THE JOURNAL OF FINANCE • VOL. XLVI, NO. 2 • JUNE 1991

The Default Risk of Swaps

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ABSTRACT

We characterize the exchange of financial claims from risky swaps. These transfers are among three groups: shareholders, debtholders, and the swap counterparty. From this analysis we derive equilibrium swap rates and relate them to debt market spreads. We then show that equilibrium swaps in perfect markets transfer wealth from shareholders to debtholders. In a simplified case, we obtain closed-form solutions for the value of the default risk in the swap. For interest-rate swaps, we obtain numerical solutions for the equilibrium swap rate, including default risk. We compare these with equilibrium debt market default risk spreads.

SWAPS HAVE BEEN ONE of the most explosive and important innovations in international capital markets in the last 10 years. In its simplest form, a swap consists of an agreement between two entities (called counterparties) to exchange in the future two streams of cash flows. In a currency swap, these streams of cash flows consist of a stream of interest and principal payments in one currency exchanged for a stream, of interest and principal payments of the same maturity in another currency. In an interest rate swap they consist of streams of interest payments of one type (fixed or floating) exchanged for streams of interest payments of the other type in the same currency. The two types of swap are shown in Appendix 1.[1]

The economic importance of swap transactions is the fact that they can be combined with debt issues to change the nature of the liability for the borrower. The sum of a bond issue and a currency swap gives a net liability stream which is equivalent to transforming the bond liability into a different currency. A floating rate note combined with an interest rate swap results in a liability equivalent to fixed rate debt.

Two issues have dominated the academic literature on swaps. The first concerns the reasons for their use. Arbitrage of imperfections in capital markets was the original cause of their appearance (Price and Henderson (1984)). Turnbull (1987) shows that lowering borrowing costs by a synthetic transaction involving a swap is not possible in a complete, integrated capital market. Subsequent authors (Wall and Pringle (1988); Arak, Estrella, Goodman, and Silver (1988); Smith, Smithson, and Wakeman (1987)) observe

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[1]For an excellent summary of swaps see Wall and Pringle (1988).

that market incompleteness or agency costs may provide an alternative explanation for the continued growth of the swap market.

The second part of the literature on swaps concerns pricing and hedging, including the pricing of default risk. Bicksler and Chen (1986) analyze the valuation of default-free interest-rate swaps and demonstrate that the swap is equivalent to holding one type of bond (fixed or floating) financed by selling short the other type. Several papers (Arak, Goodman, and Rones (1986); Federal Reserve Board and Bank of England (1987); Belton (1987)) estimate the maximum probable loss on swaps but do not attempt to value the default risk using an equilibrium model. Whittaker (1987) values the credit exposure of interest rate swaps using option pricing but does not endogenize the event triggering the swap default. His results are, therefore, the value of swap default assuming that the probability of the event triggering default is independent of the size of the default. Sundaresan (1989) models the default risk premium as a function of an "instantaneous default premium" that follows an exogenous stochastic process. This gives an equilibrium structure of default premia across different swaps but does not determine the level of the premium.

The purpose of this paper is to develop a partial equilibrium model for swap default that: a) is consistent with equilibrium rates for risky debt and enables the comparison of swap default risk with debt market default risk; b) makes clear the wealth transfers between corporate claim holders, if any, arising from swaps; and c) is applicable to both interest rate swaps and currency swaps. This analysis is important for at least three types of capital market participants. Banks holding portfolios of swaps need to be able to measure the value of these transactions net of default risk. Bank regulators require a consistent way of measuring the potential default risk so that they can set appropriate capital requirements. Corporations borrowing and using swaps to transform their liabilities should include an allowance for default risk in their comparison of the cost of direct and synthetic borrowing.

The paper is organized as follows: Section I discusses swap risk. Section II analyzes a simplified single-period swap to highlight the wealth transfers taking place. Section III discusses alternative treatments in default. Section IV analyzes a swap using a contingent claims analysis similar to the Merton (1974) model of risky debt to enable comparison of swap market spreads with debt market spreads. Section V applies the model to interest-rate swaps. Section VI contains the concluding comments.

I. Swap Risk

There are two types of risk in swap transactions: rate risk, and default risk. Rate risk arises because, during the life of the swap, exchange rates and interest rates vary so that the default-free present value of the cash flows remaining to be paid and received through the swap also varies. This rate risk can be hedged by taking offsetting positions in some combination of currency futures, bond and interest rate futures, currency forward contracts, and spot currency and bond markets.

The second type of swap risk, default risk, is much more difficult to hedge. This risk, sometimes called replacement risk, is complex to evaluate because the cost of default by the counterparty to a swap depends upon four things: the value of the swap at the default date, the event that will trigger the swap default, the relationship between the value of the swap and the event triggering default, and the rule for sharing claims in default. In swap defaults there is the added complication that the counterparty that is in default on its original debt could be due either to make or to receive a swap payment. If it is due to make a payment it will, presumably, default on the swap contract as well as on the debt if the swap payment is subordinated to the debt. Alternatively, it could be due to receive a swap payment, in which case the swap payment could be received (increasing the value of the bankrupt firm) or withheld.

Although the treatment of swaps in default is not yet certain, the consensus appears to be consistent with the following assumption (Henderson and Cates (1986)):

A1: Swaps are subordinate to debt in bankruptcy. In the event of a default on its debt by a counterparty that is owed value in a swap, the value of the swap will be paid to the bankrupt firm. The swap contract will be treated as a contract for the net cash flows due in the swap, not as an exchange of gross amounts.

This assumption is used throughout this paper except where it is explicitly stated otherwise. We assume that no collateral is held by the counterparty exposed to default risk. The assumption that the swaps are not collateralized is important in comparing the sizes of the swap spreads and the debt market spreads.[2] In the limit, collateralization could reduce the equilibrium swap spread to zero. Our analysis could be extended to include collateral, but such an extension is beyond the scope of this paper.

II. Analysis of Default Risk: Distribution-Free Results for Single-Period Swaps

In this section we analyze the impact of settlement rules on the default risk of the swap and on wealth arising from the swap. Since we are interested in the simplest possible swap that will highlight the economic substance of the analysis, we focus on single-period swaps. At this stage, we make no distinction between interest-rate swaps and currency swaps because the qualitative substance of the results is not different for the two types. The stylized single-period setting that we analyze is characterized by the following assumptions:

A2: Capital markets are perfect and competitive. There are no deadweight costs to bankruptcy.

[2] In the recent default by a Local Authority in England, collateralization did not seem to be important. By some estimates, bank counterparties in the swap contracts stand to lose around £200M.

A3: There is one risky counterparty, called firm 1. It has real assets with current value V_0. At maturity date of the debt T, the random value of the firm will be $\tilde{V}_T$.

A4: It has raised funds of amount B in a 'variable' debt market by issuing a zero coupon variable bond with face value $\tilde{X}_T$ and maturity T. The remainder of the firm is financed with equity. Equity pays no dividends prior to the debt maturity.

