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Arithmetic versus geometric mean estimators: Setting discount rates for capital budgeting

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Abstract

This paper addresses an issue central to the estimation of discount rates for capital budgeting: should the geometric mean or arithmetic mean of past data be used when estimating the discount rate? The use of the arithmetic mean ignores estimation error and serial correlation in returns. Unbiased discount factors have been derived that correct for both these effects. In all cases, the corrected discount rates are closer to the arithmetic than the geometric mean.

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JEL classification: *G*120; *G*310.

1. Introduction

In estimating the cost of capital using the capital asset pricing model (CAPM), the expected risk premium on the market plays a key role. This estimate is often obtained by the analysis of historical returns on an equity market index. There are two standard statistics used as the basis of this estimate: the arithmetic mean of historical returns or risk premia and the geometric mean. To illustrate the difference between these statistics, for the US during the period 1926–1992 the arithmetic mean real return on the equity market was 9.0%, whereas the geometric mean was 7.0% [SBBI (1993)]. For the UK in the period 1919–1994 the arithmetic real return was 10.3% whereas the geometric mean was 7.7% [BZW (1995)].

Standard references on estimating the expected return on the market differ in their advocacy of the arithmetic or geometric mean as the basis of discount rates for capital budgeting. Among the advocates of the arithmetic mean are Bodie *et al.* (1989), Brealey and Myers (1991a, b), Franks *et al.* (1985), Kolbe *et al.* (1984)

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and Ross and Westerfield (1988). Advocacy of the geometric mean is found in Copeland *et al.* (1991) and Levy and Sarnat (1986). The large difference between the two statistics means that the choice of one or the other may have a dramatic effect on the valuation of any asset with other than a short life.

A reason for favouring the arithmetic mean is given in Kolbe *et al.* (1984):

> Note that the *arithmetic* mean, *not* the geometric mean, is the relevant value for this purpose. The quantity desired is the rate of return that investors expect over the next year for the random annual rate of return on the market. The arithmetic mean, or simple average, is the unbiased measure of the expected value of repeated observations of a random variable, not the geometric mean. ... the geometric mean underestimates the expected annual rate of return.

The key point of this argument is that unbiasedness is considered to be the relevant criterion. Butler and Schachter (1989) show, however, that care must be taken to decide what unbiasedness criterion one uses. In particular, even though the arithmetic average of annual returns may be an unbiased estimate of the expected return over the next year, it is not an unbiased estimate of the expected return over periods greater than one year or of the discount factor, which is the reciprocal of the expected return.

Apart from capital budgeting, the estimate of the market risk-premium is central to the regulation of privatised utilities. Allowable returns are set using the CAPM in various countries and the choice of the market risk-premium has a significant effect. For instance, the ultimate arbiter of regulated utilities in the UK, the Monopolies and Mergers Commission, is confused about the choice of arithmetic or geometric averages [MMC (1995)]:

> Under the CAPM approach to assessing the cost of capital, the WACC depends on ... the premium required by equity holders to compensate for risk. ... Estimates of these factors cannot be precise, depending as they do on the period over which returns are calculated, whether average or geometric returns are calculated.

Thus major regulatory decisions are taken in the UK on the basis that arithmetic and geometric means of past returns have similar merit in setting expected future returns.

The purpose of this paper is to derive unbiased estimates of discount factors for use in capital budgeting. The paper is organised as follows. Section 2 shows the conditions for the arithmetic mean of past returns or risk premia to be the correct estimator for use in discounting. Section 3 explains the problems with this argument. Section 4 derives unbiased estimators for discount factors when returns are not serially correlated. Numerical examples are given in Section 5. In Section 6 the implications of the 'excess volatility' literature are discussed, and Section 7 presents the conclusions of the paper.

2. Basic theory

In capital budgeting expected future cash flows are discounted. These expected cash flows are the arithmetic means of possible cash flow outcomes. The capital asset pricing model is also formally stated in terms of arithmetic expected returns over an unspecified investment horizon. A typical use of the CAPM assumes that the expected return over one year is estimated. If the cash flow to be received occurs after N years, then this expected return is compounded over N years to give the reciprocal of the discount factor.

