

## Generating Optimization Problems with Global Variables

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### Abstract

The Optimization Problem with Global Variables (OPGV) combines objective and constraint functions belonging to a set of  $N$  different systems coupled only through a few of the variables known as *global variables*. Several decomposition algorithms have been proposed for the solution of the nonconvex OPGV. However, the development of these decomposition algorithms has been hindered by the absence of a suitable test-problem set. Although several OPGV test-problem sets have been developed in the context of the stochastic programming problem, most of them correspond to linear or convex problems. To bridge this gap, we introduce a new OPGV test problem set. Nonlinear programming as well as quadratic programming test problems can be generated. The user can choose problem characteristics such as dimension, convexity, degeneracy, and degree of coupling among systems and all local and global minimizers to the test problems are known a priori.

## 1 Introduction

Many optimization problems arising in business and engineering combine objective and constraint functions belonging to a set of weakly connected systems. In this paper we focus on problems for which only a few of the variables (known as global variables) affect the behavior of all systems, while the remainder (known as local variables) are needed only within one of the systems. We term these problems Optimization Problems with Global Variables (OPGVs), namely,

$$\begin{aligned} \min_{x, y_i} \quad & \sum_{i=1}^N F_i(x, y_i) \\ \text{s.t.} \quad & c_i(x, y_i) \geq 0, \quad i = 1:N, \end{aligned} \tag{1}$$

where  $x \in \mathbb{R}^n$  are the *global* variables,  $y_i \in \mathbb{R}^{n_i}$  are the  $i$ th system *local* variables,  $c_i(x, y_i) : \mathbb{R}^{n+n_i} \rightarrow \mathbb{R}^{m_i}$  are the  $i$ th system constraints, and  $F_i(x, y_i) : \mathbb{R}^{n+n_i} \rightarrow \mathbb{R}$  is the objective function term corresponding to the  $i$ th system.

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OPGVs arise in the design of complex systems such as an aircraft or automobile [CDF<sup>+</sup>94] and in the solution of stochastic problems such as portfolio management [Inf94, BL97].

The structure of the OPGV suggests there might be strong computational and organizational advantages in the use of decomposition algorithms to solve it. Generalized Benders decomposition [Ben62, Geo72] is widely used to solve OPGVs whose objective and constraint functions are convex in the local variables. Several decomposition algorithms have been proposed for the nonconvex OPGV [Tam87, Bra96, dM01]. However, the development of these and other decomposition algorithms for the nonconvex OPGV has been hindered by the absence of a suitable test-problem set. Although several OPGV test-problem sets [BDG<sup>+</sup>87], [Inf94, p. 47] have been developed in the context of the stochastic programming problem [BL97, Inf94], most of them correspond to linear or convex problems.

A more general test-problem set is the multidisciplinary design optimization test suite [PAG96]. For each test problem, a problem description, a benchmark solution method, sample input and output files, as well as source codes are available from the NASA Langley Research Center internet site. Test problems range from simple synthetic problems to some real engineering design problems. Unfortunately, the user has no control over important problem characteristics such as convexity and degree of degeneracy. Moreover, different test problems are given in different formats, and the implementation of a solution algorithm usually requires the modification of the FORTRAN codes used to evaluate the test problem objective and constraint functions.

Easy-to-use test-problem sets are available for several types of optimization problems related to the OPGV. Calamai and Vicente developed FORTRAN codes to generate quadratic programming [CV93b], linear-quadratic bilevel programming [CV93a], and quadratic bilevel programming test problems [CV94]. Their quadratic bilevel programming test problem set is such that the user can choose the test-problem size and the number of local and global minimizers. Moreover, all local and global minimizers are known a priori. Jiang and Ralph [JR99] developed a MATLAB code to generate mathematical programs with equilibrium constraints. Their test problems are more general than Calamai and Vicente's quadratic bilevel programs (which can be generated as a particular case) and the user can choose test-problem characteristics such as size, convexity, degeneracy, and ill conditioning. A disadvantage is that the minimizers of the test problems are not known in general.

In this paper, we introduce a new OPGV test-problem set. Quadratic programming as well as nonlinear programming test problems can be generated. To the best of our knowledge, this is the first OPGV test-problem set that allows the user control over test problem characteristics such as size, convexity, degeneracy, and degree of coupling.