A5: There is a swap counterparty Z which is riskless. Firm 1 can contract with Z a swap which pays at time T the amount $(F - \tilde{X}_T)$ from firm *1* to Z. If $(F - \tilde{X}_T)$ is negative, the amount $(\tilde{X}_T - F)$ is paid at T from Z to firm *1*.[3]

Debt in this case is zero coupon, and 'variable' debt involves a promise to pay, at the debt maturity date T, an amount which is equal to the realization of a random variable $\tilde{X}_T$. If no default occurs on the debt or the swap, the net effect of the swap is to convert the variable liability X_T to the fixed liability F. In the case of a currency swap, X_T will be equal to a fixed amount of foreign currency translated into dollars at the random exchange rate at time T (the payoff to a zero coupon foreign bond). In the case of an interest-rate swap, X_T will be equal to the principal amount in dollars multiplied by the cumulative wealth relative on a random interest rate between time zero and time T (the payoff to a rolled-up deposit). Although this treatment of the swap omits the important feature that net coupon flows occur throughout the life of the swap, introduced later in Section V, it does retain two important characteristics of swaps. The first is the fact that the swap is a net contract. The second is the economic setting characteristic of swaps, that they are most frequently used in conjunction with debt market instruments to change the characteristics of the debt.[4]

The way we analyze the net wealth transfers from the swap is to segment the future into six different states. These cover the following possibilities at the maturity date of the swap:

a)	$F \gtrless X_T$	determines who is the net payer,
b)	$X_T \gtrless V_T \gtrless F$	determines the solvency status of firm 1.

Table I columns 1–3 show the payoffs to the bondholders and shareholders of firm 1 and the shareholders of the riskless counterparty in the swap. In

[3]In each equilibrium we analyze we assume that the proceeds from the bond issue are unaffected by the swap and the swap proceeds are zero on the initiation of the swap. This means that the operations of the firms are not affected by the type of liability issued.

[4]To illustrate that net settlement is preserved in this context, consider an interest rate swap with principal amount P, a fixed amount of interest of C per \$1 of principal during the period $[0, T\}$, and variable interest of $\tilde{R}$ per \$1 of principal. In the stylized swap investigated in this section, the fixed and variable payments would be defined by: $F_T = P(1 + C)$ and $X_T = P(1 + \tilde{R})$, respectively. Thus, the contact is equivalent to a net exchange of coupons (with no principal exchange) at maturity.

this case, firm 1 issues a variable bond and swaps it for fixed with the riskless counterparty. The settlement rule for the swap requires that Z makes the swap payment even when firm 1 is insolvent. Swap default can in this case result only in Z receiving a swap payment lower than promised in the swap contract. In state 6, for instance, Z makes a swap payment $(X_T - F)$ even though firm 1 is already bankrupt. This payment is then transferred to the bondholders of firm 1 as an addition to the liquidation value of the firm, V_T. In case 2 firm 1 is solvent until it has to make the swap payment. The residual claim after the bondholders have been paid is $(V_T - X_T)$, but this is not large enough to make the full swap payment. This amount is then paid to the swap counterpart Z, and the firm is bankrupt after the swap payment has been taken into account, even though it could afford to pay its bondholders. We assume that V_T and X_T are sufficiently variable to give positive probability to each state in Table I:

A6: There is a positive probability of each state in Table I.

For most realistic distributions of V and X, this assumption is innocuous.

Table I also shows the payoffs to two other claims, a European call option on V_T with exercise price F denoted $C(V, F)$, and a European put option on the maximum of V_T and X_T with an exercise price of F denoted $PX(V, X, F)$. By inspection, the following valuation relationships hold from Table I:

$$E_S = C(V, F) \tag{1}$$

$$B_S = V_O - C(V, F) + X_0 - F_0 + PX(V, X, F) \tag{2}$$

$$Z_S = F_0 - X_0 - PX(V, X, F) \tag{3}$$

Table I

Cash Flows to Swap Participants at the Swap Maturity, Time T

Firm 1 issues a debt liability of face value X. It enters a swap with firm Z whereby the net swap payment is $(F - X)$ from firm 1 to Z. In the swap, firm 1 owes $(F - X)$ in states 1-3 and Z owes $(X - F)$ in states 4-6. The table shows the cash flows to the bondholders of firm 1 (B), the equity holders of firm 1 (E), and the riskless counterparty to the swap (Z). V is the value of the operating assets of firm 1. The table also shows the payoffs to two other claims: $C(V, F)$ a European call option on V with exercise price F, and $PX(V, X, F)$ a European put option on the maximum of V and X, with exercise price F.

	Cash Flow				
	1	2	3	4	5
State	B_S	E_S	Z_S	$C(V, F)$	$PX(V, X, F)$
1 $V > F > X$	X	$(V - F)$	$(F - X)$	$(V - F)$	0
2 $F > V > X$	X	0	$(V - X)$	0	$(F - V)$
3 $F > X > V$	V	0	0	0	$(F - X)$
4 $V > X > F$	X	$(V - F)$	$-(X - F)$	$(V - F)$	0
5 $X > V > F$	X	$(V - F)$	$-(X - F)$	$(V - F)$	0
6 $X > F > V$	$(V + X - F)$	0	$-(X - F)$	0	0

where E_S is the value of the equity of firm *1*; B_S is the value of the risky debt of firm 1; X_0 is the value of a default free claim on X_T; F_0 is the value of a default free claim on F; Z_S is the value of the claim held by the swap counterparty Z. These pricing results are distribution-free and can be used to price the stylized type of swap discussed in this section.

To facilitate the analysis, we define four more values: $E_F = C(V, F)$ and $B_F = F_0 - P(V, F)$ are the values of the equity and debt resulting if firm 1 issues zero coupon debt of face value F and does not swap it, and $E_X = C(V, X)$ and $B_X = X_0 - P(V, X)$ are the values of the equity and debt resulting if firm 1 issues zero coupon variable debt with a promised payment of X_T and does not swap it. $P(V, F)$ and $P(V, X)$ are the values of European puts on V_T with exercise prices F and X_T, respectively. In the cases of $C(V, X)$ and $P(V, X)$, the exercises prices are stochastic.

The situation where the firm does not swap its debt is important because it enables us to compare the swap claim with debt market claims. The value B_X is particularly important, since the holders of firm 1's debt do not control the decision to swap unless the debt contains a covenant to force the swap. They will, therefore, pay only the minimum of B_X (its unswapped value) or B_S (its swapped value) when the debt is issued. We now define the equilibrium swap rates and characterize the wealth transfers resulting from the swap.

The equilibrium swap rate $\bar{F}$ is the solution to the following expression:

$$Z_S(\bar{F}) = \bar{F}_0 - X_0 - PX(V, X, \bar{F}) = 0. \tag{4}$$

In a competitive swap market, this is the rate that would be set by Z in the swap settlement formula $(F - X_T)$. We now show the wealth effects resulting from firm 1 entering the swap.

PROPOSITION P1: *Any swap results in a wealth transfer to the debtholders of firm* 1.

Proof: If the debt is not swapped, its payoffs are identical to those in Column 1 of Table I except in states 5 and 6. In these states the payoff is V_T, which is lower than the payoff to the swapped debt. Using A6 proves the proposition that $B_S > B_X$.

COROLLARY C1: *An equilibrium swap results in a wealth transfer from the shareholders to the debtholders of firm* 1 *of a claim with value* $C(V, X) - C(V, \bar{F})$, *where* $\bar{F}$ *is the rate that solves expression* (4).

Proof: The value of the equity claim before the swap is $E_X = C(V, X)$. After the swap it is $E_S = C(V, \bar{F})$. Prior to the swap $E_X + B_X = V_0$. After the swap, $E_S + B_S + Z_S = V_0$. Using equation (4) and P1, $E_S < E_X$ for $F = \bar{F}$.

This result is a consequence of three important assumptions. First, the swap is subordinated to the debt and so cannot expropriate any rights of the debtholders.[5] Second, the debtholders do not control the decision to swap and will, therefore, not pay for any value added as a result of the swap, unless a covenant requires the swap. Finally, the default rule for the swap requires payment by Z even when the debt is in default, which results in the wealth transfer. Note that none of these assumptions are peculiar to the particular style of swap we are currently analyzing. This same result holds for the usual coupon-bearing swaps, but a proof for coupon-bearing swaps involves solving complex algebraic expressions without generating further insight into this particular issue.