Thus, if the cash flow to be received after N years is X, then the present value is typically estimated as:

$$V = E(X)m^{-N} \tag{1}$$

where $E(.)$ is the expectations operator and m is an estimate of the one year expected return appropriate to the risk of the investment.

The theory that motivates this is the following. Suppose that the return over the next year will be R_{T+1}, with a known arithmetic expectation of $E(R_{T+1}) = M$. Suppose now that we need the expected value of the future N-year return: $E(R_{T+1}R_{T+2} \ldots R_{T+N})$. If each return is independent with the same mean, this is equal to M^N. The expected return on the investment project is $(E(X)/V)$ and the expected return on an equivalent capital market investment is M^N. So the correct value of the project is $V = E(X)M^{-N}$ where M is the true arithmetic mean one-period return. Thus, if M is known the normal discounting formula correctly compounds M to give the n-period arithmetic expected return for use in discounting.

This argument is based on three assumptions:

A. The arithmetic expected return each period is constant.
B. Returns are serially independent.
C. The expected return is known.

The first of these assumptions can be modified to allow for changing interest rates and stated in the form of a constant risk premium. The second is based loosely on market efficiency, and corresponds to the assumption that the market is weak-form efficient with a constant expected return or risk premium. The third is untrue in most applications as the expected return is estimated with error. In general, the true mean (M) of the distribution of R is not known, and an estimate (m) is used based upon a statistic such as the arithmetic or geometric mean of past returns.

3. Problems with estimates of the expected return

To investigate the properties of the arithmetic mean and geometric mean of past returns as estimators of the discount rate, we assume that annual total return wealth relatives on the index over the past T years of $R_1, R_2, \ldots R_T$ have been observed. Then the arithmetic mean return is defined as:

$$A = \sum_i R_i / T \tag{2}$$

and the geometric mean is defined as:

$$G = \left[\prod_i R_u \right]^{1/T} \tag{3}$$

For instance, the arithmetic mean real rate of return on US equities in the period 1926–1992 was 9.0%. The geometric mean real rate of return for the same period was 7.0%.[1] A similar difference arises if one uses arithmetic or geometric average estimates of the risk premium.

There are three possible problems with the use of the arithmetic mean or geometric mean as estimates of the true expected return. These correspond to the three assumptions listed in Section 2 above. The first is that the expected return or the expected risk-premium may not be constant. The second problem is that returns in successive periods may not be independent. The third problem is that the true mean of the returns is not known. Instead an estimate based on either the geometric or the arithmetic mean of past returns is used instead of the true mean. The first two of these problems are discussed in Section 6 below. For the rest of this section and Sections 4 and 5 we assume that the expected real return is constant and returns are independent.[2]

Even if returns are serially independent, the arithmetic mean, A, is only an estimate of the true mean return, M. Any estimation error is, therefore compounded when the transform A^{-N} is used as an estimate of M^{-N}. To see this, assume that the cash flow to be discounted has expectation $E(X)$ and a beta of unity. Then the correct present value of the cash flow is $M^{-N}E(X)$. Suppose that we estimate the discount factor M^{-N} with some estimator M_N. For an unbiased estimate of the value we need:

$$E(M_N)=M^{-N} \tag{4}$$

Note that neither of the estimators A^{-N} or G^{-N} has this property. Blume (1974) shows that the arithmetic mean, A, is an unbiased estimate of the true expectation M, and the compounded geometric mean G^T is an unbiased estimate of the compounded wealth relative M^T. Thus A^{-N} and G^{-N}, which are non-linear functions of A and G^T respectively, are biased estimates of M^{-N}.

