Taxation, agency conflicts, and the choice between callable and convertible debt

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Abstract

We analyze debt choice in light of taxes and moral hazard. The model features an infinite sequence of nonzero-sum stochastic differential games between equity and debt. Closed-form expressions are derived for all contingent-claims. If equity can increase volatility without reducing asset drift, callable bonds with call premia are optimal. Although callable bonds induce risk shifting, call premia precommit equity to less frequent restructuring and are tax-advantaged. Convertible bonds mitigate risk shifting, but only induce hedging if assets are far from the default threshold. Convertibles are optimal only if risk shifting reduces asset drift sufficiently.

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1. Introduction

In pioneering papers, Ingersoll [16] and Brennan and Schwartz [4] analyze the pricing implications of embedding call and conversion options in corporate debt. More recently, Sirbu, Pikovsky and Shreve [36] and Sirbu and Shreve [37] provide a full characterization of optimal stopping times for callable-convertible bonds. Each of these models adopts the perfect financial
markets assumptions of Modigliani and Miller (MM below) [30]. Under the MM assumptions, total firm value is invariant to debt structure and the resulting stochastic differential games are zero-sum.

This paper relaxes the MM assumptions and develops a nonzero-sum stochastic differential game framework to analyze optimal design of hybrid debt with embedded call and/or conversion options. In the model, a corporation weighs tax benefits of debt against bankruptcy and four distinct agency costs. First, the model allows equity to shift risk, with volatility treated as an instantaneous control. Levered equity’s risk shifting incentive was first noted by Jensen and Meckling [17]. The second agency cost of debt is underinvestment. Underinvestment takes two distinct forms: premature default and suboptimal instantaneous capital investment. The former problem was first noted by Titman [39] and the latter by Myers [33]. Finally, levered equity may deviate from the first-best financial restructuring policy with the goal of expropriating bondholders. This agency problem was first noted by Fischer, Heinkel and Zechner [10].

In static models analyzing hybrid debt, the risk shifting problem has figured prominently. Green [14] proves a pure convertible bond (with no call feature) can restore levered equity’s incentive to implement first-best risk exposure in a one-period model with exogenous debt. Effectively, the concavity induced by the threat of conversion just balances the convexity induced by limited liability. Similarly, Barnea, Haugen and Senbet [1] argue that call provisions reduce risk shifting incentives by shortening effective maturity. Fischer, Heinkel and Zechner [10] show that call premia ensure equity will not recapitalize too frequently relative to first-best. Finally, Marshall and Yawitz [27] argue that call premia are valuable from a tax perspective since they are deductible at the corporate level and taxed as capital gains income to the bondholder.

This paper offers a unified framework for evaluating the quantitative significance of the theories of Green [14], Marshall and Yawitz [27], Fischer, Heinkel and Zechner [10] and Barnea, Haugen and Senbet [1]. The model jointly determines optimal debt coupons, call premia, and conversion ratios. The model also allows us to assess whether call and/or conversion options are effective in mitigating risk shifting in a dynamic setting where volatility is chosen instantaneously.

In the model, financial structure is fully dynamic. Growth in underlying asset value leads to endogenous upward restructuring. Endogenous default leads to an optimal recapitalization by the new owners of the firm. Between restructuring dates, a manager works in the interest of current shareholders. Incentives of the manager and bondholder are analyzed as a nonzero-sum stochastic differential game with optimal stopping and instantaneous volatility control. The solution concept is Markov Perfect Equilibrium. The two interact up to the first passage to call, conversion or default. After this, another optimal bond is issued. An infinite sequence of such games occurs over time, with anticipated future equilibria capitalized into contingent-claim prices under rational expectations.

Despite the apparent complexity of the problem, all prices are derived in closed-form, up to Nash thresholds of a single game. Variational inequalities provide analytic characterizations of the optimal stopping regions, as in Bensoussan and Friedman [2]. We show that a super contact condition is necessarily satisfied at an optimal volatility switch point. This result is complementary to that of Dumas [8], who establishes the necessity of super contact at an optimal reflecting barrier for the regulation of an Ito process under proportional transactions costs.

The benchmark model considers a firm that can change volatility without affecting underlying asset drift, e.g. by hedging or speculating using fairly-priced derivatives. In this setting, all costs
of risk shifting are endogenous. As in the model of Leland [23], high volatility has an indirect cost in that it worsens the trade-off between tax benefits and bankruptcy costs of debt.

Consistent with empirical observation, we find that pure callable bonds are generally optimal. For example, in the benchmark model the optimal bond is a pure callable bond with a call premium of 18%. By putting in a call premium, a tax wedge is captured and equity is also discouraged from calling prematurely relative to first-best.

Although a superficial reading of Green [14] may tempt one to conclude that conversion options can do no harm, the firm incurs a transaction cost on the sale of an embedded option. In addition, in a levered recapitalization, the proceeds from the bond flotation are paid out as a taxable dividend. Therefore, in order for embedded conversion options to add to net firm value, they must provide an agency benefit sufficient to offset transaction and tax costs. In the benchmark model, tax and transaction cost values are larger than the value of hedging incentives provided by conversion features.

In the benchmark model, we find that a pure callable bond does not alleviate the risk shifting problem. Intuitively, equity is long the call option, and volatility increases the value of this option. A callable-convertible can potentially induce hedging, since equity’s payoff from calling has a concave kink. A pure convertible induces the strongest hedging incentive since equity is short a call option. However, a convertible has only limited absolute effectiveness in reducing risk-shifting. In particular, a convertible only induces hedging when conversion is imminent. When asset value is low, the limited liability effect dominates and equity is risk-loving. Effectively, with instantaneous control of volatility, local concavity of the value function is needed to induce hedging and the solution proposed by Green [14] is no longer adequate.

The extended model considers a setting where the firm increases volatility using investments that reduce asset drift. In this setting, putting in conversion features is optimal if the cost of risk shifting is sufficiently high. However, we also show that embedded conversion options have a negative side-effect in that they discourage capital accumulation. Myers [33] shows that levered equity has an incentive to underinvest in bad states, when the firm is near default. We show that conversion options embedded in debt create a distinct overhang problem in good states. In particular, investment is declining on the same region where the conversion option induces hedging.

We turn next to a discussion of related literature. To date, dynamic structural models have analyzed non-convertible debt. For examples, see Brennan and Schwartz [5], Fischer, Heinkel and Zechner [11], and Ross [35]. The analysis of convertible debt is considerably more complicated since one must solve for equilibria of stochastic differential games.

Pricing models with ordinary debt have been used to quantify the ex ante benefit of committing to low volatility. For example, see Mello and Parsons [28], Leland [23], Mello, Parsons and Triantis [29], Leland [24], and Morellec and Smith [31]. Leland and Toft [25] and Leland [24] show that shortening debt maturity mitigates risk shifting, but find that it is tax-inefficient. Analyzing consol debt allows us to assess whether call and/or conversion features can substitute for shorter maturity in alleviating agency costs. Of course, the consol bond assumption serves a technical role, preserving time-homogeneity.

Section 2 presents the modeling assumptions. Section 3 analyzes pure callable bonds. Section 4 analyzes pure convertible bonds. Section 5 analyzes callable-convertible bonds. Section 6 analyzes volatility control. Section 7 compares the different classes of debt. Section 8 discusses the effect of convertibles on investment levels.
2. The benchmark model

Let \((\Omega, \mathcal{F}, \mathcal{P}, w)\) be a reference probability system where \((\Omega, \mathcal{F}, \mathcal{P})\) is a probability space, \(\mathcal{F}_t \subset \mathcal{F}\), and \(w_t\) is an \(\mathcal{F}_t\)-adapted Wiener process on \([0, \infty)\). To simplify, assume \(w\) is uncorrelated with any priced factors. The state variable \(x_t\) denotes productivity-adjusted capital assets (“assets”) of the firm. Following Cox, Ingersoll and Ross (CIR) \([6]\) and Kogan \([19]\), assets have log-normal dynamics. Letting \(i\) denote the gross rate of investment and \(\delta\) the depreciation rate, asset dynamics in the benchmark model are described by the following stochastic differential equation.

\[
dx_t = (i - \delta)x_t \, dt + \sigma_t x_t \, dw_t, \quad x_0 > 0.
\]  

(1)

Two variants of Eq. (1) are considered. We initially assume volatility is fixed by technology at \(\sigma_t = \sigma\) for all \(t \geq 0\). Subsequently, volatility is an \(\mathcal{F}_t\)-measurable control of the manager. Appendix A establishes existence of a unique solution to the stochastic differential equation (1) allowing for jumps in volatility. The informational assumption in our model is that \(x_t\) is observable but not verifiable by a court. If the entire asset path were verifiable, a court could determine whether the manager increased volatility by computing the quadratic variation of the \(x\) process. The gross investment rate \(i\) is treated as technologically fixed. However, the model accommodates investment as an instantaneous control. Such an extension is considered in Section 8.

The stochastic differential equation for productive assets allows for positive and negative shocks. Fischer, Heinkel and Zechner \([10,11]\) follow CIR in assuming free entry into all production processes. This leads to an equilibrium condition that at a restructuring date, the ratio of total firm value to asset value, denoted \(Q\), is just equal to one. In contrast, we follow Goldstein, Ju and Leland \([12]\) in assuming the firm has a monopoly over its production technology. The firm is a price-taker in the market for the asset, and its price is normalized at one. The firm receives a constant \(\pi > 0\) units of operating profit per capital asset. Since operating profits are proportional to assets, they too are log-normal. Under these assumptions, \(Q\) can exceed one at a recapitalization point. Lindenberg and Ross \([26]\) present empirical evidence consistent with both modeling approaches. In particular, \(Q\) ratios are well above one for firms with unique products, while \(Q\) ratios are near one for firms in competitive industries. See Ross \([35]\) for further discussion.

At \(t = 0\) the corporation is unlevered. The manager may issue a bond with call and/or conversion features. The proceeds from the bond issuance, net of transaction costs, finance a discrete dividend to current shareholders. The optimal financing strategy maximizes the after-tax value of the initial dividend plus the discounted value of current shareholders’ future after-tax cash flows.

The bond is a consol with coupon \(c > 0\). In order to prevent expropriation, the bond contains a covenant prohibiting additional debt issuance prior to call or conversion. Smith and Warner \([38]\) document that such covenants are common, although severity varies. In the event of default, absolute priority is obeyed and the lender has an inviolable senior claim to the asset stock.