Proposition P1 and its corollary are a stronger version of the Turnbull (1987) argument that swaps are a zero sum game in a complete market. Here, the zero sum is the total of the wealth effects on shareholders and debtholders and the swap counterparty. Without the swap, the debtholders have a claim that pays $\min[V, X]$. With the swap, the debt claim pays $\min[\max(V, V + X - F), X]$. The swap raises the value of the total assets over which the debtholders have a claim, because $\max(V, V + X - F)$ dominates V.

This analysis raises two important questions. One is why the shareholders want the debt to be swapped when it has a negative effect on their wealth. Although our model cannot directly answer this question, it suggests what circumstances are necessary for shareholders to be willing to swap. Note that the wealth effect of the swap arises when variable debt is transformed to fixed. For a firm with a preference for fixed debt liabilities, the comparison that is relevant is the choice between issuing fixed rate debt directly and issuing variable debt and swapping it. The latter could be the preferred alternative under some circumstances. If the swap is covenanted at the time the variable rate debt is issued, then there will be no wealth effect, and the shareholders will be indifferent. There may, however, be agency effects of the type suggested by Wall and Pringle (1988) and Arak, Estrella, Goodman, and Silver (1988) whereby swaps, for instance, enable the borrower to take advantage of private knowledge concerning future creditworthiness. This may give rise to a preference for swapped variable rate debt rather than fixed rate debt, and the equity holders may prefer the swap despite its negative wealth effect in the absence of such agency considerations. Finally, if there is a pricing inefficiency between the variable and fixed debt markets that is larger than the wealth transfer caused by the swap, the swapped debt could have a positive wealth effect for equity when compared with unswapped fixed debt.

The second issue is whether the second wealth effects occur if the swap is from fixed to variable rather than from variable to fixed. To see that this is,

[5]Bondholders could be hurt by the swap if the swap payments are due before the debt matures. If, for instance, a swap were designed so that the firm makes payments before the debt matures and receives payments after, bondholders would be hurt by the swap payments and receive limited benefits in exchange. The swap we analyze is, however, of the same maturity as the debt because it is explicitly tailored to match the debt and change it from floating to fixed.

indeed, the case, note that all the results so far are true (with appropriate modifications) if X and F are interchanged in Table I. Thus, the wealth effects we are considering arise from the general nature of the swap contract and not from the specific circumstances of its use.

To clarify the nature of the wealth transfers and the relationship between swap rates and debt market rates, we now consider a swap that is not an equilibrium swap, but an exchange of equal market values of debt. Suppose that the firm enters a swap to exchange floating payments for fixed that does not change the value of the equity. This swap corresponds to an exchange of X_T for a fixed amount $\hat{F}$ defined by:

$$C(V, \hat{F}) = C(V, X). \tag{5}$$

In the absence of a swap, the value of debt and equity must sum to the value of the firm, so:

$$B_{\hat{F}} = B_X. \tag{6}$$

Thus the exchange of X and $\hat{F}$ is the exchange of amounts which, if issued as unswapped promised debt payments by the firm, would have equal values. We define the rate $\hat{F}$ as the equal value swap rate, and the following proposition holds:

PROPOSITION P2: *A swap that exchanges debt payments of equal value results in a wealth transfer from the counterparty Z to the debtholders of firm* 1 *of a claim with value $PX(V, \hat{F}, X)$, where $\hat{F}$ is the rate that solves expression* (6), *and $PX(V, F, X)$ is the value of a European put on the maximum of V and F with exercise price X.*

Proof: From equation (6), the value of the equity claim is unchanged by the swap. The value of the swap claim given by (3) is shown, in Appendix 2, to be equal to:

$$F_0 - X_0 - PX(V, X, F) = C(V, X) - C(V, F) - \left[C(X, F) - M(V, X, F)\right] \tag{7}$$

where $M(V, X, F)$ is the value of a European call on the minimum of V and X with exercise price F. Using (6) gives:

$$F_0 - X_0 - PX(V, X, \hat{F}) = -\left[C(X, \hat{F}) - M(V, X, \hat{F})\right]. \tag{8}$$

From Stulz (1982), the right hand side is equal to $-PX(V, \hat{F}, X)$, which, by A6, is negative.

An equal value swap thus has negative value to the swap counterparty because the swap rate does not provide compensation for the incremental default risk arising from the way the swap is settled. This wealth transfer results from the subordination of the swap to the debt and the inability of Z to withhold the swap payment when firm 1 is in default on its debt. Default occurs if $V_T < X_T$. Firm 1 is owed a swap payment if $F < X_T$. Thus, both V_T

and F must be lower than X_T for the wealth transfer to occur, and the resulting wealth transfer takes the form of an option that pays only if both V and $\hat{F}$ are less than X. The value of this option is $PX(V, \hat{F}, X)$.

The swap that exchanges debt payments with equal values results in a wealth transfer from Z to the debtholders of firm 1. Clearly Z would not be willing to enter into this swap without promised compensation in the form of the additional amount $(\bar{F} - \hat{F})$. This is summarized by the following:

PROPOSITION P3: *The equilibrium swap rate $\bar{F}$ is greater than the swap rate $\hat{F}$ that exchanges debt payments with equal market value.*

Proof: From P1, $C(V, X) - C(V, \bar{F}) > 0$. From equation (5) $C(V, X) = C(V, \hat{F})$. Consequently, $C(V, \bar{F}) < C(V, \hat{F})$, which implies $\bar{F} > \hat{F}$.

By analogy with P3, the opposite swap where firm 1 pays floating and receives fixed will have an equilibrium rate below $\hat{F}$. Thus, there will be a bid-ask spread in the swap market arising from the requirement to compensate the bank for the fact that it subsidizes the debtholders of firm 1 if the swap takes place at $\hat{F}$, regardless of the direction of the swap.

We can now derive the relationship between swap market default spreads and debt market default spreads. For the fixed rate debt market we define the default spread in a way equivalent to Merton (1974):

$$S_F = ln[\hat{F}_0 / B_{\hat{F}}] / T \tag{9}$$

where r is the continuously compounded riskless interest rate for maturity T, and:

$$\hat{F}_0 = \hat{F}e^{-rT}. \tag{10}$$

The spread is thus the interest rate equivalent to the discount of the bond's market value from the value it would have if there were no default risk. We define the variable spread in the same way:

$$S_X = ln[X_0 / B_X] / T \tag{11}$$

where X_0 is the default-free value of the promised variable payment, and B_X is its value including default risk.

The conventional way to define a swap spread is as the rate that must be added to the riskless fixed interest rate to obtain the swap quote, with the variable payment being treated as riskless. In the swap we are considering, the riskless value of the variable side of the swap is X_0, and the swap spread S_S is defined by:

$$F = X_0 e^{(r+S_S)T}. \tag{12}$$

Rearranging:

$$S_S = ln[F_0 / X_0] / T. \tag{13}$$

Given these definitions, it is now possible to derive the relationship between the swap market spread and the debt market spreads. Combining equations (9), (11), and (13) gives:

$$S_S = S_F - S_X + ln[F_0 / \hat{F}_0] / T. \tag{14}$$

The swap spread is made up of two components. The first, $(S_F - S_X)$, is the difference between equilibrium spreads in the fixed and variable interest rate markets. If the swap was equivalent to an exact exchange of the cash flow streams arising from the two types of risky debt, the swap spread would be equal to $(S_F - S_X)$. A swap with this rate would, however, from P2, have negative value to the bank. The swap rate contains a second element, which we call the pure swap spread S'_S. This arises from the incremental default risk in the swap, given the settlement rule in bankruptcy:

$$S'_S = ln[F_0/\hat{F}_0]/T. \tag{15}$$

This is the additional rate required to compensate the swap counterparty for the value loss resulting from the way the swap is settled in bankruptcy. If the swap rate is the equilibrium rate $\bar{F}$, then from P3, $S'_S > 0$. Using equation (14) then yields, in equilibrium:

$$S_S > S_F - S_X \tag{16}$$

In equilibrium the swap spread should exceed the difference between the fixed and variable debt market spreads. The conventional test for swap arbitrage involves finding violations of the inequality (16). Such violations should not be observed in a perfect market, so the conventional arbitrage test is valid in our model, in the sense that violations of (16) indicate arbitrage opportunities.