To obtain the desired test problems, we first modify Calamai and Vicente's quadratic bilevel programming test problems [CV94] to create a quadratic programming OPGV test problem set. Then, we show how nonlinearity can be introduced in the quadratic programming OPGV test problems to obtain a

nonlinear programming OPGV test problem set.

## 2 Quadratic Programming OPGVs

### 2.1 Convex Quadratic Programming OPGVs

We propose the following convex quadratic programming OPGV:

$$\begin{aligned}
\min_{x, y_1, y_2} \quad & \frac{1}{2}k_1\|x - a\|^2 + \frac{1}{2}k_2\|y_{11} - x\|^2 + \frac{1}{2}\|y_{12}\|^2 + \\
& \frac{1}{2}k_1\|x - a\|^2 + \frac{1}{2}k_2\|y_{21} + x\|^2 + \frac{1}{2}\|y_{22}\|^2 \\
s.t. \quad & e \leq x + y_{11} \leq 2e, \\
& x - y_{11} \leq e, \\
& e \leq -x + y_{21} \leq 2e, \\
& -x - y_{21} \leq e,
\end{aligned} \tag{2}$$

where  $x \in \mathbb{R}^n$  are the global variables,  $y_i = (y_{i1}, y_{i2}) \in \mathbb{R}^{n_i}$  are the  $i$ th system local variables with  $y_{i1} \in \mathbb{R}^n$ ,  $y_{i2} \in \mathbb{R}^{n_i - n}$ ,  $k_1, k_2 \in \mathbb{R}$ , and  $e \in \mathbb{R}^n$  is the vector whose components are all ones. The feasible region of the convex quadratic programming OPGV proposed above closely resembles that of the quadratic bilevel programming test problem proposed by Calamai and Vicente [CV94]. The main differences between these two problems are that the OPGV test problem has only one objective function whereas the bilevel programming test problem has an upper-level and a lower-level objective function. We also introduce the variable vector  $y_2$  to create an OPGV problem with two systems.

For  $k_1, k_2 > 0$  the objective function Hessian corresponding to (2) is positive definite and therefore the quadratic program is strictly convex. By changing  $n$ ,  $n_1$ , and  $n_2$ , we can choose the size of the test problem. Likewise, by changing the ratios  $n_1/n$  and  $n_2/n$ , the user can control the degree of coupling among the two systems that compose (2). Finally, different degrees of degeneracy can be obtained by careful choice of  $a$ .

Note that Problem 2 can be separated into  $n + 2$  independent problems. Each of the first  $n$  problems is formed by the objective function terms and the constraints that depend on the  $r$ th component of the vectors  $x, y_{11}$ , and  $y_{21}$ , which we denote as  $x_r, y_{11r}$ , and  $y_{21r}$ . We term these  $n$  problems *three-variable convex quadratic programs*, namely,

$$\begin{aligned}
\min_{x_r, y_{11r}, y_{21r}} \quad & \frac{1}{2}k_1(x_r - a)^2 + \frac{1}{2}k_2(y_{11r} - x_r)^2 + \\
& \frac{1}{2}k_1(x_r - a)^2 + \frac{1}{2}k_2(y_{21r} + x_r)^2 \\
s.t. \quad & 1 \leq x_r + y_{11r} \leq 2, \\
& x_r - y_{11r} \leq 1, \\
& 1 \leq -x_r + y_{21r} \leq 2, \\
& -x_r - y_{21r} \leq 1.
\end{aligned} \tag{3}$$

The last two problems that compose (2) are unconstrained quadratic programs formed by the objective function terms that depend only on  $y_{12}$  or  $y_{22}$ , namely,

$$\min_{y_{r2}} \frac{1}{2}\|y_{r2}\|^2, \quad r = 1, 2. \tag{4}$$

Although these unconstrained problems may seem trivial at first glance, in Section 4 we explain how a change of variables can be used to intertwine Problems 3 and 10 into a nonseparable test problem. Moreover, these unconstrained problems allow us to control the degree of coupling among systems by changing the dimension of  $y_{12}$  and  $y_{22}$ .

### 2.1.1 Minimizers

To find the minimizer of the convex quadratic programming test problem it suffices to find the minimizer of the  $n + 2$  problems that compose it. Since the minimizers of the two unconstrained problems (10) are obviously  $y_{12}^* = 0$  and  $y_{22}^* = 0$ , it only remains to calculate the minimizers of the three-variable convex quadratic program.

