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Default risk and derivative products

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The modelling of default risk in debt securities involves making assumptions about the stochastic process driving default, the process generating the write-down in default, and risk-free interest rates. Three generic approaches have been used. The first relies on modelling the value of the assets on which the debt is written. The second involves modelling default as an arrival process. The third involves directly modelling the interest rate spreads to which default gives rise. Each of these approaches may be applied to the impact of default risk on derivative products such as swaps and options. One application is to the valuation of derivative products that may default. The other is to the new class of 'credit derivatives' that represent derivative products written on credit risk.

Keywords: default risk, credit risk, risky debt, derivative products

1. Introduction

The pricing and hedging of derivative products is an area where relatively advanced mathematical techniques have made their most obvious contribution to the practice of finance. The key insight was that of Black and Scholes: given certain assumptions derivative products can be perfectly hedged with the underlying assets on which the derivatives are written (Black and Scholes, 1973). This hedge must, being riskless, have a rate of return equal to the riskless interest rate.

Pricing and hedging strategies based on this type of no-arbitrage model have enabled financial institutions to hold enormous portfolios of derivative products while exposing themselves to minimal price risk. The mathematics involved is relatively standard, consisting of describing the evolution of the underlying asset price by a stochastic process and then characterizing the no-arbitrage condition either by a differential equation or by a risk-adjusted expectation. Different derivative products are then described by different boundary conditions. In practice, for all but the most simple cases, the p.d.e. or the expectation is solved numerically to give the value of the derivative product and the strategy whereby this can be hedged.

The technology underlying the Black–Scholes model has been used to price and hedge options on currencies, bonds, interest rates and commodities, as well as exotic options, quantos, diff swaps, index amortizing swaps, callable debt, and a plethora of other products (see, for instance, Jamshidian, 1994). The power of the technique has been enhanced by fundamental insights into the theory of no-arbitrage conditions, such as the more general analysis of Harrison and Kreps (1979), and the application to models of interest rates of Heath *et al.* (1992).

In parallel with these theoretical developments, computational procedures have been developed to enable the numerical solution of the values of a variety of options. These include the application of binomial trees, finite difference techniques, Monte Carlo methods, pseudo-Monte-Carlo methods, and other numerical integration techniques (for representative examples see Cox *et al.*, 1979; Geske and Shastri, 1985; Boyle, 1977; Barrett *et al.*, 1992; and Paskov and Traub, 1995).

1.1. *Default Risk*

All of these developments have focused on one part of the original Black Scholes analysis, the pricing of derivative products. But there was another important insight in the Black–Scholes paper, that credit risk can be priced using similar models. Thus, the debt of a firm may be viewed as the equivalent of debt that has no default risk and a contingent claim that represents the risk of default. The contingent claim component of the debt may then be valued and hedged using the same technology that has proved so productive in the pricing and hedging of derivative products such as options. Recently, however, this line of analysis has been picked up by both academics and practitioners. This has led to a flood of research and innovation based on the impact of credit risk on derivative products. These developments have three strands: the impact of default risk on rates or prices on which derivative products are written, the impact of the risk of default by the counterparties to derivative products on their prices, and the development, pricing and hedging of new derivative products to manage the impact of default risk.

The purpose of this paper is to summarize the developments in these areas. Models of the impact of default risk on derivative products are, however, almost all extensions of models of the pricing of risky debt. The reason for this is that the pricing of default risk in modern financial models is usually achieved by solving a non-arbitrage condition that takes the form of a p.d.e. The value of risky debt is the solution of this p.d.e. subject to particular boundary conditions. The value of derivatives that are related to this default risk is also a solution to the same p.d.e., subject to a different set of boundary conditions.

The paper begins, therefore, with a rather exhaustive treatment of the pricing of risky debt. After a discussion of the motivations for these models, Section 3 discusses the Merton (1974), model and its extensions. The key feature of the Merton model is that default is driven by the behaviour of the value of underlying assets on which the debt claim is written. Sections 4 and 5 then discuss models that replace this assumption either partially or completely with a more direct assumption about the stochastic process driving default. In Section 6 a rather different approach is described. This involves making direct assumptions about the process driving the spread between riskless and risky interest rates. In Section 8 the application of these models to the pricing of default risk in derivative products is discussed, and Section 9 covers the pricing of derivative products written on default risk (credit derivatives). Finally, Section 10 summarizes some of the outstanding issues.

2. The need for models of default risk that integrate derivative products

The impetus for the development of models that integrate the pricing and hedging of derivative products with the pricing and hedging of credit risk has come from three sources. The first is that many debt securities issued by risky entities contain embedded derivatives. Thus, some mortgage obligations contain the option to prepay, much corporate debt is callable, convertible bonds contain an equity option, and there is a variety of innovative debt securities containing other embedded options or forward contracts.

The second reason for interest in this topic is the massive growth in the volume of derivatives, which has led to a concern that potential default by the counterparties to these products could jeopardise the stability of some parts of the international financial system. (See, for instance, US General Accounting Office, 1994.) Regulatory requirements for the capital of financial institutions depend on an assessment of the credit risk of these instruments. Analysis of this issue requires a systematic way of modelling the impact of default risk on derivative product prices and their interaction with other values that are also sensitive to credit considerations.

The third motivation for the development of these models is the growth of markets for derivative products where the underlying asset or rate is credit-related. This category of derivative products is commonly called credit derivatives. It includes forward contracts on credit spreads, swaps whose payoff is linked to a credit index, swaps whose payoff insures the default risk on a particular debt issue or a group of debt issues, swaps where the payoff is linked to the return on a category of risky debt, and options related to these contracts. Descriptions of various credit derivatives are given in van Duyn (1995). Some uses are described in Flesaker *et al.* (1994). The pricing of these credit derivatives clearly requires models where credit risk is modelled as accurately as possible. Their hedging requires that the models must be able to encompass realistically the actual price behaviour of the rates or prices to which the derivatives are related.

3. Models of default risk with default and settlement based on underlying asset value

3.1. *The Merton model*

The first significant development of the mathematical modelling of default risk was the Black–Scholes insight that the debt of a firm can be viewed as a contingent claim on the assets of the firm (Black and Scholes, 1973). This was used by Merton to give an analysis of the behaviour of the interest rate on debt that is subject to default risk (Merton, 1974). The Merton model starts from the assumption that a firm owns a collection of assets that have value $V(t)$ at date t. This evolves, according to a diffusion-type stochastic process, with differential equation

$$dV = (\alpha V - C)\,dt + \sigma V\,dZ \qquad (1)$$

where α is the unknown instantaneous expected rate of return on the firm per unit of time, C is the net dollar payout by the firm per unit time to its shareholders, σ is a constant, and dZ is the increment to a standard Wiener process.

The debt of the firm is characterized as a claim that promises to make a single payment of a fixed amount, B, at a future date T. The actual payment to the debt will be given by the minimum of B and $V(T)$. If $V(T) \succ B$, then the firm is solvent at date T and the payment on the debt is its full promised payment B. If $V(T) \prec B$, then the firm is insolvent because it does not have sufficient assets to meet its debt claim, and the payment is $V(T)$. The value of the firm's assets is transferred to the debt holders in the model because the firm is costlessly liquidated and the proceeds given to the holders of the debt. With these assumptions, the debt may be valued as a riskless claim paying the amount B at the date T minus a put option that pays the maximum of zero and $(B - V(T))$. The put option represents the pure default risk component. The standard pricing model of Black–Scholes may then be used to value the put option.

Merton then considers the formation of a zero net investment portfolio consisting of a claim whose price is the value of the assets of the firm, the debt of the firm, and riskless debt. These are held in proportions such that the return on the portfolio is deterministic and the portfolio requires zero net investment. The expected return on the portfolio must be zero to avoid arbitrage. This condition is sufficient to derive the partial differential equation that the price $F(V, t)$ of any contingent claim on V must satisfy

$$\tfrac{1}{2}\sigma^2 V^2 F_{vv} + (rV - C)F_v - rV + F_t + C^B = 0 \tag{2}$$

where C^B is the dollar payout per unit time on the contingent security, r is the constant riskless interest rate, and subscripts denote partial derivatives. Solving this for the value of the debt, D, gives a value for the risky debt that is equal to the default-free value of the promised payment, B, minus the value of the default risk.