III. Alternative Settlement Rules

In this section we analyze three possible treatments of the swap in default. These are not the most likely cases, but, given the uncertainty about the precise treatment of swap defaults, they illustrate the size of the risk introduced by alternative settlement rules. The swap we analyze is the same swap as in Section II. We analyze the following three alternative rules: (1) swap payments are made only if both counterparties to the swap are solvent prior to the swap payment (cross default); (2) the net payment is made before any cash flows are paid to the bondholders (prior settlement); and (3) the swap is treated as an exchange of gross amounts; the counterparty Z pays X_T to firm 1 and is then a subordinated creditor for the amount F (gross settlement).

Table II shows the cash flows to the debtholders and shareholders of firm 1 and to the swap counterparty Z at the maturity of the swap for each settlement rule. To analyze the equilibrium swap rate and wealth transfers in each case, we first prove the following proposition:

PROPOSITION P4: *Under any rule, an equilibrium swap results in a net wealth change of $C(V, \hat{F}) - C(V, F^i)$ for shareholders, where F^i is the equilibrium swap rate under that rule. Debtholders wealth changes by an equal amount in the opposite direction.*

Table II

Cash Flow to Swap Participants at the Swap Maturity Time T, Alternative Settlement Rules

Firm 1 issues a debt liability of face value $\tilde{X}$. It enters a swap with firm Z whereby the net swap payment is $(F - X)$ from firm 1 to Z. In the swap, firm 1 owes $(F - X)$ in states 1-3 and Z owes $(X - F)$ in states 4-6. The table also shows the payoffs to the following claims: $C(V, X)$ is a European call option on V with exercise price X; $C(V, F)$ is a European call option on V with exercise price F; $W(V, X, F)$ is a claim that pays $(X - V)$ if $X > V > F$; F_0, X_0, and V_0 are unconditional claims on F, X_T, and V_T, respectively; $P(V + X, F)$ is a European put option on $(V + X)$ with exercise price F. In Panel A State 2, the equity holders of firm 1 pay in $(X - V)$ to prevent bankruptcy.

Panel A: Cross-default: swap payment made only if both counterparties solvent prior to the swap payment

State	B	E	Z	$C(V, X) - C(V, F)$	$W(V, X, F)$
1 $V > F > X$	X	$(V - F)$	$(F - X)$	$(F - X)$	0
2 $F > V > X$	X	0	$(V - X)$	$(V - X)$	0
3 $F > X > V$	V	0	0	0	0
4 $V > X > F$	X	$(V - F)$	$-(X - F)$	$-(X - F)$	0
5 $X > V > F$	X	$(V - F)$	$-(X - F)$	$-(V - F)$	$(X - V)$
6 $X > F > V$	V	0	0	0	0

Panel B: Prior settlement: net swap payment made before any payments to bondholders

State	B	E	Z	$(F_0 - X_0)$	$P(V + X, F)$
1 $V > F > X$	X	$(V - F)$	$(F - X)$	$(F - X)$	0
2 $F > V; F > X; V + X > F$	$(V + X - F)$	0	$(F - X)$	$(F - X)$	0
3 $F > V; F > X; V + X < F$	0	0	V	$(F - X)$	$(F - V - X)$
4 $V > X > F$	X	$(V - F)$	$-(X - F)$	$-(X - F)$	0
5 $X > V > F$	X	$(V - F)$	$-(X - F)$	$-(X - F)$	0
6 $X > F > V$	$(V + X - F)$	0	$-(X - F)$	$-(X - F)$	0

Panel C: Gross settlement: swap is treated as an exchange of gross amounts

State	B	E	Z	$(V_0 - X_0)$	$C(V, F)$
1 $V < F$	X	0	$(V - X)$	$(V - X)$	0
2 $V > F$	X	$(V - F)$	$(F - X)$	$(V - X)$	$(V - F)$

Proof: From inspection of Table II, it is clear that the value of the equity claim after the swap is $C(V, F^i)$ under all rules. The value of the equity claim before the swap is $C(V, X)$. From the definition of the equal value swap rate, $C(V, X) = C(V, \hat{F})$. So the shareholders net wealth change is $C(V, F^i) - C(V, \hat{F})$. Before the swap, the value of the debt and equity sum to V_0. After the swap, the value of the debt and equity and the swap value $Z_S(F^i)$ sum to V_0. The definition of the equilibrium swap rate is $Z_S(F^i) = 0$. So the wealth gain or loss of the equity is a transfer from or to the debt.

We now characterize the equilibrium swap rates and wealth transfers under the three alternative default rules.

PROPOSITION P5: *With cross-default, the equilibrium swap rate* F^c *solves*:

$$C(V, X) - C(V, F^c) = W(V, X, F^c) \tag{17}$$

where $W(V, X, F^c)$ *is a claim that pays* $(X_T - V_T)$ *if* $X_T > V_T > F^c$. *An equilibrium swap results in a wealth transfer from shareholders to debtholders of firm* 1 *of value* $W(V, X, F^c)$.

Proof: From Table IIA, $Z_S(F^c) = C(V, X) - C(V, F^c) - W(V, X, F^c)$. The equilibrium condition is $Z_S(F^c) = 0$. Combining these gives equation (17). The wealth transfer from equity is $C(V, F^c) - C(V, X)$, equal to $W(V, X, F^c) > 0$.

The impact of cross-default is that the swap is cancelled when the firm is in default on its debt. This enables the bank to withhold the swap payment due in state 6. In state 5, the value of the firm's assets is less than its debt market liability, but it benefits the shareholders to contribute enough to save the firm from bankruptcy and receive the swap payment.

PROPOSITION P6: *With prior settlement, the equilibrium swap rate* F^P *solves*:

$$V_0 - C(V + X, F^P) = 0 \tag{18}$$

where $C(V + X, F^P)$ *is the value of a European call on* $(V_T + X_T)$ *with exercise price* F^P. *An equilibrium swap results in a wealth transfer of* $C(V, F^P) - C(V, X)$ *between debtholders and shareholders of the firm.*

Proof: From Table IIB, $Z_S(F^P) = F_0^P - X_0 - P(V + X, F^P)$. The equilibrium condition is $Z_S(F^P)$. From put-call parity:

$$P(V + X, F^P) = C(V + X, F^P) - V_0 - X_0 + F_0^P.$$

Substitution for $P(V + X, F^P)$ in $Z_S(F^P) = 0$ yields equation (18). The wealth transfer between debt and equity cannot be signed in this case.