The direction of the biases in A^{-N} and G^{-N} as estimators of M^{-N} can be seen from the convexity of the functions A^{-N} and $(G^T)^{-N/T}$. Using Jensen's inequality:

$$E(A^{-N})>[E(A)]^{-N}=M^{-N} \qquad N>1 \tag{5}$$

Similarly:

$$E(G^{-N})=E[(G^T)^{-N/T}]>[E(G^T)]^{-N/T}=(M^T)^{-N/T}=M^{-N} \qquad T>N \tag{6}$$

Thus both A^{-N} and G^{-N} are upward biased estimators of the correct discount factor M^{-N}. As a consequence, both the arithmetic mean A and the geometric mean G are downward biased estimators of the discount rate that should be used to discount cash flows with a beta of unity. As the geometric mean is always below the arithmetic mean, it is always the more biased of the two estimates.

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[1]Throughout the paper the word 'return' is used to refer to a wealth relative, so that a rate of return of 10% corresponds to a 'return' or wealth relative of 1.1.

[2]Although the assumption of a constant expected real return is not equivalent to the assumption of a constant risk-premium, the statistical arguments are essentially the same in both cases. Inclusion of a time-varying real interest rate would, therefore, add complexity to the argument without affecting the substance of the question of whether arithmetic or geometric averaging of past data (returns of risk-premiums) is the appropriate estimation procedure.

4. Unbiased estimation of discount factors with constant mean returns

This estimation error problem is partially addressed by Blume (1974). He provides a way of using the geometric mean and the arithmetic mean of past returns to form an unbiased estimate of the expected return over any future period. The procedure he proposes is that the expected return over a horizon of N periods should be formed by a weighted average of the compounded geometric and arithmetic means G^N and A^N. He shows that this is an approximately unbiased estimate of M^N, the true expected return over N periods. Note, however, that it will not provide an unbiased estimate of M^{-N}, the discount factor which is a non-linear function of the expected return, M_N.

Blume proposes two alternative nearly unbiased estimators of the N-period expected return, the 'weighted unbiased' and the 'adjusted unbiased'. He prefers the former, which is defined as:

$$M_{NB} = aA^N + (1-a)G^N \tag{7}$$

where $a = (T-N)/(T-1)$ which forms a weighted average of A^N and G^N. When $N=1$, all the weight is on the arithmetic mean. When $N=T$, all the weight is on the geometric mean. As N drops from T to one, more and more weight is given to the arithmetic and less and less weight is given to the geometric mean. Thus, the arithmetic mean is an unbiased estimate of the short-term expected return and the compounded geometric mean an unbiased estimate of the long-term expected return. This is reasonable as one may think of the compounded geometric mean as simply the arithmetic mean over the period of length T.

This procedure leads to an approximately unbiased estimate of the expected return over N periods M^N. In capital budgeting, however, we need an unbiased estimator of the discount factor M^{-N}. The Appendix demonstrates, using analysis similar to Blume, that an approximately unbiased estimator of M^{-N} is given by:

$$\hat{D}_{N1} = bA^{-N} + (1-b)G^{-N} \tag{8}$$

where $b = (N+T)/(T-1)$.

Note that, whereas, the Blume estimator of the expected return given by (7) lies between A^N and G^N, the unbiased estimator of the discount factor lies outside the range of A^{-N} and G^{-N}. Note also that, when $N=T$, the Blume estimator is simply the compound geometric mean G^N, whereas the estimator $\hat{D}_{N1}$ is approximately $(2A^{-N} - G^{-N})$.

The estimation problem is simplified if we are prepared to specify the distribution of rates of return. If we assume that the distribution of returns is lognormal then:

$$\ln R_t \sim N(\mu, s^2) \tag{9}$$

The expected return is given by:

$$E(R_t) = \exp(\mu + s^2/2) = M \tag{10}$$

The 'true' discount factor for N periods is given by:

$$M^{-N} = \exp(-N\mu - Ns^2/2) \tag{11}$$

If the variance is constant we can get an arbitrarily good estimate of s^2 from a finite time series by chopping into arbitrarily fine intervals, so we assume henceforth that s^2 is known.