Provided  $k_1, k_2 > 0$ , (3) is a strictly convex quadratic program. Moreover, its objective function is nonnegative and hence bounded below. Therefore, for each  $a$ , there exists a unique minimizer of (3). This unique minimizer can be found by solving the KKT conditions. Here, we give the minimizer for  $a \geq 0$ . Because of the symmetry of the problem, the minimizer for  $a < 0$  is just  $(-x_r^*, y_{11r}^*, y_{21r}^*)$ , where  $(x_r^*, y_{11r}^*, y_{21r}^*)$  is the minimizer corresponding to  $|a|$ . We distinguish four cases:

**Case 1** ( $0 \leq a \leq 1/2 + 2k_2/k_1$ ): The active set is formed by the constraints  $x_r + y_{11r} = 1$  and  $-x_r + y_{21r} = 1$ . The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \frac{k_1}{k_1+4k_2}a \\ 1 - x_r^* \\ 1 + x_r^* \end{pmatrix}.$$

**Case 2** ( $1/2 + 2k_2/k_1 \leq a \leq 1 + 3k_2/k_1$ ): The active set is formed by the constraint  $-x_r + y_{21r} = 1$ . The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \frac{k_1 a - k_2}{k_1 + 2k_2} \\ x_r^* \\ 1 + x_r^* \end{pmatrix}.$$

**Case 3** ( $1 + 3k_2/k_1 \leq a \leq 3/2 + 5k_2/k_1$ ): The active set is formed by the constraints  $x_r + y_{11r} = 2$  and  $-x_r + y_{21r} = 1$ . The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \frac{k_1 a + k_2}{k_1 + 4k_2} \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}.$$

**Case 4** ( $3/2 + 5k_2/k_1 \leq a$ ): The active set is formed by the constraints  $x_r + y_{11r} = 2$ ,  $x_r - y_{11r} = 1$  and  $-x_r + y_{21r} = 1$ . The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0.5 \\ 2.5 \end{pmatrix}.$$

The set of minimizers of the three-variable convex quadratic program corresponding to  $a \in (-\infty, \infty)$  is depicted in Figure 1. The graph at the top represents  $y_{11r}^*$  as a function of  $x_r^*$  and the graph at the bottom represents  $y_{21r}^*$  as a function of  $x_r^*$ .

### 2.1.2 Degeneracy

The degree of degeneracy of the minimizer of (2) depends on the value of  $a$ . Provided  $n \geq 1$  and  $k_1, k_2 > 0$ , the following propositions give the set of values of  $a$  for which the LICQ, SCSC, and SOSC hold at the minimizer.

**Proposition 2.1** *The LICQ and SOSC hold at the unique minimizer of (2) for all  $a$ .*

*Proof:* It is easy to show from the structure of the active set at the minimizer of (3) that the LICQ holds. Also, if  $k_1, k_2 > 0$ , then the Hessian of the Lagrangian for Problem 2 is positive definite for all  $a$  and therefore the SOSC hold. ■

**Proposition 2.2** *The SCSC holds at the minimizer of (2) iff for  $i = 1:n$ ,  $a_i$  is not in the set  $\{\pm(1/2 + 2k_2/k_1), \pm(1 + 3k_2/k_1), \pm(3/2 + 5k_2/k_1)\}$ .*

*Proof:* This follows immediately from the KKT conditions of (3). ■

**Proposition 2.3** *The SLICQ holds at the minimizer of (2) iff for  $i = 1:n$ ,  $-3/2 - 5k_2/k_1 < a_i < 3/2 + 5k_2/k_1$ .*

*Proof:* This is obvious from the active set at the minimizer of (3). ■

## 2.2 Nonconvex Quadratic Programming OPGVs

We propose the following nonconvex quadratic programming OPGV:

$$\begin{aligned}
& \min_{x, y_1, y_2} && \frac{1}{2}k_1\|x - a\|^2 - \frac{1}{2}k_2\|y_{11} - (-x + be)\|^2 + \frac{1}{2}\|y_{12}\|^2 + \\
& && \frac{1}{2}k_1\|x - a\|^2 - \frac{1}{2}k_2\|y_{21} - (x + be)\|^2 + \frac{1}{2}\|y_{22}\|^2 \\
& s.t. && e \leq x + y_{11} \leq 2e, \\
& && x - y_{11} \leq e, \\
& && e \leq -x + y_{21} \leq 2e, \\
& && -x - y_{21} \leq e.
\end{aligned} \tag{5}$$

Note that the feasible regions of the convex and nonconvex quadratic programming test problems are identical. The nonconvex quadratic programming test problem is obtained from the convex test problem by replacing the objective function terms  $\frac{1}{2}k_2\|y_{11} - x\|^2 + \frac{1}{2}k_2\|y_{21} + x\|^2$  by the terms  $-\frac{1}{2}k_2\|y_{11} - (-x + be)\|^2 - \frac{1}{2}k_2\|y_{21} - (x + be)\|^2$ .