For the case of a pure convertible bond, the bondholder has the right to convert into an equity share equal to \(\theta\). In the case of the pure callable bond, the firm has the right to call the bond for \(1 + \xi\) times par value. All bonds are issued at par. The callable-convertible bond endows the bondholder with the right to convert and the firm with the right to call.

There is a linear corporate income tax equal to \(\tau_c\) times operating profits less economic depreciation less interest expense. The interest rate on a taxable risk-free government bond is \(r\). Interest income is taxed at rate \(\tau_i \in (0, \tau_c)\). We assume that asset drift \(i - \delta < r(1 - \tau_i)\), so that claim values are bounded. Cash dividends to shareholders are taxed at rate \(\tau_d \geq 0\). This stylized tax system is identical to that considered by Goldstein, Ju and Leland \([12]\). Since \(x_t\) is not ver-
ifiable by a court, contractual limitations on dividends cannot be enforced and equity is free to choose dividends ex post. Given frictionless access to external equity, and the tax disadvantage to saving within the corporate shell, levered equity finds it optimal to distribute all discretionary cash flow to shareholders.

Instantaneous dividends, denoted Δ, are distinct from discrete dividends paid at restructuring dates. The instantaneous dividend is equal to operating profits less gross investment less corporate income taxes less coupon payments:

\[ \Delta(x_t) \equiv \left[ \pi - i - \tau_c (\pi - \delta) \right] x_t - (1 - \tau_c)c. \]  

At financial restructuring points, the firm incurs a transaction cost equal to \( \phi > 0 \) times the market value of newly issued debt. At the time of each restructuring, the net proceeds from the bond issuance finance a discrete dividend.

Suppose the initial asset stock is \( x_0 > 0 \) and the manager chooses coupon \( c^* \). Let the minimum of the call and conversion thresholds be denoted \( x_u \equiv \gamma_u x_0 \) where \( \gamma_u > 1 \). At the point \( x_u \), the firm settles up with the bondholder and the assets become unlevered once again. The manager will once again maximize the value of the claim held by the current set of shareholders. The only change in the financing decision is that assets have grown to \( \gamma_u x_0 \). Therefore, if the terms of the original bond \( (c^*, \theta^*, \xi^*) \) were indeed optimal, then at this upward restructuring point the new optimal bond must feature the same conversion ratio and call premium \( (\theta^*, \xi^*) \). The only change in the bond is that the coupon must be scaled up to \( \gamma_u c^* \).

If default occurs, the new owners of the assets will also make optimal financing decisions. Suppose default occurs at \( x_d \equiv \gamma_d x_0 \) where \( \gamma_d < 1 \). Due to bankruptcy costs, the asset stock of the firm falls to \( (1 - \alpha)x_d \) at the time of default, with \( \alpha \in (0, 1) \). We here note that the existence of default costs implies that the first-best default policy would entail a commitment by equity to never default. However, as first noted by Titman [39], such a commitment is not incentive compatible ex post.

Post-default, the bondholder becomes the sole shareholder, owning an unlevered corporation with asset stock \( (1 - \alpha)\gamma_d x_0 \). Based on the reasoning presented above, the optimal bond at this point necessarily features \( (\theta^*, \xi^*) \), with the optimal coupon scaled down to \( (1 - \alpha)\gamma_d c^* \). An attractive feature of this framework is that the optimal financial policy for the entire life of the firm is uniquely determined by the vector \( (c^*, \theta^*, \xi^*) \). The numerical search for the optimal financial structure can then be conducted by evaluating the equilibrium strategies and implied security valuations for each possible vector \( (c, \theta, \xi) \).

Our valuation framework accommodates arbitrary assumptions regarding the profitability parameter \( \pi \). In the numerical analysis

\[ \pi = \frac{r(1 - \tau_i) + \delta}{(1 - \tau_d)(1 - \tau_c)} - \frac{\tau_c \delta}{(1 - \tau_c)}. \]  

Under this assumption, the value of an unlevered firm holding a single (depreciating) unit of the capital asset (i.e. \( x_0 = 1 \)) is unity. Therefore, comparison of the levered firm value with \( x \) provides a measure of the net tax shield benefit.

Finally, it is worth noting that our asset-based model is isomorphic to the EBIT model of Goldstein, Ju and Leland [12]. In particular, our model is easily converted to an EBIT framework if \( x \) is redefined as EBIT, while the parameters are set to \( \pi = 1, \delta = 0 \) and \( i = 0 \). In this paper,

1 One must then follow GJL [12] page 489 in choosing a suitable risk-neutral drift.
we opt for an asset-based valuation framework since this also allows us to analyze instantaneous investment levels.

3. Callable bonds

3.1. Valuation of callable bonds

Let \( f \) denote the value of the unlevered firm ex ante, just prior to bond issuance. Let \( b \) denote bond value. The value of the claim held by original shareholders is equal to the value of the initial discrete dividend funded by the bond flotation plus the ex-dividend value of their equity stake. The latter is denoted \( s \). We then have

\[
 f(x_0) \equiv (1 - \tau_d)(1 - \phi)b(x_0) + s(x_0). \tag{4}
\]

A convenient feature of the model is that all claim values are expressed in terms of only three primitive claims: \( p_u(\cdot) \) is the value of a claim paying one dollar at the first passage to \( x_u \), which is knocked-out if \( x_d \) is reached first; \( p_d(\cdot) \) is the value of a claim paying one dollar at the first passage to \( x_d \), which is knocked-out if \( x_u \) is reached first; and \( s_0(\cdot) \) is the value of a claim to instantaneous dividends while \( x \) occupies \((x_d, x_u)\), with the claim being knocked-out at the first passage to either boundary. Given volatility and the pair \((x_d, x_u)\), all primitive claim values are solutions to ordinary differential equations with closed-form solutions. Details are provided in Appendix B.

Call premia are taxed as capital gains income for the bondholder. The capital gains tax rate is \( \tau_g \in [0, \tau_d] \). Call premia are deductible at the corporate level and thus enjoy a large tax wedge equal to \( \tau_c - \tau_g \).

The value of a (pure) callable bond consists of three components: (1) the value of a claim \( b_0 \) to instantaneous coupons while \( x_t \in (x_d, x_u) \); (2) the value of the payment received if the bond is called; and (3) the value to becoming the sole owner in the event of default. The value of the claim to coupon payments is

\[
b_0(x) \equiv \frac{c}{r}[1 - p_u(x) - p_d(x)]. \tag{5}
\]

The price of the pure callable bond can then be expressed as

\[
b(x) = b_0(x) + p_u(x)[1 + \xi(1 - \tau_g)]b(x_0) + p_d(x)(1 - \alpha)\gamma_d f(x_0) \quad \forall x \in [x_d, x_u]. \tag{6}
\]

The first term in (6) represents the claim to coupons, the second the value of the payment received if the bond is called, and the last the value of default recoveries. Rearranging Eq. (6), the initial value of the bond is:

\[
b(x_0) = \frac{b_0(x_0) + p_d(x_0)(1 - \alpha)\gamma_d f(x_0)}{1 - p_u(x_0)[1 + \xi(1 - \tau_g)]}. \tag{7}
\]

With pure callable debt, the original shareholders have a claim to all dividends up until the time of first default. Let \( s_\infty(\cdot) \) denote the value of shareholders’ perpetual claim to instantaneous dividends prior to the first default. Let \( S(\cdot) \) denote the value of their perpetual claim to discrete dividends at each future restructuring point, with the claim being knocked-out at the time of first default. Excluding the value of the date zero dividend, the ex post value of equity is

\[
s(x) = s_\infty(x) + S(x) \quad \forall x \in [x_d, x_u]. \tag{8}
\]
Due to the scaling feature inherent in the model, \( s_\infty \) must satisfy the following recursive equation

\[
s_\infty(x) = s_0(x) + p_u(x)\gamma_u s_\infty(x_0) \quad \forall x \in [x_d, x_u].
\]  

(9)

Therefore,

\[
s_\infty(x_0) = \frac{s_0(x_0)}{1 - p_u(x_0)\gamma_u}.
\]  

(10)

Substituting (10) into (9) yields

\[
s_\infty(x) = s_0(x) + p_u(x)\left[\frac{\gamma_u s_0(x_0)}{1 - p_u(x_0)\gamma_u}\right] \quad \forall x \in [x_d, x_u].
\]  

(11)

The discrete dividend paid at the first upward restructuring date is equal to the difference between the cash in-take from the new bond flotation, less the after-tax cost of the call payment

\[
b(x_0)\left[\gamma_u (1 - \phi) - (1 + \xi(1 - \tau_c))\right].
\]  

(12)

Due to the scaling feature inherent in the model, the value of the claim to the discrete dividends at restructuring dates must satisfy the following recursive equation

\[
S(x) = p_u(x)\left\{(1 - \tau_d)b(x_0)\left[\gamma_u (1 - \phi) - (1 + \xi(1 - \tau_c))\right] + \gamma_u S(x_0)\right\} \quad \forall x \in [x_d, x_u].
\]  

(13)

Therefore,

\[
S(x_0) = p_u(x_0)\left[\frac{(1 - \tau_d)b(x_0)\left[\gamma_u (1 - \phi) - (1 + \xi(1 - \tau_c))\right]}{1 - \gamma_u p_u(x_0)}\right].
\]  

(14)

Substituting (14) into (13) yields

\[
S(x) = p_u(x)\left[\frac{(1 - \tau_d)b(x_0)\left[\gamma_u (1 - \phi) - (1 + \xi(1 - \tau_c))\right]}{1 - \gamma_u p_u(x_0)}\right] \quad \forall x \in [x_d, x_u].
\]  

(15)

Summing (11) and (15) yields an expression for the ex post equity value, expressed in terms of the primitive claims

\[
s(x) = s_0(x) + p_u(x)\left[\frac{\gamma_u s_0(x_0) + (1 - \tau_d)b(x_0)\left[\gamma_u (1 - \phi) - (1 + \xi(1 - \tau_c))\right]}{1 - \gamma_u p_u(x_0)}\right] \quad \forall x \in [x_d, x_u].
\]  

(16)

Substituting the equity and debt values into the identity (4) yields an expression for the total ex ante value of the firm

\[
f(x_0) = \frac{1 - \psi(1 - \tau_d) p_d(x_0)(1 - \alpha)\gamma_d}{1 - p_u(x_0)[1 + \xi(1 - \tau_g)]}^{-1} \times \left[\frac{s_0(x_0)}{1 - \gamma_u p_u(x_0)} + \frac{\psi(1 - \tau_d) b_0(x_0)}{1 - p_u(x_0)[1 + \xi(1 - \tau_g)]}\right].
\]  

\[
\psi = \frac{1 - \phi - p_u(x_0)[1 + \xi(1 - \tau_c)]}{1 - \gamma_u p_u(x_0)}.
\]  

(17)

At this point it is worth noting that the value of all relevant contingent-claims \( \{s, b, f\} \) have been expressed in terms of the values of only three primitive claims \( \{p_d, p_u, s_0\} \). The value of these primitive claims, in turn, depend upon \( (x_d, x_u) \), the points at which equity will find it optimal to default and call, respectively.
3.2. Optimal call policy

Let $Q$ denote total firm value at a restructuring point divided by assets

$$Q \equiv \frac{f(x_0)}{x_0}. \quad (18)$$

Each new bond flotation gives rise to a new stochastic differential game. In each particular game, the players use $Q$ as a parameter in formulating optimal strategies. Of course, in equilibrium, $Q$ hinges upon policies that will be adopted by all future players. However, in each round of the game, current players are price takers since they have no influence on strategies played in future rounds. The numerical solutions solve a fixed-point problem in $Q$.