The logarithm of the geometric mean, G, is distributed:

$$\ln G \sim N(\mu, s^2/T) \tag{12}$$

Thus:

$$\ln(G^{-N}) \sim N(-N\mu, N^2 s^2/T) \tag{13}$$

So:

$$E(G^{-N}) = M^{-N} \exp[(T+N)Ns^2/2T] \tag{14}$$

Thus an unbiased estimate of M^{-N} based on the geometric mean is given by:[3]

$$\hat{D}_{N2} = G^{-N} \exp[-(T+N)Ns^2/2T] \tag{15}$$

Alternatively, if the empirical distribution on which A and G are based is lognormal, then:

$$A = G \exp(s^2/2) \tag{16}$$

$$A^{-N} = G^{-N} \exp(-Ns^2/2) \tag{17}$$

So we can form an unbiased estimator based on the arithmetic mean by:

$$\hat{D}_{N3} = A^{-N} \exp[-N^2 s^2/2T] \tag{18}$$

We have now derived three unbiased estimators of the discount factor for N periods, M^{-N}. The first $\hat{D}_{N1}$ is a function of both the arithmetic and geometric means and the other two, $\hat{D}_{N2}$ and $\hat{D}_{N3}$ are functions of the geometric and arithmetic means respectively.

The properties of these three estimators are examined in the next section.

5. Numerical examples

Numerical examples of the three estimators are given in Table 1 for values of G, A and s^2 computed for real returns on the returns to the US stock market over the period 1926–1992.[4] The results for the UK would be very similar, as the returns series have similar properties. Indeed, the qualitative results would vertically identical.

Tables 1 and 2 show, for various horizons N, the unbiased discount factors converted to rates of return (Panel A) and annuity rates (Panel B). The three unbiased estimators are very similar and are, in all cases, much closer to the arithmetic mean return of 9.0% than the geometric mean of 7.0%. Indeed, for horizons of up to ten years, the unbiased discount rates are within 0.4% of the arithmetic mean and the annuity factors are within 0.6% of the arithmetic mean for all periods up to 30 years. Thus, although the arithmetic mean is biased, the bias is small for most practical applications and correcting this bias moves the estimator further away from the geometric mean.

[3]This cause of bias was first pointed out by Butler and Schachter (1989). They propose a correction to the estimate of the discount rate based upon a Taylor series expansion.
[4]The source of the returns is the SBBI (1993) Yearbook.

6. Serially correlated returns

An assumption of the above analysis is the serial independence of returns. A series of recent papers finds, however, that returns in equity markets are not serially independent. This is interpreted by some as meaning that markets are not efficient, and by others as meaning that the risk premium varies over time. The evidence on this point is not conclusive.[5]

If this evidence is interpreted as demonstrating time variation in risk-premiums it raises very complex questions about cost of capital estimation, which

Table 1

Approximately unbiased estimates of real discount rates for a unit beta investment assuming constant mean real returns.

N is the horizon in years. $\hat{D}_{N1}$ is an estimate of the discount factor based on a weighted average of G^{-N} and A^{-N} given by equation (8) in the text. G is the annual geometric mean real return over the period 1926–1992 and is equal to 1.0698. A is the annual arithmetic mean real return over the same period and is equal to 1.0904. $\hat{d}_{N1}$ is the annual discount rate equivalent to $\hat{D}_{N1}$. $\hat{D}_{N2}$ is an estimate of the discount factor based on a lognormal distribution given by equation (15) in the text. $\hat{d}_{N2}$ is the annual discount rate equivalent to $\hat{D}_{N2}$. $\hat{D}_{N3}$ is similarly given by (18) and $\hat{d}_{N3}$ is the corresponding discount rate. The estimated standard deviation of the annual log real return is 0.1991, based on annual data

Panel A Discount rates.