As in the convex case, the nonconvex quadratic programming test problem can be separated into  $n + 2$  independent problems. The first  $n$  problems

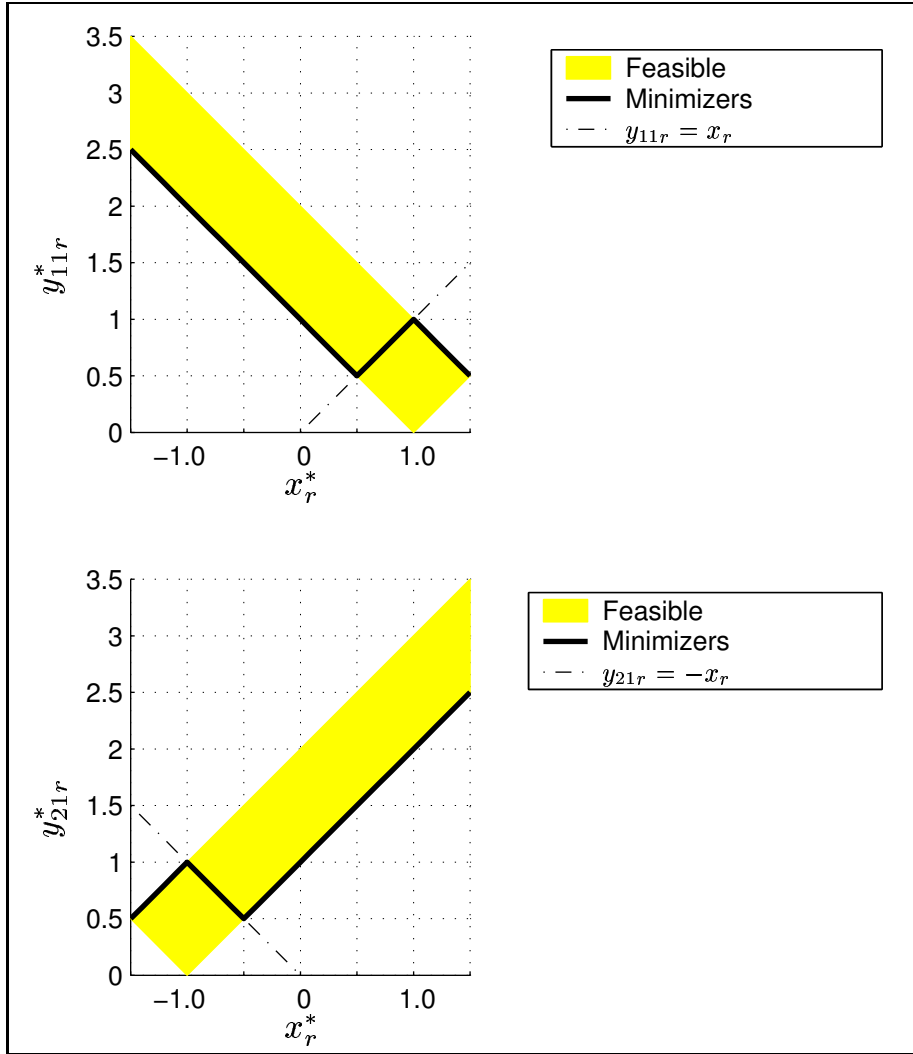


Figure 1: Minimizers to the three-variable convex quadratic program for  $a \in (-\infty, \infty)$ .

are termed *three-variable nonconvex quadratic programs* and can be written as follows:

$$\begin{aligned}
& \min_{x_r, y_{11r}, y_{21r}} \quad \frac{1}{2}k_1(x_r - a)^2 + \frac{1}{2}k_2(y_{11r} - (-x_r + b))^2 + \\
& \quad \frac{1}{2}k_1(x_r - a)^2 + \frac{1}{2}k_2(y_{21r} - (x_r + b))^2 \\
& \text{s.t.} \quad \begin{aligned} 1 &\leq x_r + y_{11r} \leq 2, \\ &x_r - y_{11r} \leq 1, \\ 1 &\leq -x_r + y_{21r} \leq 2, \\ &-x_r - y_{21r} \leq 1. \end{aligned} \tag{6}
\end{aligned}$$