A formal definition of equilibrium is now provided for the pure callable bond. We note that this definition assumes volatility is fixed. Section 6 considers instantaneous volatility control.

**Definition 1.** A pure callable bond equilibrium is a vector $(x_d^E, x_u^E, Q^E)$ satisfying:

1. $x_d^E$ and $x_u^E$ are ex post optimal default and call thresholds given the manager’s belief that $Q = Q^E$;
2. beliefs are rational, with $Q^E = f(x_0; x_d^E, x_u^E)/x_0$; and
3. total firm value ($f$) is determined by (17).

In equilibrium, stopping times are optimal given the value of the restructured firm, while the value of the firm at restructuring dates is consistent with the stopping times.

The payoff to equity if it declares default is zero and a smooth-pasting condition pins down the endogenous default threshold

$$s'(x_d^E) = 0. \quad (19)$$

In order to determine the optimal call threshold, we note first that the manager treats his payoff function as

$$Qx - b(x_0)[1 + \xi(1 - \tau_c)](1 - \tau_d). \quad (20)$$

Intuitively, the manager views himself as holding a call option on a recapitalized firm with the strike price equal to the after-tax cost of the call price paid to the bondholder. Let $x_d^m$ denote the manager’s optimal call threshold for a pure callable bond. At the optimal call threshold, the equity value function pastes smoothly to the payoff function

$$s'(x_d^m) = Q. \quad (21)$$

Appendix C provides formal verification proofs for optimal policies. As shown by Bensoussan and Friedman [2] for optimal stopping and Fleming and Soner [9] for optimal control, value functions satisfy a system of variational inequalities. From the variational inequalities in Appendix C it is possible to show that $\Delta(x_d^m) \leq 0$. Intuitively, equity is willing to inject funds in order to keep alive its call option. Further, the optimal call threshold satisfies

$$x_d^m \geq \frac{r(1 - \tau_i)b[1 + \xi(1 - \tau_c)] - c(1 - \tau_c)}{(1 - \tau_d)^{-1}Q[r(1 - \tau_i) - (i - \delta)] - [\pi - i - \tau_c]}. \quad (22)$$

Consistent with intuition, from Eq. (22) it follows that the lower bound on the optimal call threshold is increasing in the call premium, decreasing in the after-tax cost of debt service, and decreasing in the value of the recapitalized firm as measured by Tobin’s average $Q$. 
3.3. Numerical analysis of pure callable bond

In the numerical analysis, the benchmark parameter values are: $\phi = 4\%$, $\alpha = 5\%$, $\tau_c = 35\%$, $\tau_i = 25\%$, $\tau_d = 10\%$, $\tau_g = 5\%$, $\delta = 10\%$, $r = 6\%$, $i = 11\%$, and $x_0 = 100$. We choose $\phi$ to be consistent with Lee et al. [22], who estimate average underwriting costs are 3.79% for convertible bonds. Fischer, Heinkel and Zechner [11] assume bankruptcy costs constitute 5% of the face value of debt. The tax rate parameters are consistent with the estimates of Graham [13]. The depreciation rate and risk-free rate are consistent with Moyen [32]. The gross investment rate $i$ is chosen such that assets grow at a modest rate of 1%. In all cases, we use Eq. (3) in order to normalize the profit parameter $\pi$. In this subsection, asset volatility is fixed at $\sigma = 0.30$.

Fig. 1 plots the value function $b$ given in Eq. (6) for pure callable bonds that are callable at par ($\xi = 0$). The higher of the two lines is the callable bond value under the benchmark assumption that transaction costs are $\phi = 4\%$. The lower line assumes lower transaction costs of $\phi = 2\%$. In both cases, the callable bond value is non-monotonic in the value of underlying assets. This effect was first noted by Kraus [20]. The bond is issued at par, rises locally if $x$ increases above $x_0$, and falls as $x$ approaches the endogenous call threshold. For a non-callable consol bond, the value of the bond would increase monotonically in $x$. Clearly, the decision by equity to call the bond imposes a negative externality on the bondholder. Anticipating, concern over this externality provides a rationale for call premia. In particular, the ability of equity to transfer value from bondholders encourages premature recapitalizations. Call premia mitigate this incentive by forcing equity to pay a higher price in the event of a call. Finally, we note that when transaction costs are lowered, the callable bond value actually falls. Intuitively, lower

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2 MATLAB codes are available from the authors upon request.
transaction costs induce equity to recapitalize more frequently, reducing the gain to bondholders from increases in $x$.

Fig. 2 depicts the effect of increasing the call premium. For $\xi = 0$, bond value is non-monotone. However, for $\xi = 10\%$, the bond value function is strictly increasing in $x$. The two value functions are plotted over their respective $(x_d, x_u)$ intervals. Clearly, increasing the call premium reduces the frequency of financial restructuring. This suggests that call premia can serve as a credible commitment against premature recapitalizations.

As shown in Fig. 2, the value function for pure callable debt is concave. Since equity’s claim is a residual net of debt value, equity value is convex even with the embedded call option. Thus, equity remains risk-loving under pure callable bonds, contrary to the argument of Barnea, Haugen and Senbet [1]. A formal discussion of this issue is provided in Section 6.

Fig. 3 plots equilibrium total firm value ($f$) given in Eq. (17) as a function of the coupon and call premium. This allows us to determine the optimal pure callable bond. For the pure callable bond, the optimal call premium is fairly large with $\xi^* = 18\%$. This is driven by two factors. First, call premia are tax advantaged. Second, a high call premium commits equity to implementing a second-best recapitalization policy.

4. Convertible bonds

4.1. Valuation of convertible bonds

The value $(b)$ of the claim held by the owner of a pure convertible bond is composed of three pieces: (1) the value of his claim to coupons between restructuring dates presented in Eq. (5);
the value of his equity share in the event of conversion; and (3) the value of his full ownership claim in the event of default:

\[ b(x) = b_0(x) + p_u(x)\theta \gamma_u f(x_0) + p_d(x)(1 - \alpha)\gamma_d f(x_0) \quad \forall x \in [x_d, x_u]. \] (23)

Consider next the ex post value of equity \( s \) when there is a pure convertible bond outstanding. Excluding the initial discrete dividend, the value of the claim held by original shareholders is equal to the value \( s_0 \) of their claim to instantaneous dividends while \( x_t \in (x_d, x_u) \) plus the value of their \( 1 - \theta \) share of the firm if the bond is converted:

\[ s(x) = s_0(x) + p_u(x)(1 - \theta)\gamma_u f(x_0) \quad \forall x \in [x_d, x_u]. \] (24)

Substituting (23) and (24) into (4) and rearranging terms yields an expression for the total ex ante value of the firm with a pure convertible bond

\[ f(x_0) = \frac{s_0(x_0) + (1 - \tau_d)(1 - \phi)b_0(x_0)}{1 - p_d(x_0)\gamma_d(1 - \alpha)(1 - \tau_d)(1 - \phi) - p_u(x_0)\gamma_u[1 - \theta(\tau_d + \phi - \tau_d\phi)]}. \] (25)

At this stage, all the relevant claim values have been expressed in terms of primitive claim values, which, in turn, depend upon the endogenous default threshold \( x_d \) chosen by equity and the endogenous conversion threshold \( x_u \) chosen by the bondholder.

### 4.2. Optimal conversion

We now define the equilibrium concept for the pure convertible bond. Once again this definition assumes that underlying volatility is fixed. We later extend the analysis to consider endogenous volatility choice by equity.

**Definition 2.** A **pure convertible bond equilibrium** is a vector \((x_d^E, x_u^E, Q^E)\) satisfying:

1. \( x_d^E \) is the optimal default threshold given that the bond will be converted at \( x_u^E \) and the manager’s belief that \( Q = Q^E \);
In a pure convertible bond equilibrium, stopping times are a Nash equilibrium. A smooth-pasting condition (19) again determines equity’s default threshold, with the relevant equity value function presented in Eq. (24). Conversion gives the bondholder a payoff equal to $\theta Q x$. We let $x_{ub}^*$ denote the bondholder’s voluntary conversion threshold. It is optimal to convert at the point where the bond value function (23) pastes smoothly to the conversion payoff function $\theta Q x$. The smooth-pasting condition for optimal conversion is therefore

$$b'(x_{ub}^*) = \theta Q.$$  

A formal verification proof is provided in Appendix C. Using the variational inequalities for this stopping problem, it is possible to show

$$x_{ub}^* \geq \left[ r (1 - \tau_i) - (i - \delta) \right]^{-1} \left[ \frac{c(1 - \tau_i)}{Q \theta} \right].$$  

Consistent with intuition, the lower bound for the optimal conversion threshold is increasing in the debt coupon and decreasing in the conversion ratio.

4.3. Numerical analysis: pure convertible bond

This subsection adopts the benchmark parameter assumptions specified above. Again, volatility is held fixed at $\sigma = 0.30$ for these results. Fig. 4 plots the value function for a pure convertible bond, presented in Eq. (23). The straight line represents the bondholder’s payoff function from

---

**Fig. 4. Pure convertible bond value function.**
voluntary conversion, \( \theta Q^E x \). At the optimal conversion threshold, the bond value function pastes smoothly to the payoff function. For lower values of the underlying assets, the bond value function is concave, reflecting the greater relative importance of the defaultable debt component of the bond. For higher asset values, the bond value function becomes convex, reflecting the greater relative importance of the call option held by the bondholder. This foreshadows results obtained in Section 7. The fact that levered equity holds a short position in a claim that is convex for high \( x \) values is suggestive of a local hedging incentive. However, even with the pure convertible bond, equity can be seen as holding a short position in a claim that is concave for low \( x \) values. This hints at the possibility, confirmed below, that equity is risk-loving near default and risk-averse as conversion becomes more likely.