This value relationship can then be restated in terms of the promised yield on the bond, which is defined by

$$R(T) = \ln(B/D)/\tau \tag{3}$$

where $\tau = (T - t)$. Merton solves for the spread between the yield on the risky bond and the riskless interest rate

$$R(\tau) - r = -\log(N[h_2(d, \sigma^2\tau)] + N[h_1(d, \sigma^2\tau)]/d)/\tau \tag{4}$$

where $d = B\exp(-r\tau)/V$, $N(.)$ is the standard normal distribution function, $h_1 = -(0.5\sigma^2\tau - \log[d])/\sigma\sqrt{\tau}$, $h_2 = -(0.5\sigma^2\tau + \log[d])/\sigma\sqrt{\tau}$. The interest rate spread $R(\tau) - r$ is termed the risk premium and (4) defines a risk structure of interest rates, which depends on the maturity of the debt, the variance of the firm's asset value, σ^2, and the ratio, d, of the riskless present value of the promised payment, B, to the current value of the firm's assets, V.

Although the Merton model is very simple, it illustrates the various uses of such a model and the limitations of this style of analysis. As the model gives an arbitrage-based valuation of default risk it can be used to price default risk independently of the risk preferences of investors. This pricing is based on the relative price of the debt of the firm and the value of the firm's assets. It may also be stated as a relative price between the debt and the equity of the firm. Thus, the model gives a way of hedging risky debt with the equity of the same firm. The model can also describe the evolution of the price of the risky debt over time as a function of the other variables in the model, and so such models may potentially be used for pricing derivative claims on the risky debt.

One way of describing the price of the risky debt is as a spread between the interest rate promised on the debt and the interest rate it would carry if it were default-free. Thus, the model can be used to describe the structure of interest rate spreads at a point in time or their evolution over time. In principle, any claim whose payoff is contingent only on these spreads can then be valued through a no-arbitrage relationship with the underlying risky debt. As this, in turn, must satisfy a no-arbitrage condition with the value of the underlying assets of the firm and the value of the firm's equity, the Merton model has the potential to integrate the pricing of equity, risky debt, and contingent claims written on either or both of these.

The power of the Merton model is bought at the price of some strong assumptions. Apart from the standard assumptions of continuous time no-arbitrage models (continuous costless trading and short-selling, no taxes, perfectly divisible assets, price-taking investors, no borrowing–lending spread), these include:

(1) the liabilities of the firm consist only of a single class of debt;
(2) the debt has a zero coupon;
(3) the interest rate is constant;
(4) the value of the assets of the firm performs geometric Brownian motion;
(5) bankruptcy is costless;
(6) the strict priority of claims is preserved in bankruptcy;
(7) bankruptcy is triggered only at the maturity date of the debt;
(8) bankruptcy is triggered by the value of the firm's assets being below the promised payment on the debt;
(9) there are no taxes.

All of these assumptions are rather crude descriptions of reality. They largely concern the nature of the debt contract and the way that bankruptcy is triggered and settled. Much of the subsequent effort in this area has been directed at relaxing these assumptions to give more realistic models.

3.2. *Default triggering and settlement*

The stylized treatment of default in the Merton model ignores various empirical features of the way that default actually occurs and its impact on the payoff to debt. Any deviation of the payoffs on a debt instrument from their promised values will depend on the specific events triggering default, the source of underlying value from which claims are met, the set of competing claims to the assets of the firm and their relative priorities, the nature of any restructuring of these claims and the final settlement between the involved parties. Prior to actual default, there is also the possibility that financial distress will induce events that affect the value of the debt claim. Financial distress refers to situations that fall short of default, but where the probability of future default is high. In such situations, the operational decisions of the firm may be affected, changing the stochastic process of the asset value. (For empirical evidence on the bankruptcy process see Warner 1977, and Franks and Torous 1994. For a model of the effect of financial distress see Myers 1977. For a general discussion see Brealey and Myers 1991, Chapter 18.)

Because of these features of the actual default process, the assumptions in the Merton model are highly stylized versions of reality. The conditions under which default will be triggered are far more complex than the model assumes. In addition, contractual and legal settlement and priority rules only provide an approximate indication of eventual distributions to claimholders, so the final payoffs on a given claim in the event of default are unpredictable even when conditioned on the value of the firm's assets. One response to these difficulties has been to extend the Merton model to make it more realistic as a description of the way that default affects the payoff to debt claims. A 'perfect' version of the model would require at least the following extensions: multiple classes of coupon-bearing debt with realistic covenants, stochastic interest rates, the inclusion of taxes, costly bankruptcy and financial distress that can occur throughout the life of the debt, and possible violation of strict priority rules.

3.3. *Extensions of the Merton model*

Black and Cox (1976) incorporated classes of senior and junior debt, safety covenants, dividends, and restrictions on cash distributions to shareholders. Brennan and Schwartz (1977) incorporated coupons and extended the model to convertible bonds. Geske (1977) value options written on the equity of a firm with risky debt. Stochastic interest rates were introduced by Brennan and Schwartz (1980) who modelled convertible debt, and Cox *et al.* (1980) who modelled the valuation of variable rate debt. Mason and Bhattacharya (1981), incorporated a jump process for the underlying asset value, and Ho and Singer (1984) incorporated a sinking fund and different maturities of debt. A good summary of these developments can be found in Ingersoll (1987, Chapter 19). One particularly active area of the development and application of the Merton model has been mortgage debt. This literature is summarized in Kau and Keenan (1995).

The complexities of most of these additions to the model mean that the p.d.e. for the value of the risky debt no longer has a neat single equation solution. Allowing default before the final maturity of the debt introduces a boundary condition to the solution that may be very complex. This is especially so if the default choice is endogenized, so that default occurs only if it is in the interests of the shareholders of the firm. Only if the boundary where bankruptcy is triggered has a very simple form can a closed-form solution be obtained (see Black and Cox, 1976; Shimko *et al.* 1993). Otherwise, numerical procedures must be used to solve the p.d.e.

3.4. *Empirical tests of the models*

Tests of the contingent claims approach to the pricing of risky debt, based on using models of underlying asset values, have given mixed results. One successful application of the model has been the valuation of mortgage debt by Titman and Torous (1989). The examined the price quotes for mortgages on single hypothetical buildings with standard characteristics. They modelled the interest rate process as in Cox *et al.* (1985)

$$\mathrm{d}r = (\zeta - \beta r)\,\mathrm{d}t + \eta\sqrt{r}\,\mathrm{d}Z \tag{5}$$

The building value follows geometric Brownian motion. They also allowed for correlation between the two processes for the interest rate and the building value and for the optimal choice of the default date. Their model is successful in explaining the general magnitude of the default premium and a 'significant proportion' of the period-to-period changes in the premium. It does not, however, explain the maturity structure of the premium.

Titman and Tourous do not attempt, however, to apply the model to actual mortgages because of the complexity of doing so. This complexity is illustrated in Selby *et al.* (1988). They model the value of the default risk for a single firm for the purpose of valuing a loan guarantee. Another difficulty in the implementation of the model that this paper illustrates is the problem of measuring the market value of the underlying assets of the debt issuer.

Weinstein (1983) investigated the ability of the Merton model to explain the systematic risk of corporate bonds. He found that the model captures some fundamental features of the way that bond betas vary over time. His model is, however, capable of explaining only about a quarter of the time-series variation of bond returns.

Jones *et al.* (1984) investigated the contingent claims model using data on 27 firms with single capital structures. They compared the ability of the model to explain the pricing of bonds issued by these firms with a naive model reflecting the magnitude and timing of promised payments, various call provisions and sinking fund requirements. They found incremental explanatory power of the contingent claims model over their naive model for non-investment-grade bonds, but they reported that 'firm value risk is not playing a significant role in explaining investment grade bond prices'.

