

3.2 The 'value' of the convertible

The 'value' of the convertible, according to the 'cross-over' method is the share price, S, plus the present value of the excess coupon stream received until the cross-over point, T. This value depends on the discount rate used to value the excess coupon stream. It is conventional to use the risk-adjusted required rate of return on the equity, k, for the discount rate. Appendix I shows that this extra value is:

\[ E = \left( \frac{y}{k} \right) \left[ 1 - e^{-kT} \right] - S \left[ 1 - e^{-dT} \right] \]  \hspace{1cm} (2)
Thus the convertible 'value' is:

\[
CV_1 = S + E
\]  
(3)

\[
= Se^{-dT} + (y/k)[1-e^{-kT}]
\]  
(4)

The first term of this expression is the value of the equity stripped of the first \( T \) years of dividends. The second term is the value of the first \( T \) years coupon flow from the bond, valued at a discount rate of \( k \).
4. The option approach

The option approach values the straight bond component of the convertible, and adds the value of the call option embedded in the convertible.

4.1 The bond value

The straight bond component of the convertible has value:

\[ B = \left( \frac{y}{r} \right) \left[ 1 - e^{-rM} \right] + e^{-rM} \]  
(5)

The first part of this expression is the annuity value of a continuous coupon stream of \( y \) up to date \( M \). The second part is the present value of the principal to be received at date \( M \).

4.2 The option

The option component of the convertible is a call option on the underlying equity, with an exercise price equal to the value of the straight bond at the date of conversion. To make the comparison with the cross-over approach clear, we make a further assumption which will not, in general, be valid. It is, however, equivalent to the assumption made in the cross-over approach.
Assumption A5: The option to convert will be exercised, if at all, at the cross-over date, T.

Given this assumption, we can use the dividend-adjusted variant of Black and Scholes (1973) to value the option. The option component embedded in the convertible is illustrated in Figure 1. The appropriate inputs to the Black-Scholes equation are:

- Underlying asset value: \( S e^{-dT} \)
- Maturity: \( T \)
- Interest rate: \( r \)
- Volatility: \( \sigma \)
- Exercise price: \( B(T) \)

The volatility is the volatility of the underlying asset, \( S \). The exercise price, \( B(T) \), is the value that the straight bond will have at time \( T \). This is given by:

\[
B(T) = (y/r) [1 - e^{-r(T-T)}] + e^{-r(T-T)}
\]

The first term is the value, at time \( T \), of the remaining coupons on the bond. The second term is the value, at time \( T \), of the principal due at date \( T \).

4.3 The option value
FIGURE 1

THE EMBEDDED OPTION

VALUE OF UNDERLYING EQUITY

B(T)

BOND VALUE

EMBEDDED OPTION

0

0
The value of the option component of the bond is given by:

\[ C_2 = Se^{-dT}N(d_1) - B(T)e^{-rT}N(d_2) \quad (7) \]

where:

\[ N(.) \] is the standard normal distribution function

\[ d_1 = \frac{\ln(se^{-dT}/B(T)e^{-rT}) + 1/2 \sigma^2T}{\sigma \sqrt{T}} \quad (8) \]

\[ d_2 = d_1 - \sigma \sqrt{T} \quad (9) \]

4.4 The convertible value

The value of the convertible is now given by the sum of the bond value and the option value. This is:

\[ CV_2 = B + C_2 \quad (9) \]

Where the value of the call option embedded in the convertible, \( C_2 \), is given by expression (7).

By substitution of expression (7) for \( C_2 \), and expression (8) for \( B(T) \), the value of the convertible is given by:

\[ CV_2 = Se^{-dT} N(d_1) + \left( \frac{\sigma}{\sqrt{T}} \right)^2 N(d_2) + B[1-N(d_2)] \quad (10) \]
The three terms in this expression can be roughly interpreted as:

1. The equity value of the convertible, adjusted by the dividends lost up to time \( T \), multiplied by the 'probability' that the convertible will be converted.

2. The value of the coupon stream from the bond up to date \( T \) multiplied by the 'probability' that the convertible will be converted.

3. The straight bond value multiplied by the 'probability' that the convertible will not be converted.

In fact, the 'probabilities' \( N(d_1) \) and \( N(d_2) \) are not exactly equal to the true probability of conversion, since they involve adjustments for risk.
5. Comparison of the two values

The cross-over valuation starts from the equity value, $S$, and adds the present value of the excess coupon stream, $E$. The option valuation starts from the bond valuation, $B$, and adds the option value, $C_2$.

The two values for the convertible are given by:

Cross-over value:

$$CV_1 = S e^{-dT} + y A(k, T)$$

Option-based value:

$$CV_2 = S e^{-dT} N(d_1) + y A(r, T) N(d_2) + B[1-N(d_2)]$$

Where:

- $S$ is the current value of the underlying equity
- $d$ is the yield on the underlying equity
- $T$ is the cross-over date
- $N(d_1)$, $N(d_2)$ are probabilities defined by (8) and (9)
- $A(k, T)$ is the $T$-period annuity factor at rate $k$
- $B$ is the straight bond value
If we assume that the option is a long way in the money, so that $Se^{-dT}$ is significantly greater than $B(T)e^{-rT}$, then $N(d_1)$ and $N(d_2)$ are both close to one, and the option value is equal to its 'intrinsic' value:

$$C_2 = Se^{-dT} - B(T)e^{-rT}$$  \hspace{1cm} (13)

The convertible value is given by:

$$CV_2 = Se^{-dT} + yA(r,T)$$  \hspace{1cm} (14)

To maximize this, we should choose, as shown in Appendix 3, $T^*$ such that:

$$y = Sd \ e^{(r-d)T^*}$$  \hspace{1cm} (15)

The cross over method chooses $T^*$ such that:

$$y = Sd \ e^{rT^*}$$  \hspace{1cm} (16)

Thus, even if the purpose of the cross-over point is to maximize the intrinsic value of the option component, it will not achieve that purpose.