Fig. 5 plots the ex post equity value function \( s \) presented in (24) under a pure convertible bond with a conversion ratio of \( \theta = 50\% \). We choose a high conversion ratio so that curvature is easily visible. The figure lends support to the argument of Green [14] that conversion features discourage risk shifting, since equity’s value function becomes concave as \( x \) approaches the conversion threshold \( x^*_{ub} \). However, the equity value function is convex near default, suggesting that it is locally risk-loving as the likelihood of default increases.

Fig. 6 plots the total value of the firm (25) as the coupon \( c \) and conversion ratio \( \theta \) are varied. This allows us to determine the optimal bond when attention is confined to pure convertible bonds. When volatility is held fixed, the optimal convertible bond features a low conversion ratio of \( \theta^* = 6\% \) and a fairly high coupon of \( c^* = 5.7 \). To put these numbers in perspective, the voluntary conversion threshold is \( x^*_{ub} = 3014 \). Recalling that \( x_0 = 100 \), it is apparent that the optimal pure convertible bond (assuming fixed volatility) features a conversion option that is worth a negligible amount.

As was argued in the introduction, the model explains the reluctance of firms to issue valuable conversion options as resulting from concern over the transaction and tax costs associated with the sale of equity options. The sale of embedded conversion options creates additional transaction
costs up-front. In addition, in a levered recapitalization, the net proceeds from the bond flotation are paid out as a taxable dividend. Since a conversion option represents a form of backdoor equity, it is tax-inefficient to issue equity in order to finance a taxable dividend.

Numerical simulations reveal that the optimal conversion ratio for a pure convertible bond is decreasing in both $\phi$ and $\tau_d$. Intuitively, the firm is only willing to issue higher-powered conversion options if the tax and transactions costs are low. However, the sale of valuable embedded conversion options can only be justified if there is an agency benefit sufficient to offset tax/transaction costs. For the moment, we have shut-off the agency benefit of convertibles since $\sigma$ has been held fixed. We return to this issue in Section 7.

5. Callable-convertibles

The appropriate contingent-claim pricing formulas for a callable-convertible fall into one of two categories. Suppose first that the bond is called and that the bondholder finds it optimal to take cash rather than converting to an equity stake. In this case, the pricing formulas presented in Section 3 obtain. Of course, the nature of the equilibrium is endogenous. In particular, the scenario where the bondholder fails to convert is limited to cases where the conversion ratio ($\theta$) is low relative to the par value of debt (primarily determined by $c$) and the call premium ($\xi$).

Suppose next that, in equilibrium, the bondholder finds it optimal to convert his claim into an equity stake prior to call or at the time of call. Since our concern at this point is in correctly expressing the price of contingent claims in terms of $(x_d, x_u)$, it is of no concern whether the bondholder converts voluntarily or in response to the manager calling the bond. In either case, what is relevant is that at the threshold $x_u$, the bondholder holds a claim to $\theta$ times recapitalized firm value while equity holds a claim to $(1 - \theta)$ times firm value. Therefore all of the contingent claims prices formulas in Section 4 obtain in this case.
Recall that the pricing formulas in Sections 3 and 4 effectively treat \((x_d, x_u)\) as parameters, with smooth-pasting conditions pinning down the endogenous recapitalization points. Despite the fact that we are exploiting the same pricing formulas to value callable-convertibles, the equilibrium values of \((x_d, x_u)\) will generally differ.

We now define the equilibrium concept for the callable-convertible bond. Once again this definition assumes that underlying volatility is fixed. In Section 6 we extend the analysis to consider endogenous volatility.

**Definition 3.** A *callable-convertible bond equilibrium* is a vector \((x^E_d, x^E_u, Q^E)\) satisfying:

1. \(x^E_d\) is the optimal default threshold given that call/conversion occurs at \(x^E_u\) and the manager’s belief that \(Q = Q^E\);
2. \(x^E_u = \min\{x^{**}_{um}, x^{**}_{ub}\}\);
3. \(x^{**}_{um}\) is the manager’s optimal call threshold given that default occurs at \(x^E_d\) and the manager’s belief that \(Q = Q^E\);
4. \(x^{**}_{ub}\) is the bondholder’s optimal voluntary conversion threshold given that default occurs at \(x^E_d\) and the bondholder’s belief that \(Q = Q^E\);
5. if the bond is called, the bondholder converts for equity if and only if conversion value is no less than the call price (net of capital gains taxes);
6. beliefs are rational, with \(Q^E = f(x_0; x^E_d, x^E_u)/x_0\);
7. total firm value \(f\) is determined by (17) if the bondholder receives cash and by (25) if the bondholder receives equity.

Before proceeding, it is worth recalling results that have been derived under the MM assumptions. Under the MM assumptions, equity’s optimal call policy minimizes debt value. Brennan and Schwartz [4] show that under the MM assumptions, bond value should not be allowed to rise above the call price. Ingersoll [16] also analyzes callable-convertibles under the MM assumptions. His basic model assumes no dividends. In this setting, voluntary conversion never occurs, and it is optimal to call the bond as soon as the conversion value is equal to the call price. Sirbu, Pikovsky and Shreve [36], analyze perpetual callable-convertible bonds. They show that under the MM assumptions, Ingersoll’s prescribed call threshold represents an upper bound on the optimal call threshold. Earlier calls are optimal when the call price is sufficiently low relative to the coupon.

The results discussed above only pertain to optimal call/conversion policies under the MM assumptions. In our model, equity compares the after-tax cost of servicing debt against the value of the equity stake forfeited if the bondholder converts. As intuition would suggest, when the coupon is low relative to the conversion ratio, equity defers calling. When the coupon rate is high relative to the conversion ratio, equity accelerates calling, consistent with Sirbu, Pikovsky and Shreve [36].

For callable-convertibles, at

\[ x_k \equiv \frac{b(x_0)[1 + \xi(1 - \tau_g)]}{\theta Q}, \tag{28} \]

the bondholder is just indifferent between taking cash or converting in response to a call. In order to solve for the optimal call threshold \((x^{**}_{um})\), we take into account that the slope of the payoff function varies since
A callable-convertible has the potential to induce hedging, due to the concave kink in equity’s payoff function in addition to the threat of conversion. The optimal voluntary conversion point for the bondholder \((x_{um}^{**})\) satisfies the same smooth-pasting condition given for the pure convertible bond (26). Of course, this threshold is irrelevant for pricing if equity calls the bond sooner with \(x_{um} < x_{ub}^*\).

For the call threshold, a necessary condition for satisfaction of the standard smooth-pasting condition is \(x_{um}^{**} \neq x_k\). Since the call payoff function exhibits a concave kink at \(x_k\), it is often the case that in the numerical solutions \(x_{um}^{**} = x_k\). To understand why it is often optimal to call at the kink, suppose that \(x_{um}^{**} = x_k\). In this case, the equity value function \(s: \mathbb{R}_+ \rightarrow \mathbb{R}_+\) has a kink at \(x_k\) (see Appendix C). Letting \(L_t\) denote the nondecreasing local time of \(x\) at \(x_k\), the generalized Ito formula (see Theorem 7.1 of Karatzas and Shreve [18]) states:

\[
d s(x_t) = \left[ (i - \delta)x_t s'(x_t) + \frac{1}{2} \sigma^2 x_t^2 s''(x_t) \right] dt + \sigma x_t s'(x_t) dw_t - (\theta Q) dL_t. \tag{30}
\]

Of particular interest, the formula above indicates that under the callable-convertible, equity acts as if it will incur losses equal to the change in slope \((\theta Q)\) in an \(\epsilon\)-ball about \(x_k\). For this reason, equity will often find it optimal to stop at the first passage to \(x_k\). Kyle, Ou-Yang and Xiong [21] analytically prove a similar clustering of optimal stopping about the reference point under prospect theory.

The numerical procedure solves for the call threshold \((x_{um}^{**})\) in steps. First, we check whether the equity function can be smoothly pasted to the payoff function for \(x_{um} < x_k\). In this case, smooth-pasting is impossible if the slope of each resulting equity function is less than \(Q\) for each conjectured \(x_{um} < x_k\). Second, we repeat this procedure for \(x\) larger than \(x_k\). In this case, smooth-pasting is impossible if the slope of each resulting equity function is greater than \((1 - \theta)Q\) for each conjectured \(x_{um} > x_k\). If smooth-pasting is impossible in both cases, we conclude that \(x_k\) is the optimal call threshold.

For callable-convertible bonds, we obtain three types of equilibria:

\[
\begin{align*}
\text{E1: } & x_u^E = x_{um}^{**} < x_k < x_{ub}^{**}, \\
\text{E2: } & x_u^E = x_{um}^{**} = x_k < x_{ub}^{**}, \\
\text{E3: } & x_k < x_u^E = x_{ub}^{**} < x_{um}^{**}. \tag{31}
\end{align*}
\]

In E1, the call threshold is equal to that adopted under the pure callable bond \((x_{um}^{**} = x_{um}^*\)). The manager calls and the bondholder chooses to take cash rather than converting to equity. Since \(x_{um} < x_{ub}^{**}\), the bondholder would have preferred to delay and voluntarily exercised his conversion option at a higher asset value. Consistent with the inequality (22) derived above, equilibrium E1 holds for bonds with high coupons and low conversion ratios. In E2, the manager calls the bond at the point where the bondholder is just indifferent between taking cash or converting to equity. Again, since \(x_{um}^{**} < x_{ub}^{**}\), the bondholder would have preferred to delay and would have voluntarily converted at a higher asset value. Thus, E2 constitutes forced conversion. Finally, in E3 the bondholder converts voluntarily. This equilibrium holds for bonds with low coupons and high conversion ratios.
6. Endogenous volatility

For the remainder of the paper the manager is allowed to choose an $\mathcal{F}_t$-adapted volatility policy

$$\sigma_t \in \Sigma \equiv [\sigma, \bar{\sigma}] \quad \text{where} \quad 0 < \sigma < \bar{\sigma} < \infty. \quad (32)$$

To allow for direct costs of risk shifting, we assume asset drift is reduced by $\lambda \geq 0$ times the difference between $\sigma_t^2$ and the technologically feasible minimum $\underline{\sigma}^2$. Any technology with $\lambda > 0$ is value destroying and would not take place in the absence of agency conflicts between equity and debt. We rewrite the stochastic differential equation for asset value (1) as follows:

$$dx_t = \left[ i - \delta - \lambda (\sigma_t^2 - \underline{\sigma}^2) \right] x_t dt + \sigma_t x_t dW_t, \quad x_0 > 0. \quad (33)$$

The Hamilton–Jacobi–Bellman equation for this problem is

$$r(1 - \tau_i)s(x) = \max_{\sigma \in \Sigma} \left( 1 - \tau_d \right) \left[ (\pi - i - \tau_c(\pi - \delta))x - (1 - \tau_c)c \right] + \left[ i - \delta - \lambda (\sigma^2 - \underline{\sigma}^2) \right]xs'(x) + \frac{1}{2}\sigma^2 x^2 s''(x). \quad (34)$$

We thus have the following proposition.