3.5. *Limitations of the extended Merton model*

Despite the considerable efforts devoted to relaxing its assumptions, the risky debt model based on underlying asset value has had only limited success in explaining the behaviour of the rates and prices of debt instruments that are subject to credit risk. The difficulty of realistically modelling the way that bankruptcy is triggered, the way that financial distress affects asset values, and the way that claims are settled in bankruptcy makes the assumptions of this class of models necessarily simplistic. This would not matter greatly if the model were capable, despite the simplified assumptions, of explaining the empirical behaviour of the prices of securities that are subject to credit risk. Success in this area has, however, been limited. In particular, it is difficult to generate sufficient time-series variability of the default spread in interest rates using only the underlying asset value as the state variable driving the default probability.

In addition, for modelling the default risk of derivative products or derivative products written on default risk, it is important that the model be able to price the underlying default risk properly. This is usually taken as meaning that the model should be able to generate the current market structure of default risk pricing as its equilibrium. This is similar to the problem in valuing interest rate derivatives of ensuring that the current term structure of interest rates is consistent with the model. In the case of term structure models, this has led to the 'whole term structure' models of Ho and Lee (1986) and Heath *et al.* (1992). In the case of default risk it has led to attempts to use models that are less dependent on underlying asset value than the Merton model, and make more direct assumptions about the default process or its impact on the pricing of debt.

Such models may then be parameterized to recover the current structure of risky bond prices as equilibrium prices in the model. The model may then be used to concentrate on the relative pricing of derivatives given the structure of risky bond prices. Simplifying the modelling of the default process also gives more scope for introducing complexity into the stochastic process for default-free interest rates, while retaining tractability of the model.

Apart from these considerations of modelling complexity and consistency with market prices, there is another reason in many cases for using a model that does not rely on underlying asset value. Many entities that issue risky debt do not have identifiable collections of assets from which the debt claim will be met. For instance, sovereign issuers and other agencies such as municipalities cannot easily be analysed in this way. In these cases, the value of a model based on underlying asset value must necessarily be limited. So, using a set of assumptions that model the default process and the payoff to the debt in default more directly, without going through the intermediary of the behaviour of underlying asset value, becomes inevitable.

4. Hybrid models

4.1. *The Longstaff and Schwartz model*

Some of the weaknesses of the above approach are addressed in a model by Longstaff and Schwartz (1995a). Their model is built around a stopping time result for a time-varying boundary found in Buoncuore *et al.* (1987). In a way similar to Merton, they assumed that the value of the firm follows a diffusion process

$$\mathrm{d}V = \mu V\,\mathrm{d}t + \sigma V\,\mathrm{d}Z_V \tag{6}$$

where σ is a constant, μ is the rate of return on the underlying asset value and Z_V is a standard Wiener process. In contrast to Merton's assumption of constant interest rates, Longstaff and Schwartz postulated that short-term rates follow the mean-reverting Ornstein–Uhlenbeck process first used by Vasicek (1977)

$$\mathrm{d}r = (\zeta - \beta r)\,\mathrm{d}t + \eta\,\mathrm{d}Z_r \tag{7}$$

where ζ, β and η are constants and Z_r is another standard Wiener process; the instantaneous correlation between Z_V and Z_r is $\rho\,\mathrm{d}t$. The model allows for a variety of liability classes with different coupon rates, priorities and maturity dates.

Longstaff and Schwartz argue for the existence of a (constant) threshold level K which serves as a distress boundary; if the value of the assets breaches this level, default is triggered, some form of restructuring occurs and the remaining assets of the firm are allocated among the firm's claimants. Implicit in this formulation is the assumption that once this critical value is reached, default occurs on all outstanding liabilities at the same time. Thus, contrary to Merton's model, default can occur prior to maturity.

If a reorganization occurs during the life of a security, the security holder receives $(1 - w)$ times the face value of the security at maturity, where w represents the writedown on a particular security and is constant over all instruments issued by the firm. The model thus avoids the dependence of the payoff on the debt on underlying asset value. It retains a role for asset value in determining the event of default, but not the payoff. In the models discussed in Section 5 below, asset value is dispensed with altogether as a determinant of default.

Assuming perfect and frictionless markets in which trading takes place continuously, Longstaff and Schwartz derived the fundamental evaluation equation that the price of any derivative asset $H(V, r, T)$ must follow:

$$\frac{\sigma^2}{2} V^2 H_{VV} + \rho\sigma\eta V H_{Vr} + \frac{\eta^2}{2} H_{rr} + rVH_V + (\alpha - \beta r) H_r - rH = H_T \tag{8}$$

where α represents the sum of ζ and a constant representing the market price of interest-rate risk. There is a riskless bond that pays off one dollar at maturity and its price, which is a function of the interest rate process and time, is given as

$$D(r, T) = \exp(A(T) - B(T)r) \tag{9}$$

where

$$A(T) = \left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta}\right)T + \left(\frac{\eta^2}{\beta^3} - \frac{\alpha}{\beta^2}\right)(\exp(-\beta T) - 1) - \left(\frac{\eta^2}{4\beta^3}\right)(\exp(-2\beta T) - 1) \tag{10a}$$

$$B(T) = \frac{1 - \exp(-\beta T)}{\beta} \tag{10b}$$

The payoff function of a risky discount bond is given by $1 - wI(\gamma \leq T)$ where $I(.)$ is an indicator function taking the value one if the first-passage time γ of V to K is less than or equal to T and zero otherwise. Based on the above assumptions, the value of a risky zero coupon bond is

$$P(X, r, T) = D(r, T)[1 - wQ(X, r, T)] \tag{11}$$

where $X = V/K$ and $Q(X, r, T) = \tilde{E}[I(\gamma \leq T)]$ is the probability that the first passage time of $\ln X$ to zero is less than T, where the expectation is taken with respect to the risk-adjusted processes

$$\mathrm{d}\ln X = \left(r - \frac{\sigma^2}{2} - \rho\sigma\eta B(T - t)\right)\mathrm{d}t + \sigma\,\mathrm{d}Z_1 \tag{12a}$$

$$\mathrm{d}r = (\alpha - \beta r - \eta^2 B(T - t))\,\mathrm{d}t + \eta\,\mathrm{d}Z_2 \tag{12b}$$

Applying the same approach to the valuation of floating rate securities implies that the value of a floating rate coupon can be expressed as

$$F(X, r, \tau, T) = D(r, T)R(r, \tau, T) - wD(r, T)G(X, r, \gamma, T) \tag{13}$$

where

$$R(r, \tau, T) = \tilde{E}[r_\tau] \tag{14a}$$

$$G(X, r, \tau, T) = \tilde{E}[r_\tau I(\gamma \leq T)] \tag{14b}$$

The interest in this approach is that it allows stochastic interest rates and introduces an explicit role for the correlation between interest-rates and firm value. The model allows for default to occur prior to maturity. The use of a threshold level, K, triggering default is similar in spirit to Merton's approach. In the extended Merton's model, the threshold level is given either by bond covenants or by an optimality condition on the value of the equity claim. In Longstaff and Schwartz's approach K must be constructed as an aggregate indicator from the contents of the contracts of all securities the firm has outstanding. The introduction of a write-down variable, w, which is independent of the underlying asset value introduces another degree of freedom into the model so that it could, in principle, be made to fit any given level of the default spread observed in interest rates. More problematic, however, is the assumption that, regardless of seniority, all securities pay off a fraction w of their face value in the event of default.

Setting both the default trigger and the payoff fraction to constants may not be the most satisfying way of capturing either the events that precipitate a firm into bankruptcy or anticipating the outcome of the complex bargaining process that might ensue. Nevertheless, it precludes the undesirable property of simple versions of Merton's model that, before maturity, firm value can fall significantly below the face value of the bond without triggering default. The critical level K should ideally be a function of the liabilities outstanding at each point in time; w should be allowed to vary across a limited number of security classes. Undoubtedly, however, incorporation of such features is bound to compromise the model's tractability.