In general, these two expressions can be very different. The main sources of difference are:
1. The explicit option approach discounts the coupon stream from the bond at the interest rate, $r$. The cross-over method effectively discounts the coupon stream at the equity rate, $k$. This is incorrect.

2. The cross-over approach ignores the choice element of the option, which is incorporated in the option approach by $N(d_1)$ and $N(d_2)$.

3. The cross-over method assumes a non-optimal exercise policy, even if the option is certain to be exercised.

4. If the cross-over method uses an equity discount rate, $k$, that is not equal to the dividend yield plus the forecast growth rate, it makes a further error by effectively assuming that the equity is mispriced.
6. An example

To illustrate the difference between the two techniques, we use the following example:

N: 15 years = maturity
y: 7% = coupon
r: 10% = straight yield
S: 90 per 100 = underlying equity
d: 4% = dividend yield
g: 10% = expected growth
k: 14% = required equity return
σ: 30% = volatility

These inputs give the following values:

\[ T^* - \tau^* \]
- 6.6 years = cross-over date

\[ S_0e^{-dT} \]
- 67.0 = value of equity stripped of dividend

\[ (y/k)A(k,T) \]
- 30.3 = cross-over value of T years coupon

\[ (y/r)A(r,T) \]
- 34.0 = correct value of T years coupon

B
- 76.7 = bond value

B(T)
- 83.0 = bond value at date T
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Value</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_2$</td>
<td>32.1</td>
<td>value of option component</td>
</tr>
<tr>
<td>$CV_1$</td>
<td>59.3</td>
<td>cross-over value</td>
</tr>
<tr>
<td>Intrinsic</td>
<td>103.0</td>
<td>intrinsic value</td>
</tr>
<tr>
<td>$CV_2$</td>
<td>108.8</td>
<td>full value</td>
</tr>
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The table below illustrates the relationship between the two valuation methods for this example as the value of the underlying equity changes.

<table>
<thead>
<tr>
<th>S</th>
<th>Cross-over Date</th>
<th>Cross-over Value</th>
<th>Intrinsic Value</th>
<th>Full Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>60</td>
<td>10.70</td>
<td>77.9</td>
<td>85.1</td>
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<tr>
<td>65</td>
<td>9.90</td>
<td>81.2</td>
<td>87.7</td>
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<td>90.5</td>
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<td>93.4</td>
<td>100.8</td>
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<tr>
<td>80</td>
<td>7.83</td>
<td>91.8</td>
<td>96.5</td>
<td>103.4</td>
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<tr>
<td>85</td>
<td>7.22</td>
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<td>99.7</td>
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</tr>
<tr>
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<td>103.0</td>
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<tr>
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<td>103.1</td>
<td>106.4</td>
<td>111.6</td>
</tr>
<tr>
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<td>107.1</td>
<td>109.9</td>
<td>114.5</td>
</tr>
<tr>
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<td>113.6</td>
<td>117.6</td>
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<td>117.4</td>
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<td>132.4</td>
<td>133.4</td>
<td>134.6</td>
</tr>
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</table>

It is clear from this table that the cross-over method always undervalues the convertible. The undervaluation is greatest for convertibles that are out of the money.
Summary and conclusions

This paper has compared the 'cross-over' method for convertible valuation with the more modern option approach. The differences are:

1. The option approach discounts bond coupons at the debt rate, the cross-over approach discounts them at an equity rate.

2. The cross-over approach ignores the combination of uncertainty and choice of exercise strategy which give value to the option component of the convertible.

3. The cross-over approach picks the wrong exercise date, even for convertibles where the choice element is unimportant.
References


J. Cox and M. Rubinstein Options Markets, Prentice-Hall, 1985
Appendix I: Valuation of the coupon minus dividend surplus up to the cross-over date, $T$.

In the time-period $[t, t + dt]$ the surplus of the coupon over the dividend is:

$$[y - Sde^{rT}] dt$$

If the surplus is valued at the equity required rate of return, $k$, then its value is:

$$E = \int_0^T [y - Sde^{rT}] e^{-kt} dt$$

which is equal to:

$$E = (y/k)[1 - e^{-kT}] - \left[ Sd/(k-g) \right] \left[ 1 - e^{-(k-g)T} \right]$$

If, in addition, we impose the equilibrium condition (see Appendix 2):

$$k = d + g$$

then:

$$E = (y/k)[1 - e^{-kT}] - S[1 - e^{-dT}]$$
Appendix 2: The relationship between growth rate and dividend yield

We use all the assumptions in the main body of the text.

The value of the first T-year of dividends is:

$$S(0, T) = \left[ S_0 / (k - g) \right] \left[ 1 - e^{-(k-g)T} \right]$$

For equilibrium, we require that the limit of this as T gets large is S. This requires:

$$k - g = d$$

Then:

$$S(0, T) = S(1 - e^{-dT})$$

The value of the equity underlying the option, if it is to be converted at T, is:

$$S - S(0, T) = S e^{-dT}$$
Appendix 3: The optimal choice of $T$ for a way-in-the-money convertible.

The value of the option component of a way-in-the-money convertible, if exercised at time $T$ is:

$$S e^{-dT} + \left( \frac{y}{z} \right) \left[ 1 - e^{-rT} \right]$$

Differentiating with respect to $T$ gives:

$$-dS e^{-dT} + y e^{-rT} = 0$$

where $T$, the optimal conversion date, is given by:

$$y = S d \left( r - d \right)^{-1}$$