**Proposition 1.** Equity volatility has a bang–bang solution, with

$$\frac{xs''(x)}{s'(x)} \geq 2\lambda \quad \Rightarrow \quad \sigma^*(x) = \bar{\sigma},$$

$$\frac{xs''(x)}{s'(x)} < 2\lambda \quad \Rightarrow \quad \sigma^*(x) = \underline{\sigma}. \quad (35)$$

The ratio $xs''(\cdot)/s'(\cdot)$ is levered equity’s coefficient of relative risk appetite, analogous to an investor’s coefficient of relative risk aversion. Since $\lambda \geq 0$, local convexity of $s$ is a necessary condition for the manager to choose $\bar{\sigma}$. Higher costs of risk ($\lambda$) mitigate risk appetite. However, Proposition 2 shows that equity’s risk appetite is insatiable when the firm is near default.

**Proposition 2.** For each type of hybrid debt, there exists a right neighborhood of the optimal default threshold where $\sigma^* = \bar{\sigma}$.

**Proof.** The function $s \in C^2$ on the region considered. Smooth-pasting demands $\lim_{x \downarrow x_d} s'(x) = 0$. Since $s > 0$ on the continuation region, it must be the case that $s'' > 0$ on some right neighborhood of $x_d$. Hence, $\lim_{x \downarrow x_d} xs''(x)/s'(x) = \infty$. The optimal policy follows from Proposition 1. \(\square\)

Proposition 2 shows vividly the speculative incentive induced by limited liability. The next proposition illustrates equity’s speculative incentive near optimal call thresholds. Intuitively, at an optimal call threshold, the equity value function pastes smoothly to the payoff line from above, which implies convexity.

**Proposition 3.** If $\lambda = 0$, then for all pure callable bonds and all callable-convertibles called (in equilibrium) at a point other than $x_k$, there exists a left neighborhood of the optimal call threshold where $\sigma^* = \bar{\sigma}$.
Proof. Solving the HJB equation for \( s'' \) and taking limits as \( x \) converges to the call threshold, satisfaction of the relevant variational inequality implies \( \lim_{x \uparrow x_{um}} s''(x) \geq 0 \). The optimal policy follows from Proposition 1. \( \square \)

Remark. If optimal volatility has at most one switch threshold, then under the conditions in Proposition 3, \( \sigma^*(x) = \bar{\sigma} \) for all \( x \in (x_d^*, x_u^*) \).

Due to the complexities involved, the numerical analysis assumes that under optimal policies volatility has at most one switch threshold. From Proposition 2 we know that for all classes of hybrid debt, the firm will choose \( \bar{\sigma} \) for low asset values. The question is whether hybrid debt can induce equity to switch from high to low volatility as the asset value increases. Proposition 3 shows that pure callable bonds and a broad set of callable-convertibles fail to do so.

When volatility switches, additional boundary conditions must be invoked in order to determine the ex post value of equity and debt. Let the subscripts \( l \) and \( r \) denote claim values to the left and right of \( x_s \), respectively. Dixit [7] presents an informal argument that whenever parameters of the underlying process change at a transitional boundary, correct valuation demands value matching and smooth-pasting. More formally, Fleming and Soner [9] show that the value function must be sufficiently smooth to invoke the Dynkin formula. Therefore, we impose

\[
\begin{align*}
    s_l(x_s) &= s_r(x_s), \\
    s'_l(x_s) &= s'_r(x_s), \\
    b_l(x_s) &= b_r(x_s), \\
    b'_l(x_s) &= b'_r(x_s).
\end{align*}
\]

(36)

It should be stressed that the smooth-pasting condition applied to the equity value function at the transitional boundary is not an optimality condition. The result from Dixit [7] obtains even if the parameters of the underlying process change exogenously. Lemma 1, proved in Appendix D, states the appropriate optimality condition. We label this condition a super contact condition.

Lemma 1. If \( x_s^* \) is an optimal volatility switch point, then \( s''_l(x_s^*) = s''_r(x_s^*) \).

Closed-form solutions for all claim values were obtained above, with the solutions expressed in terms of the endogenous thresholds \( (x_d, x_s, x_u) \). Smooth-pasting conditions pin down \( (x_d, x_u) \) while the super contact condition from Lemma 1 pins down \( x_s \).

7. The optimal bond

This section compares the alternative debt instruments: pure callable bonds, pure convertible bonds, and callable-convertible bonds. In the numerical analysis, the benchmark assumptions are: \( \phi = 4\% \), \( \alpha = 5\% \), \( \tau_c = 35\% \), \( \tau_l = 25\% \), \( \tau_g = 5\% \), \( \delta = 10\% \), \( r = 6\% \), \( i = 11\% \), and \( x_0 = 100 \). In order to assess incentive effects, we will be interested in analyzing how the different securities lead to endogenous differences in \( \sigma \). In addition, we will be interested in exploring how variation in the parameters \( (\tau_d, \lambda) \) affects the choice of debt instruments.

We begin by evaluating security design for a firm that does not face any risk shifting problem whatsoever. In particular, in Table 1 it is assumed that volatility is fixed at \( \sigma = 0.30 \). We begin with panels A and B, where it is assumed that \( \tau_d = 0\% \). Here, the pure callable bond is the dominant security. For example, a pure callable bond with a call premium of 10% attains firm
Table 1
Results for fixed volatility model

<table>
<thead>
<tr>
<th></th>
<th>$x_d$</th>
<th>$x_u$</th>
<th>$s(x_0)$</th>
<th>$b(x_0)$</th>
<th>$f(x_0)$</th>
<th>$c^*$</th>
<th>$\theta^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Panel A: $\tau_d = 0%$, $\xi = 0%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pure callable</td>
<td>19.178</td>
<td>180.88</td>
<td>56.215</td>
<td>39.760</td>
<td>103.68</td>
<td>2.961</td>
<td>n/a</td>
</tr>
<tr>
<td>Pure convertible</td>
<td>35.757</td>
<td>1129.5</td>
<td>38.356</td>
<td>69.261</td>
<td>104.85</td>
<td>5.384</td>
<td>0.19</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>19.178</td>
<td>180.88</td>
<td>56.215</td>
<td>39.760</td>
<td>103.68</td>
<td>2.961</td>
<td>$\leq 0.20$</td>
</tr>
<tr>
<td>Panel B: $\tau_d = 0%$, $\xi = 10%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pure callable</td>
<td>32.763</td>
<td>233.58</td>
<td>43.524</td>
<td>63.149</td>
<td>106.14</td>
<td>5.115</td>
<td>n/a</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>32.763</td>
<td>233.58</td>
<td>43.524</td>
<td>63.149</td>
<td>106.14</td>
<td>5.115</td>
<td>$\leq 0.35$</td>
</tr>
<tr>
<td>Panel C: $\tau_d = 10%$, $\xi = 0%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pure callable</td>
<td>15.761</td>
<td>192.76</td>
<td>74.729</td>
<td>48.754</td>
<td>135.08</td>
<td>3.538</td>
<td>n/a</td>
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<td>Pure convertible</td>
<td>25.415</td>
<td>3013.9</td>
<td>70.53</td>
<td>75.185</td>
<td>135.49</td>
<td>5.661</td>
<td>0.06</td>
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<tr>
<td>Callable convertible</td>
<td>15.761</td>
<td>192.76</td>
<td>74.729</td>
<td>48.754</td>
<td>135.08</td>
<td>3.538</td>
<td>$\leq 0.15$</td>
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<tr>
<td>Panel D: $\tau_d = 10%$, $\xi = 10%$</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Pure callable</td>
<td>22.716</td>
<td>284.00</td>
<td>58.846</td>
<td>68.249</td>
<td>136.79</td>
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<tr>
<td>Callable convertible</td>
<td>22.716</td>
<td>284.00</td>
<td>58.846</td>
<td>68.249</td>
<td>136.79</td>
<td>5.111</td>
<td>$\leq 0.25$</td>
</tr>
</tbody>
</table>

Value of $f = 106.14$ while the optimal pure convertible only obtains a firm value of $f = 104.85$. In fact, it is possible for the firm to do even better, with $f^* = 106.75$ using the optimal pure callable bond featuring a call premium of $\xi^* = 21\%$ and a coupon of $c^* = 7.89$. Relative to the pure convertible bond, total firm value increases by 1.8% when the firm uses the optimal pure callable bond.

The fact that the pure convertible bond is dominated in this setting is not surprising. By assumption, the convertible bond has no agency benefit, since volatility is here assumed to be fixed. Since the convertible bond features embedded backdoor equity, it is costly to issue in light of tax and transactions costs. The dominance of the callable bond with the high call premium is due to the tax advantaged treatment of call premia and the beneficial effect of the call premium in terms of discouraging excessively frequent recapitalizations. For example, when the bond is callable at par, the firm carries a low coupon of $c = 2.961$ and the manager calls at the first passage to $x_{um}^c = 180.88$. When the call premium is raised to $\xi = 10\%$, for example, the firm carries a heavier debt coupon of $c = 5.115$ and delays calling until the first passage to $x_{um}^c = 233.58$.