N	$\hat{d}_{N1}$ (%)	$\hat{d}_{N2}$ (%)	$\hat{d}_{N3}$ (%)
1	9.1	9.2	9.1
2	9.1	9.2	9.1
3	9.2	9.2	9.1
4	9.2	9.2	9.2
5	9.2	9.3	9.2
10	9.4	9.4	9.3
15	9.6	9.6	9.5
20	9.9	9.7	9.7
25	10.2	9.9	9.8
30	10.6	10.0	10.0

Panel B Real annuity rates equivalent to the discount rates in Panel A ($\hat{a}_{Ni}$ is the annuity rate corresponding to the unbiased annuity factor given by $\hat{D}_{Ni}$.

N	$\hat{a}_{N1}$ (%)	$\hat{a}_{N2}$ (%)	$\hat{a}_{N3}$ (%)
5	9.2	9.2	9.2
10	9.3	9.3	9.2
15	9.4	9.4	9.3
20	9.5	9.5	9.4
25	9.6	9.5	9.4
30	9.7	9.6	9.5

[5] A comprehensive discussion can be found in Kleidon (1986).

Table 2

Approximately unbiased estimates of discount rates for the index assuming returns are independent over different intervals.

The variance ratios are from Poterba and Summers (1988) Table 4. $\hat{D}_{N4}$ is the estimator of the N period discount factor given by (19), and $\hat{d}_{N4}$ is the corresponding discount rate. The figures along the top are the differencing intervals applied to the data to compute the variance ratios.

Interval (years)	1	2	3	4	5	6	7	8
Variance ratio	1.000	0.814	0.653	0.656	0.696	0.804	0.803	0.800
$\hat{d}_{1,4}$ (%)	9.2	8.8	8.4	8.4	8.5	8.7	8.7	8.7
$\hat{d}_{10,4}$ (%)	9.5	9.0	8.6	8.6	8.7	9.0	9.0	9.0
$\hat{d}_{30,4}$ (%)	10.1	9.5	9.0	9.0	9.2	9.5	9.5	9.5

have not been addressed in the literature. Required rates of return at any date would have to be estimated conditional on the set of variables that predict the risk-premium using the appropriate model. Similar issues to those addressed in this paper would then arise in terms of using the estimated parameters of the model to form estimates of the discount factors.

If, on the other hand, serial correlation is caused by transient disequilibrium we can see the likely size of the effect on discount rates if we maintain the assumption that returns are lognormally distributed, but allow for the fact that the variance of the distribution of returns changes as the horizon alters. Suppose that we use n-period differencing intervals for the data and maintain the assumption that n-period returns are independent and lognormal. Denoting the n-period return by R_n and the n-period geometric mean by G_n, we assume:

$$\ln R_n \sim N(\mu_n, s_n^2)$$

and derive:

$$\ln G_n \sim N(\mu_n, s_n^2/(T/n))$$

Then we can form an unbiased estimator of the N-period discount rate similar to that given by (15):

$$\hat{D}_{N4} = G_n^{-N/n} \exp[-[(T/n)+(N/n)](N/n)s_n^2/(2T/n)] \tag{19}$$

Using the fact that $G^{-N} = G_n^{-N/n}$ gives:

$$\hat{D}_{N4} = G^{-N} \exp[-(T+N)N(s_n^2/n)/2T] \tag{20}$$

Thus the estimator is the same as $\hat{D}_{N2}$ except that the estimated variance of the annual log return is given by (S_N^2/N), which is the annualised n-period variance. If the differencing interval is long enough to make the n-period returns serially independent, then this estimator eliminates transient effects in prices when constructing the discount rate.

In their study of mean reversion in stock prices, Poterba and Summers (1988) give estimates of the variance ratio $[(s_n^2/n)/s^2]$. These are shown in Table 2 for values of n up to eight years. Corresponding estimates of the discount rates for different horizons are also given. These are based on the estimator $\hat{D}_{N4}$ given in equation (20) computed for the relevant differencing interval (shown at the top of the table) and horizon. The estimates are shown as annual percentage discount rates. Thus the final column shows that using data differenced by eight years gives an annualised return variance equal to 80% of that computed usually annually differenced data. The discount rate equivalent to an unbiased discount factor for a one year horizon is then 8.7%, compared with the arithmetic mean of annual returns of 9% and the geometric mean annual returns of 7%.