5. The default intensity approach

5.1. *Jarrow and Turnbull*

A third approach, which explicitly considers the possibility of default of the risky asset, has recently been proposed by Jarrow and Turnbull (1995). They incorporate stochastic interest rates, but the processes for bankruptcy and the payoff on the risky debt conditional on default are specified exogenously. Stipulating that payoffs to the risky security are made in nominal terms in a risky currency which they call '*XYZ*-dollars', the imposition of no-arbitrage conditions yields the value of risky bonds in terms of the value of the riskfree bond and a conversion factor used to translate riskless dollars into risky '*XYZ*-dollars'. In their continuous-time model, bankruptcy is modelled as a jump process $N(t)$, which is defined as

$$N(t) \equiv I(t \geq \tau_1^*) \equiv \begin{cases} 1 & \text{if} \quad t \geq \tau_1^* \\ 0 & \text{if} \quad t \leq \tau_1^* \end{cases} \tag{15}$$

where τ_1^* is the stopping time representing the first time of bankruptcy of the firm which has issued the risky debt. This stopping time is assumed to be exponentially distributed with parameter λ, a constant.

This assumption for the default process achieves two effects. The first is that it removes the dependence on the value of underlying assets, so that the model may be applied to situations where this is not observable. Secondly, it cures a technical problem with models of the Merton type. This is that, for such models, default becomes predictable when the value of the underlying assets approaches the default boundary. This may lead to implausible behaviour of credit spreads for short maturities (Pitts and Selby, 1983). The assumption of a jump process makes the default time unpredictable and avoids this technical difficulty.

Following the approach of Heath *et al.* (1992) bond prices are modelled by specifying the stochastic processes for forward rates. The default-free forward rate $f_0(t, T)$ is assumed to follow the process

$$\mathrm{d}f_0(t, T) = \alpha_0(t, T)\,\mathrm{d}t + \sigma(t, T)\,\mathrm{d}W(t) \tag{16}$$

The forward rate of the risky bond then evolves according to the process

$$\mathrm{d}f_1(t, T) = \alpha_1(t, T)\,\mathrm{d}t + \sigma(t, T)\,\mathrm{d}W(t) + \theta(t, T)\,I(t \leq \tau_1^*)[\mathrm{d}N(t) - \lambda\,\mathrm{d}t] \tag{17}$$

This is identical to the risk-free process, with the exception that at the time of bankruptcy, a jump of magnitude $\theta(t, T)$ occurs. The source of uncertainty common to both riskless and risky forward rate processes is the standard Brownian Motion $W(t)$. Default free and risky spot interest rates are defined by $r_0(t) \equiv f_0(t, t)$ and $r_1(t) \equiv f_1(t, t)$ while money-market accounts accruing at these respective rates are given as

$$B_0(t) = \exp\left\{ \int_0^t r_0(s)\, \mathrm{d}s \right\} \tag{18a}$$

$$B_1(t) = \exp\left\{ \int_0^t r_1(s)\, \mathrm{d}s \right\} \tag{18b}$$

The conversion between risky and riskless currency occurs at a rate of unity up to the point of bankruptcy, at which time the rate falls to $\exp(-\delta)$ (<1). Consequently, the model allows for default to occur before the maturity date of the risky claim. Nevertheless, default is triggered by an exogenously specified process that does not depend on the modelling of underlying asset value. Settlement of the risky asset occurs at maturity at the face value, written down by a fixed fraction if default has occurred.

Jarrow and Turnbull then impose the condition that the market is complete. It follows that no arbitrage opportunities exist and all assets can be valued as though they appreciate at the riskless rate $r(t)$. The probabilistic equivalent of this condition is the assumption of the existence of a unique equivalent measure $\tilde{Q}$ under which the price processes for assets which have been divided by the price of a numeraire asset accruing at $r(t)$, $v_1(t,T)/B(t)$, $p_0(t,T)/B(t)$ and $B_1(t)e_1(t)/B(t)$, are martingales, independent of the asset's maturity.

A limitation of the model is that the parameters of the Poisson process for bankruptcy are assumed to be constant over time, effectively imposing independence of the bankruptcy process from the default-free interest-rate process. By the definition of a martingale, it then follows that

$$\begin{aligned} v_1(t,T) &= \tilde{E}_t(e_1(T)/B(T))\,B(t) \\ &= [I(t \geq \tau_1^*)\,\mathrm{e}^{-\delta} + I(t \geq \tau_1^*)(\mathrm{e}^{-\lambda\mu(T-t)} + \mathrm{e}^{-\delta}(1 - \mathrm{e}^{-\lambda\mu(T-t)}))]p_0(t,T) \end{aligned} \tag{19a}$$

$$v_1(t,T) = \tilde{E}_t(e_1(T))p_0(t,T) \tag{19b}$$

where $\tilde{E}_r(.)$ is the expectation under probability measure $\tilde{Q}$ conditional on information at time t. The last equation shows the price of the default-risky bond as being composed of the price of the riskless bond of identical maturity, multiplied by the exchange rate expected to prevail at time T used to convert risky into riskless currency units. This decomposition is possible because the default process and the process of the riskless bond are assumed to be uncorrelated. At the same time, the writedown in the event of default is fixed over time. Relaxation of any one of these features is likely to invalidate the elegant form of (19).

This above approach has the merit that it avoids the modelling of underlying asset value in deriving the spread between risky and riskfree bonds. It is, therefore, applicable to situations where the underlying assets either do not exist or their value is unobservable. The model could be parameterized by fitting to the prices of risky debt, so that the value of other securities or derivative products that are dependent on the same underlying default process could be derived relative to the debt spread. One weakness of the model is that its simplest form assumes independence of the interest rate process from the default process. There is nothing in the structure of the model that demands this, and the model would be numerically computable in a more general form. Another, possible greater, weakness of the model in the form presented by its authors is that it can generate only limited time-series variability in the spread between riskless and risky debt. This is caused by the assumed constancy of the arrival rate of the default process in the simplest form of the model. Many of the credit derivatives discussed below in Section 8 are very sensitive to the stochastic process for the credit spread, so this is a serious shortcoming of the model for certain purposes.

5.2. *Duffie and Singleton*

A similar approach is pursued by Duffie and Singleton (1995). Their model assumes a multi-factor square-root process for the riskless interest rate and a Poisson process for default with state-dependent values for the hazard rate and the loss in default. They demonstrated some general properties of the model. In particular, valuation under the risk-adjusted probability measure may be executed by discounting the default-free payoff on the debt by a discount rate that is adjusted for the parameters of the default process. Thus, the valuation procedure is the same as for riskless claims, with an adjustment to the interest rate for the impact of default risk. This is a possible motivation for a series of more *ad hoc* models that simply assume a process for the spread and then use this in a way similar to that derived by Duffie and Singleton. These are discussed in the next section.

6. Models of the spread

6.1. *Direct spread models*

An alternative approach, introduced by Ramaswamy and Sundaresan (1986), uses a direct assumption about the stochastic process followed by the interest rate default spread. The default-free interest rate is assumed to evolve according to the square root model of Cox *et al.* (1985), and the Local Expectation Hypothesis, according to which expected returns on all default-free bonds over an infinitesimally short period of time are equal to the riskless rate, is assumed to hold

$$E[\mathrm{d}P(r,t,T)] + c(t)\,\mathrm{d}t = P(r,t,T)\,r(t)\,\mathrm{d}t \tag{20}$$

where $P(r,t,T)$ is the price of a riskless bond and $c(t)$ is the coupon rate. For default-free bonds issued at par, $c(t)$ defines the risk-free rate corresponding to the particular maturity of the bond and cannot deviate from $r(t)$.