In Table 2, we consider optimal bond indentures for a firm whose manager can use derivatives to alter the firm’s instantaneous risk exposure at cost $\lambda = 0$. Following Leland [24], the manager can choose $\sigma \in [0.15, 0.30]$. The lower bound on volatility can be viewed as resulting from the manager hedging to the maximum extent possible using traded instruments. The upper bound on

---

3 Firm value also changes due to the fact that the $\pi$ parameter changes based upon (3).
Table 2
Results for endogenous volatility model with costless risk shifting

<table>
<thead>
<tr>
<th>Thresholds</th>
<th>Value</th>
<th>Optimal parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Equity $s(x_0)$</td>
<td>Debt $b(x_0)$</td>
</tr>
<tr>
<td>$x_d$</td>
<td>$x_s$</td>
<td>$x_u$</td>
</tr>
<tr>
<td>Pure callable</td>
<td>n/a</td>
<td>233.58</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>n/a</td>
<td>233.58</td>
</tr>
<tr>
<td>Pure callable</td>
<td>32.763</td>
<td>n/a</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>32.763</td>
<td>n/a</td>
</tr>
<tr>
<td>Pure callable</td>
<td>15.761</td>
<td>n/a</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>15.761</td>
<td>n/a</td>
</tr>
<tr>
<td>Pure convertible</td>
<td>24.747</td>
<td>963.60</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>24.747</td>
<td>963.60</td>
</tr>
</tbody>
</table>

The manager chooses volatility optimally each instant, maximizing the ex post value of the claim held by current shareholders. Table 2 adopts the assumption that $\lambda = 0$. That is, there are no direct costs associated with risk shifting, although indirect costs are present due to the existence of bankruptcy costs.

volatility can be viewed as the highest volatility the manager can choose without violating bond covenant prohibitions on the holding of particularly risky derivatives.

Consistent with Proposition 3, callable bonds fail to induce hedging. Pure convertible bonds do induce hedging, but only when the asset process is sufficiently far from the default boundary. For example, when $\tau_d = 0\%$, the optimal pure convertible bond results in the firm choosing $\sigma = .30$ for $x \in (35.378, 302.38)$ and shifting to $\sigma = .15$ for $x \in (302.38, 658.50)$.

The pure convertible bond is still the dominant security when there is endogenous volatility provided that speculation is done in a way that is fairly-priced ($\lambda = 0$). For example, when $\tau_d = 10\%$, the maximum value attainable with the pure convertible bond is $f = 135.43$. By way of contrast, $f^* = 137.02$ using the optimal pure callable bond featuring a call premium of $\xi^* = 18\%$ and a coupon of $c^* = 5.661$. Relative to the pure convertible bond, total firm value increases by 1.2% when the firm uses the optimal pure callable bond.

This analysis does not imply that valuable conversion options should never be included in a bond indenture. Rather, it suggests that the costs of risk shifting must be sufficiently high to justify them. To illustrate this point, Table 3 evaluates optimal financial structure for a firm facing direct costs of risk shifting with $\lambda = 0.2$. For lower values of $\lambda \in \{0.05, 0.10, 0.15\}$, pure callable debt was still the dominant security.
Table 3
Results for endogenous volatility model with costly risk shifting

<table>
<thead>
<tr>
<th>Panel</th>
<th>( \tau_d = 0% ), ( \xi = 0% )</th>
<th>( \tau_d = 0% ), ( \xi = 10% )</th>
<th>( \tau_d = 10% ), ( \xi = 0% )</th>
<th>( \tau_d = 10% ), ( \xi = 10% )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Pure callable</td>
<td>Pure convertible</td>
<td>Callable convertible</td>
<td>Pure callable</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Panel A</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pure callable</td>
<td>17.73</td>
<td>56.88</td>
<td>211.8</td>
<td>66.86</td>
</tr>
<tr>
<td>Pure convertible</td>
<td>22.345</td>
<td>61.283</td>
<td>327.929</td>
<td>56.719</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>22.116</td>
<td>62.582</td>
<td>297.94</td>
<td>59.110</td>
</tr>
<tr>
<td>Panel B</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pure callable</td>
<td>17.39</td>
<td>60.55</td>
<td>382.5</td>
<td>67.25</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>22.148</td>
<td>62.391</td>
<td>328.67</td>
<td>58.779</td>
</tr>
<tr>
<td>Panel C</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pure callable</td>
<td>16.93</td>
<td>54.34</td>
<td>215.2</td>
<td>89.75</td>
</tr>
<tr>
<td>Pure convertible</td>
<td>20.968</td>
<td>58.81</td>
<td>870.61</td>
<td>79.585</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>20.138</td>
<td>56.69</td>
<td>458.42</td>
<td>81.84</td>
</tr>
<tr>
<td>Panel D</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Pure callable</td>
<td>16.58</td>
<td>57.82</td>
<td>387.0</td>
<td>90.31</td>
</tr>
<tr>
<td>Callable convertible</td>
<td>20.147</td>
<td>56.645</td>
<td>503.23</td>
<td>81.705</td>
</tr>
</tbody>
</table>

The manager chooses volatility optimally each instant, maximizing the ex post value of the claim held by current shareholders. Table 3 adopts the assumption that \( \lambda = 0.2 \). That is, there are direct costs associated with risk shifting, although indirect costs are also present due to the existence of bankruptcy costs.

Bond indenture due to the fact that equity is encouraged to hedge over a larger interval of asset values.

8. Convertible bonds and investment levels

The debt overhang problem posited by Myers [33] has figured prominently among agency costs of debt, along with risk shifting. In our asset-based framework, one can also analyze the effect of convertible debt on the investment level. Anticipating, we argue that attempts to remedy risk shifting through convertibles exacerbate underinvestment. The intuition is simple. As argued by Myers, levered equity underinvests relative to first-best when the probability of default is high, e.g. when asset value is near the default threshold. This is because a portion of the return to capital accrues to lenders in the event of default. A similar externality problem is created by conversion options (in sharp contrast to call provisions). When contemplating investment in good states, current shareholders recognize they will share a percentage of the respective project cash flow with new shareholders in the event of conversion. In light of this anticipated dilution, current shareholders may reduce investment.

We now show the underinvestment problem associated with conversion options is inextricably linked with the hedging incentive provided by those same options. To illustrate the argument, we extend the set of instantaneous controls to include investment in addition to asset volatility, assuming \( \lambda = 0 \). In particular, suppose the manager chooses \( \mathcal{F}_t \)-adapted controls \( i_t \in I \equiv (-\infty, \infty) \) and \( \sigma_t \in \Sigma \). Following Hayashi [15], assume installing or removing capital assets disrupts business operations. In particular, assume capital adjustment costs are convex with operating profit per asset equal to \( \pi - \eta^2 \), where \( \eta > 0 \). The Hamilton–Jacobi–Bellman equation is
\[ r(1-\tau_i)s(x) = \max_{i \in I, \sigma \in \Sigma} (1-\tau_d)[(\pi - \eta i^2 - i - \tau_c(\pi - \eta^2 - \delta)]x - (1-\tau_c)c + (i - \delta)x s'(x) + \frac{1}{2} \sigma^2 x^2 s''(x). \] (37)

The optimal controls are

\[ s''(x) \geq 0 \Rightarrow \sigma^*(x) = \sigma, \]
\[ s''(x) < 0 \Rightarrow \sigma^*(x) = \sigma, \]
\[ i^*(x) = \frac{s'(x) - (1-\tau_d)}{2\eta(1-\tau_c)(1-\tau_d)}. \] (38)

Thus, optimal investment is increasing in \( s'(\cdot), \) the shadow price of capital. Further, the firm will only hedge if \( s''(x) < 0. \) But concavity of the value function is equivalent to a declining shadow price of capital. This leads to the following proposition, sharply illustrating the dark side of the provision of hedging incentives.

**Proposition 4.** The optimal investment rate is decreasing (increasing) over any interval of the state-space where conversion options induce equity to choose minimum (maximum) volatility.

Conversely, Proposition 4 reveals an agency benefit of pure callable debt. Recall from Proposition 3 that equity value is necessarily convex in the neighborhood of the optimal call threshold. From Proposition 4 it follows that call options encourage capital accumulation. For many firms, encouraging capital accumulation may be of greater concern than discouraging risk shifting. This result, in conjunction with the tax advantages of call premia, may help to explain the relative popularity of pure callable debt.

9. Conclusions

This paper contributes to the structural corporate finance literature by developing a contingent-claims model that can be used to analyze the choice between hybrid debt instruments when financial markets are imperfect. In addition, the paper contributes to the nonzero-sum stochastic differential games literature, since equilibrium prices are shown to depend upon the Markov Perfect Equilibria of an infinite sequence of such games.

The model yields predictions broadly consistent with empirical observation. We predict that the majority of firms should exhibit a preference for pure callable bonds, despite the fact that only convertible bonds induce hedging. Additionally, we showed that the value of the hedging incentive provided by convertibles is limited since they only induce hedging when asset value is far from the default threshold. Finally, we showed that convertibles can have unintended agency costs since forcing equity to share upside with bondholders simultaneously discourages risk shifting and capital accumulation.

Acknowledgments

We thank Patrick Bolton, Dmitry Livdan, Jacob Sagi, Mete Soner, and Richard Stanton for helpful suggestions. We are grateful to seminar participants at Princeton University, the 2005 European Finance Association Annual Meeting, Moody’s KMV, Rice, and Texas A&M.
Appendix A. Existence of unique solution to stochastic differential equation

In the case of constant $\sigma$, $x$ is a geometric Brownian motion and the solution to the SDE is well known. A technical difficulty arises when $\sigma$ jumps, as may be the case under an optimal control policy. Here, the standard linear growth condition is not satisfied and we must invoke the following result due to Nakao [34].

Suppose $y(0) > 0$ and
\[
dy = m(y(t)) \, dt + \sigma(y(t)) \, dw_t.
\]
(39)
If $\{m, \sigma\}$ are bounded and Borel measurable, $\sigma$ is bounded below by a positive constant, and $\sigma$ is of bounded variation on any compact interval, then there exists a pathwise unique solution to the SDE defining $y$.

We need only reformulate the problem as originally stated into an equivalent problem that allows us to make use of Nakao’s result. To this end, let
\[
x_t \equiv \exp(y_t),
\]
\[
dy_t \equiv \left[ i - \delta - \frac{1}{2} \sigma_t^2 \right] dt + \sigma_t \, dw_t.
\]
(40)

Appendix B. Valuation of primitive claims

Let
\[
\mu(\sigma) \equiv i - \delta - \lambda(\sigma^2 - \sigma)^2.
\]
(41)
A contingent claim generating an instantaneous payment equal to $mx + k$ satisfies the following equilibrium condition
\[
\frac{1}{2} \sigma^2 x^2 g''(x) + \mu(\sigma) x g'(x) - r(1 - \tau_i) g(x) + mx + k = 0.
\]
(42)
The solution is
\[
g(x) = A x^{a(\sigma)} + Z x^{z(\sigma)} + \frac{mx}{r(1 - \tau_i) - \mu(\sigma)} + \frac{k}{r(1 - \tau_i)},
\]
(43)
where $a(\sigma) < 0$ and $z(\sigma) > 1$ solve the quadratic equation:
\[
\frac{1}{2} \sigma^2 \kappa^2 + \left( \mu(\sigma) - \frac{1}{2} \sigma^2 \right) \kappa - r(1 - \tau_i) = 0.
\]
(44)
Unknown constants $(A, Z)$ are determined by appropriate boundary conditions. We price claims in two cases, constant volatility and switching from $\tilde{\sigma}$ to $\overline{\sigma}$ when $x$ reaches $x_s$.