The equilibrium swap rate with prior settlement F^P is lower than the equilibrium swap rate $\bar{F}$. The equilibrium conditions are:

$$\bar{F}_0 - X_0 - PX(V, X, \bar{F}) = 0 \tag{19}$$

$$F_0^P - X_0 - P(V + X, F^P) = 0 \tag{20}$$

From inspection of Tables I and IIB, $P(V + X, F^P) < PX(V, X, \bar{F})$. Thus:

$$\bar{F}_0 - X_0 - P(V + X, \bar{F}) > 0 \tag{21}$$

From the proof of P6:

$$V_0 - C(V + X, \bar{F}) > 0 \tag{22}$$

The value of the call in equation (22) is decreasing in F. The condition for F^P is (18). Thus:

$$F^P < \bar{F}.$$

Prior settlement is advantageous to the swap counterparty and enables him to offer a lower swap rate than the equilibrium swap rate $\bar{F}$. Prior settlement also potentially results in a wealth transfer from debt to equity, since it appropriates part of the debtholders prior claim on a firm's assets. If debtholders believe that prior settlement is possible, however, they will price their debt claim accordingly, and the resulting wealth transfer from swapping will again be from the equity to the debt.

With gross settlement of the swap, shown in Table IIC, the claims held are extreme simple. The gross payment by Z of the variable cash flow due in the swap effectively fully guarantees the debt of firm 1. The swap counterparty Z then holds a fixed claim of face value F on the assets of the firm. This results in the following equilibrium:

PROPOSITION P7: *With gross settlement, the equilibrium swap rate F^G solves*:

$$V_0 - X_0 - C(V, F^G) = 0 \tag{23}$$

A swap at this rate results in wealth transfer of $P(V, X)$ from the shareholders to the debtholders of firm 1.

Proof: Equation (23) is obvious from inspection of Table IIC. The wealth change for the shareholders is $C(V, F^G) - C(V, \hat{F}) = C(V, F^G) - C(V, X)$. Substituting (23) gives a wealth change of $V_0 - X_0 - C(V, X)$. From put-call parity, this is equal to $-P(V, X)$.

This wealth transfer results from the fact that the swap effectively guarantees the debt. The debtholders will not, however, pay for this guarantee, since they may not receive it if the swap does not occur. Thus, when the swap occurs, the price of guaranteeing the debt, $P(V, X)$, is effectively paid by the equity holders.

IV. Determinants of the Equilibrium Swap Spread

The results in the previous sections were presented in terms of general contingent claims without making specific assumptions about the stochastic processes followed by the value of the firm V and the variable swap payment X. In this section, we specialize the analysis to the case of currency swaps. In Section V, interest-rate coupon swaps are analyzed in more detail.

The variable payment in a currency swap is a fixed amount of foreign currency, with a corresponding value in dollars solely dependent on the exchange rate. We assume that the value of the firm's assets V and the present value in dollars of the foreign currency payment X follow joint geometric Brownian processes:

$$\mathrm{d}V/V = \mu_V\,\mathrm{d}t + \sigma_V\,\mathrm{d}z_V \tag{24}$$

$$\mathrm{d}X/X = \mu_X\,\mathrm{d}t + \sigma_X\,\mathrm{d}z_X \tag{25}$$

where dz_V and dz_X are increments to standard Wiener processes with $dz_V\,dz_X = \rho\,dt$. At earlier dates $t \in [O, T]$, X_t is defined as the value of a default-free claim on X_T. This is the value in dollars of a fixed amount of foreign currency to be received at time T. The final assumptions are:

A7: Trading in assets takes place continuously.
A8: The continuous interest rate in dollars, r, is constant.[6]

Given these assumptions, all of the contingent claims arising from the default-risk of the swap are continuously spanned by a combination of the riskless security, the asset V_t, and the asset X_t. The swap can be valued using the results in Black and Scholes (1973), Margrabe (1978), and Stulz (1982). The value of the swap is, from Appendix 2:

$$Z_S = C(V, X) - C(V, F) - C(X, F) + M(V, X, F) \tag{26}$$

The values of the components of this expression are (see Appendix 3):

$$C(V, X) = V_0 N(\mathrm{d}_W + \bar{\sigma}_W) - X_0 N(\mathrm{d}_W) \tag{27}$$

$$C(V, F) = X_0 N(\mathrm{d}_V + \bar{\sigma}_V) - F_0 N(\mathrm{d}_V) \tag{28}$$

$$C(X, F) = X_0 N(\mathrm{d}_X + \bar{\sigma}_X) - F_0 N(\mathrm{d}_X) \tag{29}$$

$$\begin{aligned} M(V, X, F) = {} & V_0 N_2(\mathrm{d}_V + \bar{\sigma}_V, -(\mathrm{d}_W + \bar{\sigma}_W), \rho_k) \\ & + X_0 N_2(\mathrm{d}_X + \bar{\sigma}_X, \mathrm{d}_W, \rho_W) - F_0 N_2(\mathrm{d}_X, \mathrm{d}_V, \rho) \end{aligned} \tag{30}$$

with:

$$F_0 = Fe^{-rT}; \quad \bar{\sigma}_V = \sigma_V \sqrt{T}; \quad \bar{\sigma}_X = \sigma_X \sqrt{T}; \quad \bar{\sigma}_W^2 = \bar{\sigma}_V^2 + \bar{\sigma}_X^2 - 2\rho\bar{\sigma}_V\bar{\sigma}_X;$$
$$\rho_W = (\rho\sigma_V - \sigma_W)/\sigma_W; \quad \rho_k = (\rho\sigma_X - \sigma_V)/\sigma_W$$

and:

$$\mathrm{d}_X = [ln(X_0/F_0) - \bar{\sigma}_X^2/2]/\bar{\sigma}_X \tag{31}$$

$$\mathrm{d}_V = [ln(V_0/F_0) - \bar{\sigma}_V^2/2]/\bar{\sigma}_V \tag{32}$$

$$\mathrm{d}_W = [ln(V_0/X_0) - \bar{\sigma}_W^2/2]/\bar{\sigma}_W \tag{33}$$

Where $N_2[\mathrm{d}_1, \mathrm{d}_2, \rho]$ is the standard bivariate normal distribution function with correlation ρ evaluated at d_1 and d_2. $N[\mathrm{d}]$ is the standard univariate normal distribution function evaluated at d.

When the swap contract is to exchange debt payments with equal market value, the claim that is transferred from the swap counterparty to the debtholders of firm 1 is equal to:[7]

$$\begin{aligned} PX(V, \hat{F}, X) = {} & X_0[\mathrm{d}_X - \bar{\sigma}_X] - \hat{F}_0 N[\mathrm{d}_X] - X_0 N_2[\mathrm{d}_X + \bar{\sigma}_X, \mathrm{d}_W, \rho_W] \\ & + \hat{F}_0 N_2[\mathrm{d}_X, \mathrm{d}_V, \rho] - V_0 N_2[\mathrm{d}_X + \rho\bar{\sigma}_V, \mathrm{d}_V + \bar{\sigma}_V, \rho] \\ & + V_0 N_2[\mathrm{d}_X + \rho\bar{\sigma}_V, \mathrm{d}_W + \bar{\sigma}_W, \rho_W] \end{aligned} \tag{34}$$

[6]This assumption is, of course, unsustainable for interest-rate swaps. Currency swaps are, however, those where the default risk problem is greater.

[7]There is a corresponding set of expressions for a risky counterparty issuing fixed-rate debt and swapping it for floating with a riskless counterparty.