Due to credit risk, corporate bonds will sell at a discount relative to government bonds, the size of which depends on the probability of default, the contractual provisions that define payoffs contingent on bankruptcy and the premium demanded in the market for similar instruments. Ramaswamy and Sundaresan propose, as instrumental variables, default premia on newly issued instruments with the same maturity and from the same risk class (in terms of probability of default and covenant specification) required by investors. Since these instruments are close substitutes for the newly issued risky bonds, it is argued that the premia observed in the market should provide good proxies for the required premium on the risky instrument which has already been trading for some time.

Assuming that the risky variable-rate bond pays a continuous coupon with a spread π over the riskless rate (where π is set at the time of issuance), their pricing condition adjusted for the possibility of default takes the following form

$$E[\mathrm{d}P(r,t,T)] + (r(t)+\pi)\,\mathrm{d}t = P(r,t,T)[r(t)+p(t)]\,\mathrm{d}t \tag{21}$$

where $p(t)$ is the expected market premium on newly issued obligations in the same risk class. Assuming that both the risk-free rate and the expected market premium follow the mean-reverting square-root processes

$$\mathrm{d}r = \kappa(\mu - r)\,\mathrm{d}t + \sigma\sqrt{r}\,\mathrm{d}z \tag{22a}$$

$$\mathrm{d}p(t) = \kappa_p(\mu_p - r)\,\mathrm{d}t + \sigma_p\sqrt{p}\,\mathrm{d}z_p \tag{22b}$$

the valuation equation that must be satisfied by the price of any default-risky bond is shown to be

$$\tfrac{1}{2}F_{rr}\sigma^2 r + \tfrac{1}{2}F_{pp}\sigma_p^2 p + F_{rp}\sigma_{rp}\sqrt{r}\sqrt{p} + F_r\kappa(\mu - r) + F_p\kappa_p(\mu_p - p) + (r+\pi) = F_\tau \tag{23}$$

subject to the boundary condition $F(r,p,0,\tau) = 1$.

The required return differential approach has been criticized on a number of accounts. Most obviously, the boundary condition is not adequate for instruments whose payoff will be reduced by default. Also, the pricing condition (21) indicates that a risky bond, given that it has not defaulted, should yield the same return as a newly issued bond of the same type. This is clearly contradictory when applied over the whole life of the instrument.

Solnik (1990) argued that at the time of issue, the spread π is added to the riskfree rate to compensate both for the expected capital loss (CL) due to default and to provide a higher expected return on a risky bond due to risk aversion (RA) and can thus be written as the sum of these two components

$$\pi = \pi^{\mathrm{RA}} + \pi^{\mathrm{CL}} \tag{24}$$

If investors could diversify the default risk, risk premia should be expected to be zero but the premium required due to the possibility of default should remain. For the case of risk neutrality, the expected return on the bond should thus be the riskfree interest rate and $p(t)$ should not appear in the pricing equation

$$E[\mathrm{d}P(r,t,T)] + (r(t)+\pi^{\mathrm{CL}})\,\mathrm{d}t = P(r,t,T)\,r(t)\,\mathrm{d}t \tag{25}$$

If agents were risk-averse and the risk could not be diversified, Ramaswamy and Sundaresan's approach would hold with

$$E[\mathrm{d}P(r,t,T)] + (r(t)+\pi^{\mathrm{RA}}+\pi^{\mathrm{CL}})\,\mathrm{d}t = P(r,t,T)[r(t)+p(t)]\,\mathrm{d}t \tag{26}$$

Similar in spirit is the approach taken by Longstaff and Schwartz (1995b) where the risk-adjusted process for the log of the credit spread is modelled as an Ohrnstein–Uhlenbeck process, and the riskless interest rate is as in Vasicek (1977). The model is then used to value credit derivatives of the type discussed in Section 8 below.

The limitations of these models that make direct assumptions about the spread is that they are not generally derived by writing down the true process that rates follow and then showing how the no-arbitrage condition may be used to switch this into a risk-adjusted form. Instead, a risk-adjusted process for the spread and a risk-adjusted valuation procedure are simply assumed.

This means that the models are not derived from fundamental assumptions about the default process itself and, as such, the link with the pricing of other securities that depend on this process is necessarily complete. Thus, the models may be viewed as extremely partial relative pricing models which could, in principle, be made consistent with a more general representation of the default process. They do, however, have the merit that they can be parameterized fairly directly to match the actual process followed by interest rate credit spreads, thus overcoming a fundamental weakness of many of the other default risk models (see Longstaff and Schwartz 1995a).

6.2. *Markov credit-rating models*

An alternative way of modelling spread behaviour has been pursued by Jarrow *et al.* (1994), and Fons (1994). This involves characterizing ratings changes as Markov transitions between categories. With appropriate assumptions, the true probabilities of the transitions may be transformed into risk-adjusted probabilities and used to value contingent payoffs. The weakness of the approach is, however, that ratings categories do not represent homogeneous discrete groups of bonds. There is considerable variation in the credit quality within a rating group, so that this type of model assumes too much discreteness in the structure of creditworthiness.

7. Default risk of derivative products

7.1. *Introduction*

The massive growth of derivative products has led to growing concern about the impact of default risk on their values and the consequence of any default for the financial system (see US General Accounting Office, 1994). Researchers have developed models to measure and price the default risk of these products. Some of these models simply assume stochastic processes for the variables driving default and do not necessarily attempt to make their models consistent with equilibrium or arbitrage-free pricing. These models are commonly used for Monte-Carlo analysis of the possible range of losses faced by a holder of the security and the way that these combine in portfolios. This literature is extensive and is not summarized here. Excellent examples of the approach can be found in Iben and Brotherton-Ratcliffe (1994) and Giberti *et al.* (1993).

The focus here is on the applications of the arbitrage-free models of default risk to the analysis of derivative products. This literature is concerned with the impact of default risk on the pricing of derivative products and the pricing of derivative products written on prices or rates that reflect default risk. The former is discussed in this section, and the latter in Section 8.

7.2. *Default risk and swaps*

The largest literature on the impact of default risk on the price of derivative products concerns swaps. This is because other derivatives, such as futures and options, are protected against the impact of default by their design, options because they involve only one-way payments at maturity and futures because they are margined. In contrast, forward contracts and swaps are not protected in this way and so the potential impact of default risk is greater. This, combined with the size of the market, makes the swap market a favourite focus of regulators concerned about default risk.

In its most general form, a swap consists of an agreement between two parties to exchange two streams of cash flows. It is the definition of these cash flows that defines the particular swap. In an interest rate swap, streams based on one interest rate index (fixed or floating) are exchanged for streams based on another index in the same currency. In a currency swap, flows based on an index in one currency are exchanged against flows based on an index in another currency. A key feature of swaps is that the payments due on any date in these two-way flows are usually contractually netted against each other in the swap settlement formula before payment is made.

If there is no default risk, a fixed/floating interest rate swap can be characterized as an exchange of two riskless bonds. The transaction is identical to the party agreeing to make fixed payments exchanging a fixed coupon default-free bond (with coupon dates and maturity equal to that of the swap) for a default-free floating rate note (with index, payment dates and maturity identical to that of the swap) issued by the party having agreed to make floating payments (see Bicksler and Chen, 1986).

In practice, the counterparties to swaps are not riskless, so the possibility of default risk affects the pricing of the swap. The price of this default risk depends on the events triggering the default of the swap and the way that the swap is settled in bankruptcy. Conditions under which swap agreements are terminated are set out in the ISDA Master Agreement. Although counterparty default is only one of a number of contingencies triggering early termination in swap agreements, studies of the credit risk of swaps have focused on the loss incurred if a counterparty defaults.

A number of studies (Federal Reserve and Bank of England, 1987; Giberti *et al.*, 1991; Group of Thirty, 1993) used simulation approaches to quantify the default risk of swaps. These studies assumed that the payment due to the solvent party in the event of default is the higher of its default-free market value and zero. Thus, if the market value of the insolvent party, measured as the discounted expected net payments under the contract, is positive, its claim is zero. If the market value is positive, that amount represents the claim of the solvent counterparty. An alternative settlement provision in the swap master provides for the party with a position of positive market value to be compensated by the other party, independently of which party has actually defaulted. In practice, this provision is not frequently used.