Consider first the primitive claims $p_u$ and $p_d$. If volatility is constant, the boundary conditions for the upward hitting claim are
\[
p_u(x_d) = 0,
\]
\[
p_u(x_u) = 1.
\]
(45)
It follows that
\[
p_u(x) = \frac{-x_d^{z(\sigma)} x^{a(\sigma)} + x_d^{a(\sigma)} x^{z(\sigma)}}{x_d^{a(\sigma)} x^{z(\sigma)} - x_u^{a(\sigma)} x^{z(\sigma)}}.
\]
(46)
The boundary conditions for the default hitting claim are

\[ p_d(x_d) = 1, \]
\[ p_d(x_u) = 0. \]  
(47)

It follows that

\[ p_d(x) = x_u^{a(\sigma)} x_a(\sigma) - x_d^{a(\sigma)} x_z(\sigma), \]  
(48)

If volatility switches, the upward hitting claim value function is

\[ p_u(x) = \hat{p}_u(x) \equiv \hat{A} x_a(\hat{\sigma}) + \hat{Z} x_z(\hat{\sigma}) \quad \forall x \in [x_d, x_s) \]
\[ = \tilde{p}_u(x) \equiv \tilde{A} x_a(\tilde{\sigma}) + \tilde{Z} x_z(\tilde{\sigma}) \quad \forall x \in [x_s, x_u]. \]  
(49)

The four unknown constants \{\hat{A}, \tilde{A}, \hat{Z}, \tilde{Z}\} are determined using

\[ \hat{p}_u(x_d) = 0, \]
\[ \tilde{p}_u(x_u) = 1, \]
\[ \hat{p}_u(x_s) = \tilde{p}_u(x_s), \]
\[ \hat{p}'_u(x_s) = \tilde{p}'_u(x_s). \]  
(50)

The default hitting claim has the same functional form, with the unknown constants \{\hat{A}, \tilde{A}, \hat{Z}, \tilde{Z}\} being determined by the boundary conditions

\[ \hat{p}_d(x_d) = 1, \]
\[ \tilde{p}_d(x_u) = 0, \]
\[ \hat{p}_d(x_s) = \tilde{p}_d(x_s), \]
\[ \hat{p}'_d(x_s) = \tilde{p}'_d(x_s). \]  
(51)

Since the boundary conditions represent a system of four linear equations in the four unknown constants, the hitting claim value functions have closed-form solutions, omitted in the interest of brevity.

Next, we derive the value of \( s_0 \), the dividend primitive claim. When the control policy is constant, the solution is

\[ s_0(x) = A x^{a(\sigma)} + Z x^{z(\sigma)} + \frac{(1 - \tau_d)[\tau - i - \tau_c(\tau - \delta)] x}{r(1 - \tau_i) - \mu(\sigma)} - \frac{(1 - \tau_d)(1 - \tau_i)c}{r(1 - \tau_i)}, \]  
(52)

with the unknown constants \((A, Z)\) determined by the boundary conditions:

\[ s_0(x_d) = 0, \]
\[ s_0(x_u) = 0. \]  
(53)

If the volatility switches, the solution is

\[ \hat{s}_0(x) = \hat{A} x^{a(\hat{\sigma})} + \hat{Z} x^{z(\hat{\sigma})} + \frac{(1 - \tau_d)[\tau - i - \tau_c(\tau - \delta)] x}{r(1 - \tau_i) - \mu(\hat{\sigma})} - \frac{(1 - \tau_d)(1 - \tau_i)c}{r(1 - \tau_i)} \quad \forall x \in [x_d, x_s) \]
\[ = \tilde{s}_0(x) \equiv \tilde{A}x_0(x) + \tilde{Z}x(x) + \frac{(1 - \tau_d)[\sigma - i - \tau_c(\sigma - \delta)]x}{r(1 - \tau_i) - \mu(\sigma)} - \frac{(1 - \tau_d)(1 - \tau_c)c}{r(1 - \tau_i)} \forall x \in [x_s, x_u], \] 

with the unknown constants \{\tilde{A}, \hat{A}, \hat{Z}, \tilde{Z}\} determined by value matching and smooth-pasting conditions:

\[ \begin{align*}
\hat{s}_0(x_d) & = 0, \\
\tilde{s}_0(x_u) & = 0, \\
\hat{s}_0(x_s) & = \tilde{s}_0(x_s), \\
\hat{s}'_0(x_s) & = \tilde{s}'_0(x_s).
\end{align*} \] 

(55)

Appendix C. Verification

This appendix states sufficient conditions such that each player attains his maximum feasible payoff. We begin by recalling a useful result from Brekke and Oksendal [3]. Let \( A_\sigma \) denote the infinitesimal generator corresponding to the Ito process \( x \) indexed by its volatility \( \sigma \). A continuous function \( s \) is said to be stochastically \( C^2 \) on domain \( D \) with respect to the diffusion \( x \) if the following generalized Dynkin formula holds:

\[ s(x_0) = E \left[ \int_0^T e^{-\rho t} \left[ \rho s(x_t) - A_\sigma(s(x_t)) \right] dt \bigg| \mathcal{F}_0 \right] + E \left[ e^{-\rho T} s(x_T) \bigg| \mathcal{F}_0 \right]. \] 

(56)

for all stopping times \( T \leq \inf \{ t > 0; \ x_t \notin D \} < \infty \). Brekke and Oksendal [3] prove that if a function \( s \) is \( C^1 \) on \( D \) and \( C^2 \) outside a thin set with respect to Green measure, then \( s \) is stochastically \( C^2 \) on \( D \).

C.1. Pure callable bond

Let \( t_u \) denote a stopping time (for call of bond), \( t_d \) a stopping time (for default), and \( T \equiv t_u \land t_d \). Let \( t_u^* \) denote the first passage from below to \( x_u^* \) and \( t_d^* \) the first passage from above to \( x_d^* \). With \( \sigma \) an admissible control policy, let

\[ J(x_0; \sigma, t_d, t_u) = E \int_0^T e^{-\rho t} (1 - \tau_d) \Delta(x_t) dt + e^{-\rho T} \chi(t_u < t_d) \Psi(x_t^u) \bigg| \mathcal{F}_0 \],

\[ \Psi(x) \equiv Qx - (1 - \tau_d)[1 + \xi(1 - \tau_c)]b, \]

\[ \rho \equiv r(1 - \tau_i), \]

\[ V(x_0) \equiv \max_{\sigma, t_d, t_u} J(x_0; \sigma, t_d, t_u). \] 

(57)

The candidate function \( s \) is \( C^1 \) on the positive real line and \( C^2 \) except at the points \( x_d^* \) and \( x_u^* \), which have measure zero. Further, the following system is satisfied:

\[ \begin{align*}
\hat{s}_0(x_d) & = 0, \\
\tilde{s}_0(x_u) & = 0, \\
\hat{s}_0(x_s) & = \tilde{s}_0(x_s), \\
\hat{s}'_0(x_s) & = \tilde{s}'_0(x_s).
\end{align*} \]
\[ \rho s(x) = (1 - \tau_d)\Delta(x) + \max_{\sigma} A^\sigma(s(x)) \quad \forall x \in (x^*_d, x^*_u), \]

\[ \sigma^*(x) \in \arg \max_{\sigma} A^\sigma(s(x)) \quad \forall x, \]

\[ \forall x \in (0, x^*_d] \quad s(x) = 0, \]

\[ \forall x \geq x^*_u \quad s(x) = \Psi(x), \]

\[ s(x) \geq \max\{0, \Psi(x)\} \quad \forall x, \]

\[ \forall x \notin (x^*_d, x^*_u) \quad \rho s(x) \geq (1 - \tau_d)\Delta(x) + \max_{\sigma} A^\sigma(s(x)). \quad (58) \]

We claim:

(a) \[ s(x_0) \geq J(x_0; \sigma, t_d, t_u) \quad \forall (\sigma, t_d, t_u), \]

(b) \[ s(x_0) = J(x_0; \sigma^*, t^*_d, t^*_u) = V(x_0). \quad (59) \]

Proof. We first prove (a). Let \( t_d \) and \( t_u \) denote arbitrary stopping times and \( \sigma \) an arbitrary admissible control policy. Since \( s \) is stochastically \( C^2 \) the generalized Dynkin formula applies and

\[ s(x_0) = E \left[ \int_0^T e^{-\rho t} \left( \rho s(x_t) - A^\sigma(s(x_t)) \right) dt \mid \mathcal{F}_0 \right] + E \left[ e^{-\rho T} \chi(t_u < t_d) s(x_{tu}) \mid \mathcal{F}_0 \right] \]

\[ \geq E \left[ \int_0^T e^{-\rho t} (1 - \tau_d)\Delta(x_t) dt + e^{-\rho T} \chi(t_u < t_d) \Psi(x_{tu}) \mid \mathcal{F}_0 \right] \equiv J(x; \sigma, t_d, t_u). \quad (60) \]

The greater than sign follows from the HJB equation and the last two inequalities. For part (b), the inequality becomes equality under the conjectured optimal policies.  \( \Box \)

C.2. Pure convertible: equity policy

Let \( t^*_u \) denote the first passage to the bondholder conversion threshold \( x^*_u \). Let \( t_d \) denote an arbitrary stopping time adopted by equity and \( t^*_d \) denote the first-passage to the conjectured optimal default threshold \( x^*_d \). Let \( T \equiv t^*_u \wedge t_d \). For an admissible \( \sigma \), let

\[ J(x_0; \sigma, t_d) \equiv E \left[ \int_0^T e^{-\rho t} (1 - \tau_d)\Delta(x_t) dt + e^{-\rho T} \chi(t_u^* < t_d) \Psi(x_u^*) \mid \mathcal{F}_0 \right], \]

\[ \Psi(x_u^*) \equiv (1 - \theta)Qx_u^*, \]

\[ V(x_0) \equiv \max_{\sigma, t_d} J(x_0; \sigma, t_d). \quad (61) \]

The candidate function \( s \) is \( C^1 \) on \((0, x^*_u)\) and \( C^2 \) except at the point \( x^*_d \) which has measure zero. Further, the following system is satisfied:

\[ \rho s(x) = (1 - \tau_d)\Delta(x) + \max_{\sigma} A^\sigma(s(x)) \quad \forall x \in (x^*_d, x^*_u), \]

\[ \sigma^*(x) \in \arg \max_{\sigma} A^\sigma(s(x)) \quad \forall x, \]

\[ \forall x \in (0, x^*_d] \quad s(x) = 0, \]
∀x ≤ x_d^* \quad ρs(x) \geq (1 − τ_d)Δ(x) + \max_{σ} A^σ(s(x)),

s(x_d^*) = Ψ(x_d^*).