Table III shows representative values of the swap spread. Panel A shows the effect of changing the leverage of the risky counterparty for the cases where the assets of firm 1 are uncorrelated with the variable debt payment and the case where default spreads are equal in the fixed and floating markets. Increased leverage increases spreads in both debt markets and in the swap market. Similarly, in Panel B where the maturities of the debt and swap are varied, a longer maturity raises all spreads. Panel C shows the effect of changing the correlation of the assets of the firm with the payment due on the variable debt. As this correlation rises the pure swap spread falls. The total swap spread is compensation to the counterparty Z for the default

Table III

Equilibrium Swap Spreads in Basic Points per Annum

Firm 1 issues a debt liability of face value $\tilde{X}$. It enters a swap to exchange this for a fixed liability F. S_X and S_F are default spreads in the variable and fixed rate debt markets. S_S is the equilibrium swap spread quoted as a rate per annum by which the default free value of F exceeds the default free value of X. $S'_S = S_S - (S_F - S_X)$ is the pure swap spread which compensates the swap counterparty for the incremental default risk arising from the difference between the way that the swap is settled and the way that risky debt is settled. σ_V is the volatility of the firm's assets, σ_X the volatility of $\tilde{X}$, and ρ the correlation between V and X. r is the riskless interest rate, T the maturity of the debt and the swap, and B/V the market value of debt divided by the market value of assets of the firm.

Panel A: $\sigma_V = 0.3$; $\sigma_X = 0.1$; $r = 0.1$; $T = 5$						
	B/V	0.10	0.20	0.30	0.40	0.50
$\rho = 0$	S_X	1	12	49	117	222
	S_F	0	8	36	93	186
	S_S	0	2	10	25	50
	S'_S	1	4	23	49	66
$\rho = 0.167$	S_X	0	8	36	94	186
	S_F	0	8	36	94	186
	S_S	0	3	14	33	62
	S'_S	0	3	14	33	62
Panel B: $\sigma_V = 0.3$; $\sigma_X = 0.1$; $r = 0.1$; $B/V = 0.40$; $\rho = 0$						
	T	1	3	5	10	20
	S_X	3	59	117	207	289
	S_F	1	44	93	174	250
	S_S	0	12	25	52	88
	S'_S	2	27	49	85	127
Panel C: $\sigma_V = 0.3$; $\sigma_X = 0.07$; $r = 0.1$; $T = 5$						
B/V	0.30	0.30	0.30	0.50	0.50	0.50
ρ	0.25	0.00	−0.25	0.25	0.00	−0.25
S_X	30	42	56	166	204	242
S_F	36	36	36	186	186	186
S_S	12	8	4	51	37	24
S'_S	6	14	24	31	55	80

by firm 1 when it owes a swap payment. This occurs if both V and X are low. As the correlation between V and X rises, this is more likely, and the swap spread rises. Note, however, that the total swap spread rises because the decline in the variable spread dominates the decline in the pure spread in the expression $S_S = S_F - S_X + S'_S$. In all cases, the swap spread is lower than the debt market spreads, often by a considerable margin. This results from the different amounts of exposure in the two contracts. The debt contract exposes the entire repayment to default. The swap, however, only exposes the net difference between the cash flows due on two debt contracts.[8]

Clearly, the inclusion of coupons in the model would reduce default spreads. This will, however, affect both the bond and the swap contract. Thus, it will not necessarily reduce the swap spread relative to the debt spread. To do so it would be necessary for the reduction of default risk due to the coupon stream to be greater for a swap contract than for a debt contract. Since the swap is subordinated to the debt, this may well be the case, as part of the cash flow to the swap will occur prior to part of the cash flow to the debt when there is a coupon stream.

V. Swaps of Coupon-Bearing Debt

In this section we examine interest rate swaps of debt with coupons. We replace equation (24) with the following:

$$\mathrm{d}V = (\alpha V - C)\,\mathrm{d}t + \sigma_V V\,\mathrm{d}z_V \tag{35}$$

The coefficient α is the instantaneous expected rate of return on the firm, and C represents the total dollar payout by the firm per unit of time.

[8] With two risky counterparties engaged in the swap, there is a possibility of default by either. The claims exchanged in the swap now depend upon the values of the assets of both firms engaged in the swap. Against a risky counterparty firm 1 can now swap at an apparently more favourable rate (a lower value of F) than with a riskless counterparty. This is, however, simply compensation for the extra default risk. With two risky counterparties we can split the change in the value of the equity resulting from the swap contract into two components. The first component is the gain that would arise from swapping with a riskless counterparty. The second component, which is always negative, is the loss from the fact that the counterparty is risky. We have solved for these values by Monte Carlo simulation. The Monte Carlo results indicate that the gain to the equity holders of a firm swapping with a risky counterparty is increasing in the asset value of the counterparty firm, decreasing in ρ_{VX} of the counterparty firm if the counterparty issued variable rate debt and is paying fixed in the swap, and increasing in ρ_{VX} of the counterparty firm of the counterparty issued fixed rate debt and is paying variable in the swap. The effect of the correlation between the counterparty asset value and the variable payment occurs because X determines who is due to pay in the swap. If the counterparty is due to pay when X is high and has assets highly correlated with X, this makes the swap more valuable. If the counterparty is due to pay when X is low and has assets highly correlated with X, this makes the swap less valuable.

Instead of A8 we now assume that the term structure of interest rates is fully specified by the instantaneous riskless rate r. Its dynamics are given by:[9]

$$dr = m(\mu - r)\,dt + \sigma_r r\,dz_r \quad m > 0,\ \mu > 0,\ dz_r\,dz_V = \rho_{Vr}\,dt \tag{36}$$

The instantaneous drift term $m(\mu - r)$ represents a force that pulls the interest rate towards its long-term value μ with magnitude proportional to the current deviation.

For any security issued by the corporation, its market value at any point in time can be written as a function of V and r alone. For a fixed rate corporate bond, with value B, the dynamics are, using Ito's Lemma:

$$\begin{aligned} dB = {} & \big[B_V(\alpha V - C) + B_r m(\mu - r) + B_t + 1/2\big(B_{VV}\sigma_V^2 V^2 + B_{rr}\sigma_r^2 r^2 \\ & + 2B_{Vr} V r \sigma_V \sigma_r \rho_{Vr}\big)\big]\,dt + B_V V \sigma_V\,dz_V + B_r r \sigma_r\,dz_r \end{aligned} \tag{37}$$

Using the traditional argument that results in a no-arbitrage condition for the price gives:

$$\begin{aligned} & B_V\big[(\alpha V - C) - K_1\sigma_V V\big] + B_r\big[m(\mu - r) - K_2\sigma_r r\big] + B_t \\ & \quad + 1/2\big(B_{VV}\sigma_V^2 V^2 + B_{rr}\sigma_r^2 r^2 + 2B_{Vr} V r \sigma_V \sigma_r \rho_{Vr}\big) + C - rB = 0 \end{aligned} \tag{38}$$

where $K_1 = (\alpha - r)/\sigma_V$.

We assume that the market is risk-neutral with respect to nominal interest rate risk, so the Local Expectation Hypothesis of the term structure holds, and $K_2 = 0$. The corporate bond price must then satisfy the following equation at all times:[10]

$$\begin{aligned} & B_V(rV - C) + B_r m(\mu - r) + 1/2\big(B_{VV}\sigma_V^2 V^2 + B_{rr}\sigma_r^2 r^2 + 2B_{Vr} V r \sigma_V \sigma_r \rho_{Vr}\big) \\ & \quad + C - rB = -B_t \end{aligned} \tag{39}$$

The terminal condition is:

$$B(V, r, T, C) = \min(F, V) \tag{40}$$

We assume that coupons are paid semiannually and that the firm will default on the coupon if the value of the firm at any coupon date falls below the coupon payment. In that case the value of the firm's assets V is paid to the debtholders, and the firm is liquidated. Thus $B(V, r, T, C) = V$ if $V \le C$, $t < T$, and t is a coupon date.

To define the interest rate spreads, we also need the coupons on default-free government bonds. For a coupon-paying government bond with value

[9]These dynamics are similar to those used by Brennan and Schwartz (1982). An alternative interest rate process is used in Ramaswamy and Sundaresan (1986). These papers contain discussion of the relative merits of different processes.