Another strand of analysis has sought to embed the valuation of risky swaps in the general literature on the pricing of risky debt. Sundaresan (1991) used the differential required return approach formulated by Ramaswamy and Sundaresan (1986) to take account explicitly of default risk in swaps. The swap is identical to an exchange of a fixed coupon bond for a floating rate note, the default premia on which are modelled as functions of an exogenously specified instantaneous default premium. This allows for the derivation of an equilibrium structure of default premia across different swaps but does not determine the level of the premium.

Babbs (1991) also used the differential required return approach to construct a model for the joint evolution of the default-free term structure and the term structure of swap rates. He assumed a continuous trading environment in which riskless bonds issued by the government and default-risky bonds issued by private agents are traded simultaneously. Agents have access to information summarized by the filtration $\{F_t t \in [0, T]\}$ generated by two Brownian motions $Z_p(t)$ and $Z_g(t)$. He assumed that the term structure dynamics for bonds issued by borrowers of the two risk-classes can be described by the following Ito processes

$$\frac{\mathrm{d}B_g(M,t)}{B_g(M,t)} = (r_g(t) + s(M,t) + \theta_g(t)\sigma_g(M-t,t))\,\mathrm{d}t + \sigma_g(M-t,t)\,\mathrm{d}Z_g(t) \tag{27a}$$

$$\frac{\mathrm{d}B_p(M,t)}{B_p(M,t)} = (r_p(t) + s(M,t) + \theta_p(t)\sigma_p(M-t,t))\,\mathrm{d}t + \sigma_p(M-t,t)\,\mathrm{d}Z_p(t) \tag{27b}$$

where $B_g(M,t)$ and $B_p(M,t)$ denote the prices at time t of pure discount bonds maturing at time $M > t$ issued by 'government' and 'private' sector borrowers, respectively, and θ_p and θ_g relate to the market price of risk for the two types of bonds.

Due to the possibility of default, investors demand a positive yield spread between the bonds issued by risky 'private' borrowers and bonds issued by riskless 'government' borrowers. The spread consists of two components. The first

$$s(M,t) = \int_t^M \beta(m,t)\,\mathrm{d}m \tag{28}$$

reflects the maturity-dependent return differential between riskless and default-risky bonds. The other component of the return differential that does not depend on maturity is reflected in the spread between the instantaneous spot interest rates $r_g(t)$ and $r_p(t)$.

Abken (1993) uses results from option theory to analyse default risk between a swap dealer and a risky counterparty in the framework of Cox *et al.* (1985). Default is triggered if the counterparty is unable to meet payments on the swap. Abken proposed the decomposition of a given swap into two yield options. The position of the fixed-rate payer is equivalent to a long position in a yield call on the reference yield with strike rate equal to the fixed swap rate. The position of the floating rate payer is identical to a long position in a yield put, written by the fixed-rate payer, on the reference yield with strike rate equal to the fixed swap rate. This decomposition into yield options allows the two-sided default risk inherent in swaps to be separated into one-sided risk borne by two separate instruments. Abken uses the yield option model derived by Longstaff (1990) to value the swap as the difference between a series of caps and floors on the reference yield of the swap. Equilibrium swap rates in Abken's model are identified through an iterative search that equates the value of the sum of the yield calls and the sum of the yield puts using Monte Carlo simulation.

Other somewhat *ad hoc* models of swap default risk are given in Hull (1989), Sorensen and Bollier (1994) and Solnik (1990). They assumed that the valuation of the default risk on the swap can be separated between a probability of default and a risk-adjusted expected loss conditional on default. They then valued the expected loss using standard option models in a way similar to Abken. This independence is questioned by Usmen (1993), who used a state pricing contingent claims model. Unfortunately, however, he assumed risk neutrality, which limits the interest of the model. Similarly Duffee (1995a), questioned the independence assumption.

The first Merton-style equilibrium pricing model for the credit risk of swaps is Cooper and Mello (1991). They used the model to analyse equilibrium swap rates and compared them with the rates on the debt of the counterparties to the swap. Another example of the use of the Merton model in this way can be found in Rendleman (1992). Cooper and Mello analysed a swap between a riskless and a risky counterparty and showed the relationship between the credit spread that must be paid in the swap and the credit spreads that the risky borrower would pay on the two debt instruments that represent the two legs of the swap.

They analysed both currency and interest rate swaps. In the case of interest rate swaps, they considered a firm with either variable or fixed rate debt that can enter into an interest rate swap. They assumed that the value of the assets of the risky firm follows the stochastic differential equation

$$\mathrm{d}V = (\alpha V - C)\,\mathrm{d}t + \sigma_V V\,\mathrm{d}z_V \tag{29}$$

where α is the instantaneous expected rate of return on the firm and C represents the total dollar payout by the firm per unit time. It is further assumed that the term structure of interest rates is fully specified by the instantaneous riskless rate r, the dynamics of which are given by

$$\mathrm{d}r = m(\mu - r)\,\mathrm{d}t + \sigma_r r\,\mathrm{d}z_r \quad m > 0,\ \mu > 0,\ \mathrm{d}z_r\,\mathrm{d}z_V = \rho_{Vr}\,\mathrm{d}t \tag{30}$$

Under the assumption that the Local Expectation Hypothesis holds, the market price for interest rate risk is zero and the price of a corporate bond paying semi-annual coupons must satisfy the following equation

$$B_V(rV - C) + B_r m(\mu - r) + \tfrac{1}{2}(B_{VV}\sigma_V^2 V^2 + B_{rr}\sigma_r^2 + 2B_{Vr}Vr\sigma_V\sigma_r\rho_{Vr}) + C - rB = -B_t \tag{31}$$

It is assumed that a coupon s is paid semi-annually and that the firm will default on the coupon if the value of the firm at any coupon date falls below the coupon payment. If this occurs, the value of the firm's assets is paid to the debtholders and the firm is liquidated.

The spread for risky variable bonds is defined by the fixed mark-up to the coupon rate that reflects the credit risk of a particular bond. This spread S_X is defined by $B(V, r, 0, (r + S_X)F) = F$. The spread for the default risk of fixed coupon bonds is given by the coupon rate differential S_F such that $B(V, r, 0, (g + S_F)F) = F$. The swap spread is defined as the fixed coupon that must be added to the default free fixed rate offered in exchange for the default free variable rate r. This is solved numerically using the no-arbitrage condition between the swap and the risky and riskless debt.

The swap spread is composed of the difference between the equilibrium spreads in the fixed and variable debt markets plus a specific swap spread. The specific swap spread is added to the debt market spread differential to compensate the riskless counterparty for the fact that the swap is not exactly equivalent to an exchange of the two types of risky bond. The size of this spread is shown to be very sensitive to the assumed priority rules and settlement procedures for the swap.

The default intensity approach has been applied to risky swaps by Duffie and Huang (1995). The model is an extension of Duffie and Singleton (1995). The model is capable of valuing a swap between two risky counterparties and can encompass a variety of assumptions about the way that swap default is triggered and settled. The model appears to explain quite well the actual structure of swap prices.

7.3. *Options and other derivatives*

Various default risk models have been applied to value the impact of credit risk on the pricing of options. Johnson and Stulz (1987), and Hull and White (1995), use extended Merton models with two state-variables: the price on which the option is written and the value of the underlying assets of the firm writing the option. Johnson and Stulz solve their model analytically. Hull and White show how their model can be solved numerically, and the impact of default risk on the value of 'vulnerable' options written by risky counterparties.

Litterman and Iben (1991), fit market yield spreads with a probabilistic model of conditional default. They then use the model to value the embedded call option in callable debt issued by a risk firm, combining a logarithmic interest rate model with a maturity-dependent conditional default probability. While this model has a structure similar to a default intensity model, it is not completely specified as such by the authors.