These estimates are generally much closer to the arithmetic mean of past annual returns rather than the geometric mean. The attenuation of the variance to allow for 'overreaction' in share prices over short horizons does not result in estimates close to the geometric mean return. Indeed, it can be seen from equation (20) above that the condition for the geometric mean to be an unbiased discount rate is that $(s_n^2/n) = 0$. Even if equity markets overreact, the limit of (s_n^2/n) as n gets large will not be zero. From the Poterba and Summers results, this limit looks to be about 80% of the one year variance, indicating that the geometric mean is a significantly downward biased estimate of discount rates even when 'market overreaction' is taken into account.

7. Summary and conclusions

This paper has addressed an issue central to the estimation of discount rates for capital budgeting: should the geometric mean or arithmetic mean of past data be used when estimating the discount rate? The use of the arithmetic mean ignores estimation error and serial correlation in returns. Unbiased discount factors have been derived that correct for both these effects. In all cases, the corrected discount rates are closer to the arithmetic than the geometric mean.

It may be that the correct model of returns is more complex than that analysed here. If so, then estimation of discount rates would involve more complicated analysis than looking at the means of past returns. Some progress in this direction has been made by Brennan (1993). It may also be that a more complex criterion than unbiasedness is correct. Past average returns are, however, the most commonly cited statistics in estimating the market risk premium, so an understanding of the relative properties of geometric and arithmetic means as estimators is essential until a more sophisticated procedure is adopted.

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Appendix: An approximately unbiased discount rate with constant mean returns

We derive an approximation to A^{-N} as follows:

$$R_t = M + e_t$$

$$A = M + \sum_{t=1}^{T} e_t/T = M + h$$

$$A^{-N} = (M+h)^{-N}$$

$$E(A^{-N}) = E(M^{-N}(1+h/M)^{-N})$$

Expanding $(1+h/M)^{-N}$ as a Taylor series and keeping only terms of order h^2 or less:

$$E(A^{-N}) \simeq M^{-N}E(1 - Nh/M + (N+1)Nh^2/2M^2)$$

Let var $(e) = \sigma^2$, then:

$$E(h^2) = \sigma^2/T$$

$$E(h) = 0$$

$$E(A^{-N}) \simeq M^{-N}(1+(N+1)K) \tag{A1}$$

where:

$$K = N\sigma^2/2TM^2$$

Similarly, an approximation to G^{-N} is given by:

$$G = \left[\prod_{t=1}^{T}(M+e_t)\right]^{1/T}$$

$$G^{-N} = \prod_{t=1}^{T}(M+e_t)^{-N/T}$$

$$E(G^{-N}) = M^{-N}E\left[\prod_{t=1}^{T}(1+e_t/M)^{-N/T}\right]$$

Expanding $(1+e_t/M)^{-N/T}$ as a Taylor series and keeping only terms of e^2 or less:

$$E(G^{-N}) \simeq M^{-N} E\left[\prod_{t=1}^{T}(1 - Ne_t/TM + ((N/T+1)Ne_t^2/2TM^2)\right]$$

Using the independence of e_t, and e_τ:

$$= M^{-N} E\left[1 - N\sum_{t=1}^{T} e_t/TM + (\mathrm{N}/T+1)N\sum_{t=1}^{T} e_t^2/2TM^2\right]$$

Using:

$$E(e_t^2) = \sigma^2 \quad \text{and} \quad E(e_t) = 0$$

$$E(G^{-N}) \simeq M^{-N}(1 + (N/T+1) + NT\sigma^2/2TM^2) = M^{-N}(1 + (T+N)K) \tag{A2}$$

Using (A1) and (A2) gives:

$$E(bA^{-N} + (1-b)G^{-N}) = M^{-N}$$

where:

$$b = (N+T)/(T-1)$$