[10]As Dothan (1978) points out, the term structure of interest rates will, in general, depend upon the risk preference parameter K_2. Equilibrium swap spreads will also depend upon this parameter. We leave open the question of how large this dependence is.

$G(r, t, C)$ the corresponding PDE is:

$$G_r[m(\mu - r)] + 1/2 G_{rr}\sigma_r^2 r^2 + C - rG = -G_t \tag{41}$$

with terminal condition $G(r, T, C) = F$.

To determine the value of the default risk spread in the fixed and variable markets we use the following procedure. First, we determine the fixed coupon rate that makes a default-free bond sell at a price equal to its face value F. For variable coupon bonds this is simply r_t. For fixed rate bonds it is the coupon rate g, such that $G(r, O, g, F) = F$. We then determine the coupon rate required for a bond subject to default risk to sell at par at the date of issuance. Bond prices depend on the short term interest rate, the value of the firm, time to maturity, and how the processes z_V and z_r are correlated. Accordingly, the default risk spreads will reflect these characteristics. The spread for risky variable bonds is defined by the fixed mark-up to the coupon rate that reflects the credit risk of a particular bond. This spread S_X is defined by $B(V, r, O, (r + S_X)F) = F$. Similarly, the spread for the default risk of fixed coupon bonds is given by the coupon rate differential S_F such that $B(V, r, O, (g + S_F)F) = F$. To determine these spreads we use the Hopscotch method. This method, developed by Gourley (1970) can be applied to solve parabolic and elliptic equations in two dimensions with a mixed derivative term (see Gourley and McKee (1977)).[11]

In Table IV we present the equilibrium spreads for both fixed coupon and variable coupon bonds. The parameters for the interest rate process are

Table IV

Fixed Coupon Spreads and Variable Coupon Spreads due to Default-Risk in Basis Points per Annum

A firm has assets of value V and a single debt issue of value B. B/V is the market-value leverage; ρ_{Vr} is the correlation between the firm value and the interest rate; S_F is the default spread in the fixed debt market; and S_X is the default spread in the floating rate market. The volatility of firm value is 30% per annum. Bonds pay coupons twice a year and have a 5-year maturity. The instantaneous interest rate is r. Its dynamics are given by: $\mathrm{d}r = m(\mu - r)\,\mathrm{d}t + \sigma_r r\,\mathrm{d}z_t$. The parameter values are: $m = 0.72$, $\mu = 0.09$, and $\sigma_r^2 = 0.006$.

	$B/V = 0.30$			$B/V = 0.50$		
ρ_{Vr}	0.25	0.00	−0.25	0.25	0.00	−0.25
S_F	16	18	20	66	69	72
S_X	12	19	27	57	71	87
$S_F - S_X$	+4	−1	−7	+9	−2	−15

[11]More specifically, we have used the "line" version of the Hopscotch method, appropriate in applications with mixed derivative terms and variable coefficient (see Gourlay and McKee (1977)). In the first time step we solved explicitly for bond values at alternating values of **r** for all firm values. Then we used these calculated values to set up the tridiagonal equation system of the remaining values of r and solved it by elimination. The second step is solved implicitly in the direction of V. For any particular grid point, explicit and implicit calculations are used at alternating time steps. At coupon dates implicit replacements were used.

$m = 0.72$, $\mu = 0.09$, and $\sigma_r^2 = 0.006$. The instantaneous variance of the return on the firm is $\sigma_V^2 = 0.09$.[12] Table IV reports the spreads for a current value of $r = 10\%$ and different values of the debt to firm value ratio and the correlation parameter ρ_{Vr}. All bonds have 5 years to maturity. For both fixed and variable bonds, the spread increases with the debt to firm value ratio. A lower value of the correlation parameters also raises the floating rate spread, indicating a higher credit risk. The spread in the variable rate market is more sensitive to changes in the value of the correlation parameter. Thus, for $\rho_{Vr} = -0.25$, the spread in the variable rate market is bigger than the spread in the fixed rate market, and the opposite occurs for $\rho_{Vr} = 0.25$.

To compute the swap spread in a way that is consistent with the debt market spreads, we use the following procedure. Suppose that the firm issues bonds that promise to pay a coupon rate of $(r_t + S_X)$, and at the same time it agrees to a coupon swap of $(r_t + S_X)$ for $(g + S_F)$ with a riskless counterparty. If, at coupon dates, $(r_t + S_X) < (g + S_F)$ and the firm is solvent, it pays the counterparty a net amount of $[g + S_F - r_t - S_X]F$. Conversely, if $(r_t + S_X) > (g + S_F)$, the bank will pay the firm an amount equal to $[r_t + S_X - g - S_F]F$. This swap corresponds to the swap of equal values of debt discussed in Section II. Once the firm enters the swap it promises to pay a total coupon rate of $(g + S_F)$, to be divided between the bondholders and the swap counterparty. Ex post, shareholders of the firm are worse off if the firm enters the swap and happens to be the net payer and are better off if the firm enters the swap and becomes the net receiver. Ex ante, since the present values of two bonds issued by the firm, one paying a fixed coupon rate of $(g + S_F)$ and other paying a variable coupon rate of $(r_t + S_X)$ would be equal, shareholders are indifferent when the firm agrees to this swap contract. Bondholders, however, gain because the cash flow to service the debt will be increased by the counterparty payment when the firm is due to receive on the swap. Consequently, whereas the straight floating rate notes sell at par F the swapped floating rate notes will be worth more than par. At any point in time the swapped price must satisfy the following equation:

$$B_V\left[rV - (g + S_F)F\right] + B_r m(\mu - r) + 1/2\left(B_{VV}\sigma_r^2 V^2 + B_{rr}\sigma_r^2 r^2 + 2B_{Vr}Vr\sigma_V\sigma_r\rho_{Vr}\right) + (r + S_X)F - rB = -B_t \tag{42}$$

Because there is no swap cash flow at the maturity date, the terminal condition (40) still holds.

At any time the firm will default on the bond if it cannot pay the promised amount of $[r_t + S_X]F$ to the debtholders. But now the cash flows to service the debt can come from two different sources: the firm's assets with value V and the claim on the assets of the swap counterparty. We assume, therefore,

[12] Except for m, the speed of reversion, we have used parameter values similar to those in Ramaswamy and Sundaresan (1986). The low rate of mean reversion was used so that short rate behavior resembles a random walk.

a default boundary on the debt $t < T$ which is given by:

$$B(V, r, r + S_X, t) = \begin{cases} \min[V + (r_t + S_X - g - S_F)F, (r_t + S_X)F] \\ \qquad \text{if } r_t + S_X > g + S_F \\ \min[V, (r_t + S_X)F] \\ \qquad \text{if } r_t + S_X \leq g + S_F \end{cases} \tag{43}$$

The first inequality refers to the case where the firm is receiving a swap payment. It then defaults on its debt if its assets plus the swap payments at the current rate will not cover the coupon payments on the debt. The second applies when the firm is due to make a swap payment. It then defaults on its debt when its assets will not cover the coupon payments on the debt.

This equal value swap is not an equilibrium transaction, since it is has a negative value for the riskless counterparty. As a result, the firm must increase the fixed coupon it offers in exchange for $r_t + S_X$. The equilibrium swap rate is the agreement to exchange $r_t + S_X$ for a fixed coupon rate c_s, such that the swap claim has zero value. By deviating from a total cash flow per unit time of $g + S_F$, the value of the equity will fall. Similarly, by paying a higher coupon rate on the swap, c_s, the value of the debt will decrease relative to the equal value swap, both because the likelihood of the firm being the net receiver will go down and because the net amount received in that swap will drop.