8. Credit derivatives

In response to concerns about default risk and changes in the regulation of investments subject to credit risk, a series of products have been recently innovated that are derivative claims on variables that reflect default risk. (See Arak, 1992, for a description of regulatory changes.) Among these 'credit derivatives' are forward contracts on credit spreads, swaps of the total return on credit-risky instruments for default-free returns, credit risk insurance products, and options on credit spreads. (See Smith, 1993; Flesaker *et al.*, 1994; Smithson, 1995; and Van Duyn, 1995, for descriptions.) The proposed uses of these products include the control of credit risk, the separation of credit decisions from other portfolio decisions, and improving the liquidity of a portfolio.

Because of their newness, the mathematical modelling of these innovations is in its infancy. Flesaker *et al.* presented a default intensity model with a stochastic interest rate. Jarrow and Turnbull (1995) gave a more general version of this type of model. Longstaff and Schwartz (1995a) presented a model based on a direct assumption about the behaviour of yields. It is too early to say whether any of these models can capture the empirical features of the pricing of these products. It remains a significant challenge to do so within a general model that can price all securities that are subject to default risk.

9. Conclusions

Significant progress has been made in modelling credit risk since the pioneering work of Black and Scholes and Merton. The early models based on underlying asset value are now giving way to models based upon more direct assumptions about the default process. These models can be used to value derivative products that are affected by default risk simultaneously with the default risk itself. Most importantly, they can be parameterized to fit the current structure of prices of risky bonds. Considerable challenges remain, however. The numerical implementation of these models often requires an independence assumption. Either risk-free interest rates are assumed to be independent of the process driving default or the process driving the incidence of default is independent of the write-down in default. Both these assumptions are unsatisfactory.

Another concern is the inability of the models to explain the time-series behaviour of credit spreads or the relative levels of spreads in different parts of the market (see Cooper and Mello, 1988). Many of the models cannot generate sufficient time-series variability in the spread to match actual rates. The behaviour of actual spreads is very complex and, as yet, no model adequately captures this complexity (see Brown *et al.*, 1994). Unless a model can do this, it will not be useful in determining the relative prices of the new credit derivatives, some of which are extremely sensitive to the time-series properties of the spread.

In addition, there are other credit derivatives that cannot be directly modelled as claims on the default of an individual firm. For instance, claims that make payments related to the value of an index of firms with a given credit quality are essentially claims on an ever-changing portfolio. This raises the whole question of how the impact of default risk on portfolios should be modelled, when including a default variable for each security in the portfolio would soon explode the dimensionality of the problem beyond reasonable bounds. (See Rowe, 1993; Lucas, 1995b, for a discussion of portfolio issues.)

The other development that needs modelling is the innovation of design features. Concern about potential default risk in swaps and other derivative products has led to innovations in design that are designed to limit this risk. The netting of cash flows in most swaps can be viewed in this way, and the margining and marking-to-market of futures contracts performs a similar role. Innovations that have been used in the swap market to limit credit risk are marking-to-market and margining, the use of forward swaps, conducting transactions through special AAA subsidiaries, and downgrade provisions (see Brown and Smith, 1993; Lucas, 1995a, and Wilson, 1994).

Another important unresolved issue is the continuing paradox of using no-arbitrage models in which derivative products are redundant securities to model their prices. This raises the whole question of why such products exist (see Turnbull, 1987; Arak *et al.*, 1988; Wall, 1989; Wall and Pringle, 1989; Campbell and Kracaw, 1991; Litzenberger, 1992; Titman, 1992; Grinblatt, 1994). Some of the explanations for the existence of derivative products rely on market incompleteness, which suggests that incomplete-market models such as Artzner and Delbaen (1994), may be necessary to understand fully their pricing.

Despite these unresolved issues, there appears to be a growing trend in the literature on default risk modelling towards the use of default intensity models in parallel with stochastic models of the default-free term structure. This is the approach followed by Madan and Unall (1992), Duffie *et al.* (1995) and Jarrow and Turnbull (1995). These relatively general models have similar structure. The challenge is to find the particular form of this general approach that comes closest to fitting the actual behaviour of prices and rates that are exposed to credit risk.

References

Abken, P.A. (1993) Valuation of default-risky interest rate swaps, *Advances in Futures and Options Research*, **6**, 93–116.

Arak, M., Estrella, A., Goodman, L. and Silver, A. (1988) Interest rate swaps: an alternative explanation, *Financial Management*, **17**, 12–18.

Arak, M. (1992) The effect of the new risk-based capital requirements on the market for swaps, *J. Financial Services Res.*, **6**, 25–36.

Artzner, P. and Delbaen, F. (1994) Default risk insurance and incomplete markets, *Mathematical Finance*, **5**, 187–95.

Babbs, S.H. (1991) Interest rate swaps and default-free bonds: a joint term structure model, *FORC Preprint* 91/27, University of Warwick.

Barrett, J., Moore, G. and Wilmott, P. (1992) Inelegant efficiency, *Risk*, **5.9**, 82–84.

Bicksler, J. and Chen, A.H. (1986) An economic analysis of interest rate swaps, *J. Finance*, **41**, 645–56.

Black, F. and Cox, J.C. (1976) Valuing corporate securities: some effects of bond indenture provisions, *J. Finance*, **31**, 351–67.

Black, F. and Scholes, M. (1973) The pricing of options and corporate liabilities, *J. Political Economy*, **81**, 637–59.

Boyle, P.P. (1977) Options: a Monte Carlo approach, *J. Financial Economics*, **4**, 3223–38.

Brealey, R.A. and Myers, S.C. (1991) *Principles of Corporate Finance*, McGraw-Hill.

Brennan, M.J. and Schwartz, E.S. (1977) Convertible bonds: valuation and optimal strategies for call and conversion, *J. Finance*, **32**, 1699–715.

Brennan, M.J. and Schwartz, E.S. (1980) Analysing convertible bonds, *J. Financial and Quantitative Analysis*, **15**, 907–29.

Brown, K.C. and Smith, D.J. (1993) Default risk and innovations in the design of interest rate swaps, *Financial Management*, **22**, 94–105.

Brown, K.C., Harlow, W.V. and Smith, D.J. (1994) An empirical analysis of interest rate swap spreads. *J. Fixed Income*, March, 61–78.

Buoncuore, A., Nobile, A.G. and Ricciardi, L.M. (1987) A new integral equation for the evaluation of first-passage time probability densities, *Advances in Applied Probability*, **19**, 784–800.

Campbell, T.S. and Kracaw, W.A. (1991) Intermediation and the market for interest rate swaps, *J. Financial Intermediation*, **1**, 362–84.

Cooper, I.A. and Mello, A.S. (1988) Default spreads in the fixed and in the floating interest rate markets: a contingent claims approach, *Advances in Futures and Options Res.*, **3**, 269–89.

Cooper, I.A. and Mello, A.S. (1991) The default risk of swaps, *J. Finance*, **46**, 597–620.

Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1980) An analysis of variable rate loan contracts, *J. Finance*, **15**, 389–403.

Cox, J.C., Ingersoll, J.E. and Ross, S.A. (1985) A theory of the term structure of interest rates, *Econometrica*, **53**, 385–407.

Cox, J.C., Ross, S.A. and Rubinstein, M. (1979) Option pricing: a simplified approach, *J. Financial Economics*, **7**, 229–263.

Duffee, G.R. (1995) On measuring credit risks of derivative instruments. Working Paper.

Duffie, D. and Singleton, K. (1995) Econometric modelling of the term structure of defaultable bonds. Working Paper, Graduate School of Business, Stanford University.

Duffie, D., Schroder, M. and Skiadas, C. (1995) Recursive valuation of defaultable securities and the timing of the resolution of uncertainty. Working Paper, Graduate School of Business, Stanford University.

Federal Reserve Board and Bank of England (1987) Potential credit exposure on interest rate and foreign exchange related instruments.

Flesaker, B., Hughston, L., Schreiber, L., and Sprung, L. (1994) Taking all the credit, *Risk*, **7.9**, 104–108.