(62)

We claim:

(a) s(x_0) ≥ J(x_0; σ, t_d) \quad ∀(σ, t_d)

(b) s(x_0) = J(x_0; σ^*, t_d^*) = V(x_0).

(63)

Proof. We first prove (a). Let t_d denote an arbitrary stopping time and σ denote an arbitrary admissible control. Since s is stochastically C^2 the generalized Dynkin formula applies and

\begin{align*}
    s(x_0) &= E\left[\int_0^T e^{-ρt} [ρs(x_t) - A^σ(s(x_t))] dt \mid F_0\right] + E\left[ e^{-ρT} \chi(t_u^* < t_d) s(x_u^*) \mid F_0\right] \\
    &≥ E\left[\int_0^T e^{-ρt} (1 − τ_d)Δ(x_t) dt + e^{-ρT} \chi(t_u^* < t_d)Ψ(x_u^*) \mid F_0\right] \\
    &≡ J(x; σ, t_d).
\end{align*}

The greater than sign follows from the HJB equation and the variational inequality. This establishes (a). For part (b), the inequality becomes equality under the conjectured optimal policies.

C.3. Optimal conversion policy

Let t_d^* denote the first passage to the default threshold x_d^*. Let t_u denote an arbitrary stopping time and T ≡ t_d^* ∧ t_u. Let t_u^* denote the first passage to the conjectured optimal conversion threshold x_u^*. Let

\begin{align*}
    J(x_0; t_u) \\
    &≡ E\left[\int_0^T e^{-ρt} c(1 − τ_i) dt + e^{-ρT} \left[ \chi(t_u^* < t_u)(1 − α)Qx_u^* + \chi(t_u < t_d^*)Ψ(x_u) \right] \mid F_0\right], \\
    Ψ(x) &≡ \theta Qx, \\
    V(x_0) &≡ \max_{t_u} J(x_0; t_u).
\end{align*}

(65)

The candidate function b is C^1 on (x_d^*, ∞) and C^2 except at the points x_u^* and x_d^* which have measure zero. Further, the following system is satisfied:

\begin{align*}
    ρb(x) &= c(1 − τ_i) + A^σ^*(b(x)) \quad ∀x ∈ (x_d^*, x_u^*), \\
    b(x_d^*) &= (1 − α)Qx_d^*, \\
    b(x_u^*) &= Ψ(x) \quad ∀x ≥ x_u^*, \\
    ρb(x) &≥ c(1 − τ_i) + A^σ^*(b(x)) \quad ∀x ≥ x_u^*.
\end{align*}

(66)

We claim:

(a) b(x_0) ≥ J(x_0; t_u) \quad ∀t_u,

(b) b(x_0) = J(x_0; t_u^*) = V(x_0).

(67)
Proof. We first prove (a). Let \( t_u \) denote an arbitrary stopping time. Since \( b \) is stochastically \( C^2 \) the generalized Dynkin formula applies and

\[
b(x_0) = E \left[ \int_0^T e^{-\rho t} \left[ \rho b - A^\sigma (b) \right] dt + e^{-\rho T} \left[ \chi (t_u < t_d) (1 - \alpha) Q x_d^* \right. \right. \\
+ \left. \left. \chi (t_u < t_d^*) b(x_u) \right] \bigg| \mathcal{F}_0 \right]\]

\[
\geq E \left[ \int_0^T e^{-\rho t} c(1 - \tau_d) dt + e^{-\rho T} \left[ \chi (t_d^* < t_u) (1 - \alpha) Q x_d^* + \chi (t_u < t_d^*) \Psi (x_u) \right] \bigg| \mathcal{F}_0 \right] \\
\equiv J(x_0; t_u).
\]

The greater than sign follows from the system of inequalities. This establishes (a). For part (b), the inequality becomes equality under the conjectured optimal policy. \( \square \)

C.4. Callable-convertible: equity policy

In this particular verification proof, we consider the case in which it is optimal for equity to call at the kink, with \( x_{um}^{**} = x_k \). Verification for the other cases follows the same line of argument. Let \( t_u \) and \( t_d \) denote arbitrary stopping times adopted by equity for call and default. Let \( t_d^* \) and \( t_d^* \) denote the first-passage to the conjectured optimal call and default thresholds, denoted \( x_{um}^{**} \) and \( x_d^* \), respectively. Let \( T \equiv t_u \wedge t_d \).

For an admissible \( \sigma \), let

\[
J(x_0; \sigma, t_d, t_u) \equiv E \left[ \int_0^T e^{-\rho t} (1 - \tau_d) \Delta (x_t) dt + e^{-\rho T} \chi (t_u < t_d) \Psi (x_u) \bigg| \mathcal{F}_0 \right],
\]

\[
\Psi (x) \equiv 0 \quad \forall x \in \left( 0, (1 - \tau_d)(1 + \xi (1 - \tau_c)) b/Q \right]
\]

\[
\equiv Q x - (1 - \tau_d)(1 + \xi (1 - \tau_c)) b \quad \forall x \in \left[ (1 - \tau_d)(1 + \xi (1 - \tau_c)) b/Q, x_k \right]
\]

\[
\equiv (1 - \theta) Q x \quad \forall x > x_k,
\]

\[
V(x_0) \equiv \max_{\sigma, t_d, t_u} J(x_0; \sigma, t_d, t_u).
\]

The candidate function \( s \) has the following properties. First, \( s'' \) is continuous on \([0, \infty)\) except at the points \( x_d^* \) and \( x_{um}^{**} \), but has well-defined one-sided derivatives at those two points. Further, \( s' \) is continuous on \([0, \infty)\) except at \( x_{um}^{**} \), but has well-defined one-sided derivatives at that point. Further, the following system is satisfied:

\[
\rho s(x) = (1 - \tau_d) \Delta (x) + \max_{\sigma} A^\sigma (s(x)) \quad \forall x \in (x_d^*, x_{um}^{**}),
\]

\[
\sigma^*(x) \in \arg \max_{\sigma} A^\sigma (s(x)) \quad \forall x,
\]

\[
\rho s(x) \geq (1 - \tau_d) \Delta (x) + \max_{\sigma} A^\sigma (s(x)) \quad \forall x \notin (x_d^*, x_{um}^{**}),
\]

\[
\forall x \in \left( 0, x_d^* \right] \quad s(x) = 0,
\]

\[
\forall x \geq x_{um}^{**} \quad s(x) = (1 - \theta) Q x,
\]

\[
s(x) \geq \Psi (x) \quad \forall x.
\]
We claim:

(a) \( s(x_0) \geq J(x_0; \sigma, t_d, t_u) \) \( \forall (\sigma, t_d, t_u) \)

(b) \( s(x_0) = J(x_0; \sigma^*, t^*_d, t^*_u) = V(x_0) \). (71)

**Proof.** We first prove (a). Let \( t_d \) and \( t_u \) denote arbitrary stopping times and \( \sigma \) denote an arbitrary admissible control. Since \( s \) has a kink, we must use the following generalized Dynkin formula, stated as Theorem 7.1 and Corollary 7.2 in Karatzas and Shreve [18]:

\[
\begin{align*}
  s(x_0) &= E \left[ \int_0^T e^{-\rho t} \left[ \rho s(x_t) - A^\sigma (s(x_t)) \right] dt \bigg| F_0 \right] + E \left[ e^{-\rho T} \chi(t_u < t_d) \Psi(x_{tu}) \bigg| F_0 \right] + \theta Q L^d_T \\
  &\geq E \left[ \int_0^T e^{-\rho t} (1 - \tau_d) \Delta(x_t) dt + e^{-\rho T} \chi(t_u < t_d) \Psi(x_{tu}) \bigg| F_0 \right] \\
  &\equiv J(x; \sigma, t_d, t_u).
\end{align*}
\]

Above, the term \( L^d_T \) denotes the discounted local time of \( x \) at \( x_k \). The greater than sign follows from the HJB equation, the variational inequality, and \( s \geq \Psi \). This establishes (a). For part (b), the inequality becomes equality under the conjectured optimal policies. \( \square \)

**Appendix D. Proof of super contact condition**

We here demonstrate necessity of the super contact condition at the ex post optimal volatility switch point. Suppose that \( x^*_s \) is the optimal point at which volatility switches from \( \bar{\sigma} \) to \( \bar{\sigma} \). Since the value function is stochastically \( \mathbb{C}^2 \) we know it is \( \mathbb{C}^1 \) across the switch threshold

\[
\begin{align*}
  s_l(x^*_s) &= s_r(x^*_s), \\
  s_l'(x^*_s) &= s_r'(x^*_s).
\end{align*}
\]

Further, the HJB equations are

\[
\begin{align*}
  \rho s_l(x^-_s) &= (1 - \tau_d) \left[ (\pi - i - \tau_c (\pi - \delta)) x^-_s - (1 - \tau_c) c \right] \\
  &\quad + [i - \delta - \lambda (\sigma^2 - \bar{\sigma}^2)] x^-_s s_l'(x^-_s) + \frac{1}{2} \sigma^2 (x^-_s)^2 s_l''(x^-_s), \\
  \rho s_r(x^+_s) &= (1 - \tau_d) \left[ (\pi - i - \tau_c (\pi - \delta)) x^+_s - (1 - \tau_c) c \right] + [i - \delta] x^+_s s_r'(x^+_s) \\
  &\quad + \frac{1}{2} \sigma^2 (x^+_s)^2 s_r''(x^+_s). \quad (74)
\end{align*}
\]

Taking limits and differencing one obtains

\[
\frac{x^-_s s_l''(x^-_s)}{s_l'(x^-_s)} - 2\lambda = \left( \frac{\sigma}{\bar{\sigma}} \right)^2 \left[ x^+_s s_r''(x^+_s) - 2\lambda \right]. \quad (75)
\]

If \( x^*_s \) is indeed an optimal switch point, Lemma 1 tells us that satisfaction of the HJB equation demands that the left side is weakly positive and the right is weakly negative. Therefore, both sides must equal zero in the limit and

\[
\begin{align*}
  s_l''(x^*_s) &= s_r''(x^*_s). \quad (76)
\end{align*}
\]
References