To solve for the equilibrium swap rate we use the Hopscotch algorithm referred to above. First, we solve for the value of the bonds according to equation (42) for a given debt to value of the firm ratio B/V. We then increase the fixed spread by S'_S, such that $c_s = g + (S_F + S'_S)$ and solve for new values of the debt and equity. We repeat this procedure until, for a particular value of the pure swap spread S'_S, $V = \overline{B} + \overline{E}$. The corresponding coupon rate c_S is the equilibrium swap quote that makes the value of the swap claim zero, since the values of the swapped debt, the equity, and the swap itself must sum to the value of the firm.

We then compare the equilibrium spread in the swap market with the bond market spreads. The equilibrium swap is an offer to exchange c_S for $r_t + S_X$. This is equivalent, because of net settlement, to an offer to exchange $c_S - S_X$ for r_t. We, therefore, define the swap spread as:

$$S_S = c_S - S_X - g \tag{44}$$

This is the coupon that must be added to the government bond rate in an equilibrium swap offered in exchange for r_t. The fixed swap coupon c_S is equal to $g + S_F + S'_S$, so that:

$$S_S = S_F - S_X + S'_S \tag{45}$$

This expression is identical to equation (14) for the zero coupon swap analyzed in Section 3.

Table V reports the values of swap market spreads in six different situations. As in the currency swap case, the pure swap spread S'_S is inversely

Table V

Interest Rate Swap Market Spreads in Basis Points per Annum

A firm has assets of value V and a single debt issue of value B. B/V is the market-value leverage; ρ_{Vr} is the correlation between the firm value and the interest rate; S_F is the default spread in the fixed debt market; and S_X is the default spread in the floating rate market. The volatility of firm value is 30% per annum. Bonds pay coupons twice a year and have a 5-year maturity. The instantaneous interest rate is r. Its dynamics are given by: $dr = m(\mu - r)\,dt + \sigma_r r\,dz_r$. The parameter values are: $m = 0.72$, $\mu = 0.09$, and $\sigma_r^2 = 0.006$. The firm swaps from a floating rate debt issue to a fixed rate, paying C_S fixed in exchange for $(r + S_X)$. A default free fixed rate bond pays a coupon of **g**. The equilibrium swap spread is $S_S = C_S - S_X$ quoted as a rate annum by which the default free value of the fixed side of the swap exceeds the default free value of the floating side. The pure swap spread is S'_S which compensates the swap counterparty for the incremental default risk arising from the difference between the way that the swap is settled and the way that risky debt is settled.

	$B/V = 0.30$			$B/V = 0.50$		
ρ_{Vr}	0.25	0.00	−0.25	0.25	0.00	−0.25
S_X	12	19	27	57	71	87
c_S-g	18	23	30	77	89	104
S_S	6	4	3	20	18	17
S'_S	2	5	10	11	20	32

proportional to the correlation parameter ρ_{Vr}. The total swap spread is an increasing function of ρ_{Vr} because a rise in the correlation of the firm's assets with the variable payment reduces the variable spread S_X. This effect dominates the rise in the pure swap spread in expression (45). The order of magnitude of the relationship between the pure swap spread S'_S, the swap spread S_S, and the debt market spreads S_F and S_X is similar in Table V to the relationship in Table III for zero coupon currency swaps. Although the two tables are not strictly comparable, both suggest that the equilibrium swap spread is significantly lower than debt market spreads.

VI. Summary and Conclusions

We have characterized the exchange of financial claims arising from risky swaps. These transfers are between three groups: shareholders, debtholders and the swap counterparty. Swaps generally result in wealth transfers from shareholders to debtholders.

We have derived equilibrium swap rates and related them to debt market spreads. In the case of geometric Brownian motion we obtained closed form solutions for the value of the default risk in the swap. These values are of a significantly lower order of magnitude than debt market spreads. We extended the analysis to include coupon payments and to analyze swaps between two risky counterparties. It remains to be tested whether this model can explain such empirical features of swap markets as the level of swap spreads and their relationship to the variables that determine them in equilibrium. Other extensions would be to include the effects of collateralizing the swap and then to embed this analysis in a model with agency effects or other imperfections that can provide a motivation for the swap transaction.

Appendix 1

Net cash flow for \$1 of principal due to the receiver of fixed rate dollars in a swap

A: Interest-rate swap

Time	1	$\cdots$	$N-1$	N
Cash Flow	$(C - \tilde{R}_1)$		$(C - \tilde{R}_{N-1})$	$(C - \tilde{R}_N)$

B: Currency swap

Time	1	$\cdots$	$N-1$	N
Cash Flow	$C - C_F\tilde{E}_1$		$C - C_F\tilde{E}_{N-1}$	$(C+1) - (C_F + P_F)\tilde{E}_N$

Time is in units equal to the coupon frequency.

C: Coupon in dollars per \$1 principal value
R_i: Variable rate at time i (random)
E_i: Exchange rate in dollars per unit of foreign currency at time i (random)
C_F: Coupon in foreign currency per \$1 principal value
P_F: Principal in foreign currency per \$1 principal value
N: Maturity of swap.

Appendix 2

From Stulz (1982) equations (14), (12), and (11'):

$$PX(V, X, F) = F_0 - MX(V, X, O) + MX(V, X, F) \qquad \text{(A2.1)}$$

$$MX(V, X, F) = C(V, F) + C(X, F) - M(V, X, F) \qquad \text{(A2.2)}$$

$$M(V, X) = V_0 - C(V, X) \qquad \text{(A2.3)}$$

Where $MX(V, X, F)$ is a European call option on the maximum of V and X with an exercise price of F.

Substituting (A2.2) in (A2.1) and using $C(V, O) = V_0$, $C(X, O) = X_0$, and (A2.3) gives:

$$PX(V, X, F) = F_0 - X_0 - C(V, X) + C(V, F) + C(X, F) - M(V, X, F) \qquad \text{(A2.4)}$$

Rearranging gives equation (7). Using (4) gives:

$$Z_S = C(V, X) - C(V, F) - C(X, F) + M(V, X, F) \qquad \text{(A2.5)}$$

Appendix 3

From Stulz (1982) equation (11), with the correction noted in Johnson (1987):

$$\begin{aligned} M(V, X, F) = {} & X_0 N_2\big(\gamma_1 + \sigma_x\sqrt{T}, \big(ln(V_0 / X_0) \\ & - \tfrac{1}{2}\sigma_W^2 T\big)/\sigma_W\sqrt{T}, (\rho\sigma_V - \sigma_X)/\sigma_W\big) \\ & + V_0 N_2\big(\gamma_2 + \sigma_V\sqrt{T}, \big(ln(X_0 / V_0) \\ & - \tfrac{1}{2}\sigma_W^2 T\big)/\sigma_W\sqrt{T}, (\rho\sigma_X - \sigma_V)/\sigma_W\big) \\ & - F_0 N_2(\gamma_1, \gamma_2, \rho) \end{aligned}$$

where:

$$\begin{aligned} \gamma_1 &= \big(ln(X_0/F_0) - \tfrac{1}{2}\sigma_X^2 T\big)/\sigma_X\sqrt{T}, \\ \gamma_2 &= \big(ln(V_0/F_0) - \tfrac{1}{2}\sigma_V^2 T\big)/\sigma_V\sqrt{T}, \\ \sigma_W^2 &= \sigma_V^2 + \sigma_X^2 - 2\rho\sigma_V\sigma_X, \end{aligned}$$

substituting for $\gamma_1, \gamma_2, \bar{\sigma}_V, \bar{\sigma}_X, \bar{\sigma}_W, \rho_W, \rho_X, \mathrm{d}_X, \mathrm{d}_V, \mathrm{d}_W$ gives equation (30).

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