Fons, J.S. (1994) Using default rates to model the term structure of credit risk, *Financial Analysts J.*, September–October, 25–32.

Franks, J.R. and Torous, W.N. (1994) A comparison of financial recontracting in distressed exchanges and chapter 11 reorganizations, *J. Financial Economics*, **35**, 349–70.

Geske, R. (1977) The valuation of corporate liabilities as compound options, *Journal of Financial and Quantitative Analysis*, **12**, 541–52.

Geske, R. and Shastri, K. (1985) Valuation by approximation: a comparison of alternative option valuation techniques, *J. Financial and Quantitative Analysis*, **20**, 45–71.

Giberti, D., Mentini, M. and Scabellone, P. (1993) The valuation of credit risk in swaps: methodological issues and empirical results. Working Paper, Centre for Research in Finance-IMI Group.

Grinblatt, M. (1994) An analytic solution for interest rate swaps. Working Paper, Anderson Graduate School of Management, UCLA.

Group of Thirty (1993) Working Paper of the Credit Risk Measurement and Management Subcommittee.

Harrison, H.J. and Kreps, D.M. (1979) Martingales and arbitrage in multiperiod securities markets, *J. Economic Theory*, **20**, 381–408.

Heath, D., Jarrow, R.J. and Morton, A. (1992) Bond pricing and the term structure of interest rates: a new methodology for contingent claims valuation, *Econometrica*, **60**, 77–105.

Ho, T. and Singer, R.F. (1984) The value of corporate debt with a sinking-fund provision, *J. Business*, **57**, 315–36.

Ho, T.S.Y. and Lee, S. (1986) Term structure movements and pricing interest rate contingent claims, *J. Finance*, **41**, 1011–29.

Hull, J. (1989) Assessing credit risk in a financial institution's off-balance sheet commitments. *J. Financial and Quantitative Analysis*, **24**, 489–501.

Hull, J. and White, A. (1995) The impact of default risk on the prices of options and other derivative securities. *J. Banking and Finance*, **19**, 299–322.

Iben, B. and Brotherton-Ratcliffe, R. (1994) Credit loss distributions and required capital for derivatives portfolios, *J. Fixed Income*, **3**, 6–14.

Ingersoll, J.E. (1987) *Theory of Financial Decision Making*, Rowman and Littlefield.

Jamshidian, F. (1994) Hedging quantos, diff-swaps and ratios, *Applied Mathematical Finance*, **1**, 1–20.

Jarrow, R.A. and Turnbull, S.M. (1995) Pricing derivatives on financial securities subject to credit risk, *J. Finance*, **50**, 53–85.

Jarrow, R.A., Lando, D. and Turnbull, S.M. (1994) A Markov model for the term structure of credit risk spreads. Working Paper.

Johnson, H. and Stulz, R. (1987) The pricing of options with default risk, *J. Finance*, **48**, 267–80.

Jones, E.P., Mason, S.P. and Rosenfeld, E. (1984) Contingent claims analysis of corporate capital structures: an empirical investigation, *J. Finance*, **39**, 611–27.

Kau, J.B. and Keenan, D.C. (1995) An overview of the option-theoretic pricing of mortgages, *J. Housing Res.*, **6**, 217–44.

Lando, D. (1994) Three essays in contingent claims pricing. PhD Thesis, Cornell University.

Litzenberger, R.H. (1992) Swaps: plain and fanciful, *J. Finance*, **47**, 831–50.

Litterman, R. and Iben, T. (1988) Corporate bond valuation and the term structure of credit spreads, *Financial Analysts' J.*, Spring 1991, 52–64.

Longstaff, F.A. (1991) The valuation of options on yields, *J. Financial Economics*, **26**, 97–121.

Longstaff, F.A. and Schwartz, E.S. (1995a) A simple approach to valuing risky and floating rate debt, *J. Finance*, **50**, 789–819.

Longstaff, F.A. and Schwartz, E.S. (1995b) Valuing credit derivatives, *J. Fixed Income*, **5.1**, 6–12.

Lucas, D.J. (1995a) The effectiveness of downgrade provisions in reducing counterparty risk, *J. Fixed Income*, **4**, 32–41.

Lucas, D.J. (1995b) Default correlation and credit analysis, *J. Fixed Income*, **4**, 76–87.

Madan, D.B. and Unal, H. (1992) Pricing the risks of default. Working Paper, University of Maryland.

Mason, S.P. and Bhattacharya, S. (1981) Risky debt, jump processes, and safety covenants, *J. Financial Economics*, **9**, 281–307.

Merton, R.C. (1974) On the pricing of corporate debt: the risk structure of interest rates, *J. Finance*, **29**.

Myers, S.C. (1977) Determinants of corporate borrowing, *Journal of Financial Economics*, **5**, 146–75.

Paskov, S. and Traub, J. (1995) Faster valuation of financial derivatives. Working Paper, Department of Computer Science, Columbia University.

Pitts, C. and Selby, M. (1983) The pricing of corporate debt: a further note, *J. Finance*, **38**, 1311–13.

Ramaswamy, K. and Sundaresan, S.M. (1986) Valuation of floating rate instruments, *J. Financial Economics*, **17**, 251–272.

Rendleman, R.J. (1992) How risks are shared in interest rate swaps, *J. Financial Services Res.*, 7, 5–34.

Rowe, D. (1993) Curves of confidence, *Risk* **6.11**, 52–55.

Selby, M.J.P., Franks, J.R. and Karki, J.P. (1988) Loan guarantees, wealth transfers and incentives to invest, *J. Industrial Economics*, **37**, 47–65.

Shimko, D.C., Tejima, N. and Van Deventer, D.R. (1993) The pricing of risky debt when interest rates are stochastic, *J. Fixed Income*, **3**, 58–65.

Smith, T. (1993) The new credit derivatives, *Global Finance*, **7**, 109–10.

Smithson, C. (1995) Credit derivatives, *Risk*, 8.12.

Solnik, B. (1990) Swap pricing and default risk: a note, *J. International Financial Management and Accounting*, **2**, 79–91.

Sorensen, E.H. and Bollier, T.F. (1994) Pricing swap default risk, *Financial Analysts' J.*, May–June, 23–33.

Sundaresan, S.M. (1991) Valuation of swaps. In Khoury, S. (ed.) *Recent Developments in Banking and Finance*, (Vols. IV and V) Amsterdam: North Holland.

Titman, S. (1992) Interest rate swaps and corporate financing choices, *J. Finance*, **47**, 1503–16.

Titman, S. and Torous, W. (1989) Valuing commercial mortgages: an empirical investigation of the contingent-claims approach to pricing risky debt, *J. Finance*, **44**, 345–73.

Turnbull, S.M. (1987) Swaps: a zero sum game? *Financial Management*, **16**, 15–21.

US General Accounting Office (1994) *Report to the US Congress: Financial Derivatives, Actions Needed to Protect the Financial System.*

Usmen, N. (1993) Currency swaps, financial arbitrage and default risk. Working Paper, Rutgers University.

Vasicek, O. (1977) An equilibrium characterization of the term structure. *J. Financial Economics*, **5**, 177–88.

Van Duyn, A. (1995) Credit risk for sale. Any buyers? *Euromoney*, April, 41–43.

Wall, L.D. (1989) Interest rate swaps in an agency theoretic model with uncertain interest rates, *J. Banking and Finance*, **18**, 261–270.

Wall, L.D. and Pringle, J.J. (1989) Alternative explanations of interest rate swaps: a theoretical and empirical analysis. *Financial Management*, **18**, 59–73.

Warner, J.B. (1977) Bankruptcy costs: some evidence. *J. Finance*, **32**(2), 337–48.

Weinstein, M.I. (1983) Bond systematic risk and the option pricing model, *J. Finance*, **38**, 1415–29.

Wilson, D. (1994) In good company, *Risk*, **7.8**, 35–39.

Yawitz, J.B., Maloney, K.J. and Ederington, L.H. (1985) Taxes, default risk and yield spreads, *J. Finance*, **50**(4), 1127–40.