The last two problems that compose the nonconvex quadratic programming test problem are the following unconstrained optimization problems:

$$\min_{y_{r2}} \frac{1}{2} \|y_{r2}\|^2, \quad r = 1, 2. \tag{7}$$

### 2.2.1 Minimizers

Since the minimizers of the two unconstrained problems (10) are obviously  $y_{12}^* = 0$  and  $y_{22}^* = 0$ , we only need to calculate the minimizers of the three-variable nonconvex quadratic program. Because of the symmetry of the problem it suffices to compute the local minimizers for  $a > 0$ . Here, we give the local minimizers for  $k_1 > 2k_2 > 0$  and  $b = 1.5$ . We distinguish five cases:

**Case 1** ( $0 \leq a \leq 1$ ): There exist four local minimizers that are also global:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} a \\ 1 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 1 - x_r^* \\ 2 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 2 + x_r^* \end{pmatrix}.$$

**Case 2** ( $1 < a \leq 1 + (b - 1)k_2/k_1$ ): There exist two global minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} a \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 2 + x_r^* \end{pmatrix},$$

and two local minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

**Case 3** ( $1 + (b - 1)k_2/k_1 \leq a < 1.25$ ): There exist two global minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} a \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 2 + x_r^* \end{pmatrix},$$

and two local minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \frac{k_1 a - (1+b)k_2}{k_1 - 2k_2} \\ x_r^* - 1 \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} \frac{k_1 a - (1+b)k_2}{k_1 - 2k_2} \\ x_r^* - 1 \\ 2 + x_r^* \end{pmatrix}.$$

**Case 4** ( $1.25 \leq a \leq 1.5$ ): There exist two global minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} a \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 2 + x_r^* \end{pmatrix}.$$

**Case 5** ( $1.5 \leq a$ ): There exist two global minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0.5 \\ 2.5 \end{pmatrix}, \begin{pmatrix} 1.5 \\ 0.5 \\ 3.5 \end{pmatrix}.$$

The set of minimizers of the three-variable nonconvex quadratic program for  $k_1 > 2k_2 > 0$  and  $b = 1.5$  is depicted in Figure 2 for  $a \in (-\infty, \infty)$ . The graph at the top represents  $y_{11r}^*$  as a function of  $x_r^*$  and the graph at the bottom represents  $y_{21r}^*$  as a function of  $x_r^*$ .

### 2.2.2 Degeneracy

Provided  $n \geq 1$ ,  $k_1 > 2k_2 > 0$  and  $b = 1.5$  the following propositions give the particular values of  $a$  for which the LICQ, SCSC, and SOSC hold at the minimizer of (5).

**Proposition 2.4** *The LICQ and SOSC hold at all local minimizers of (5) for all  $a$ .*

**Proposition 2.5** *The SCSC holds at all local minimizers of (5) iff for  $i = 1:n$ ,  $a_i$  is not in the set  $\{\pm 1, \pm 1.5\}$ .*

**Proposition 2.6** *The SLICQ holds at the minimizer of (2) iff for  $i = 1:n$ ,  $-3/2 < a_i < 3/2$ .*

## 3 Nonlinear Programming OPGVs

In some occasions, the quadratic programming nature of the test problems introduced so far might limit our ability to perform a fair comparison among different decomposition algorithms. This is the case when one of the decomposition algorithms compared has a relative advantage when applied to the solution of *quadratic* programs. To overcome this limitation, we introduce the following nonlinear programming OPGV:

$$\begin{aligned} \min_{x, y_1, y_2} \quad & \frac{1}{2}k_1 \|x - a\|^2 + \frac{1}{2}k_2 \|y_{11} - x\|^2 + \frac{1}{2}\|y_{12}\|^2 + \\ & \frac{1}{2}k_1 \|x - a\|^2 + \frac{1}{2}k_2 \|y_{21} - (6 - x)\|^2 + \frac{1}{2}\|y_{22}\|^2 \\ \text{s.t.} \quad & e \leq x \cdot y_{11} \leq 9e \\ & 4e \leq (6 - x) \cdot y_{11} \\ & e \leq (6 - x) \cdot y_{21} \leq 9e \\ & 4e \leq x \cdot y_{21} \end{aligned} \tag{8}$$



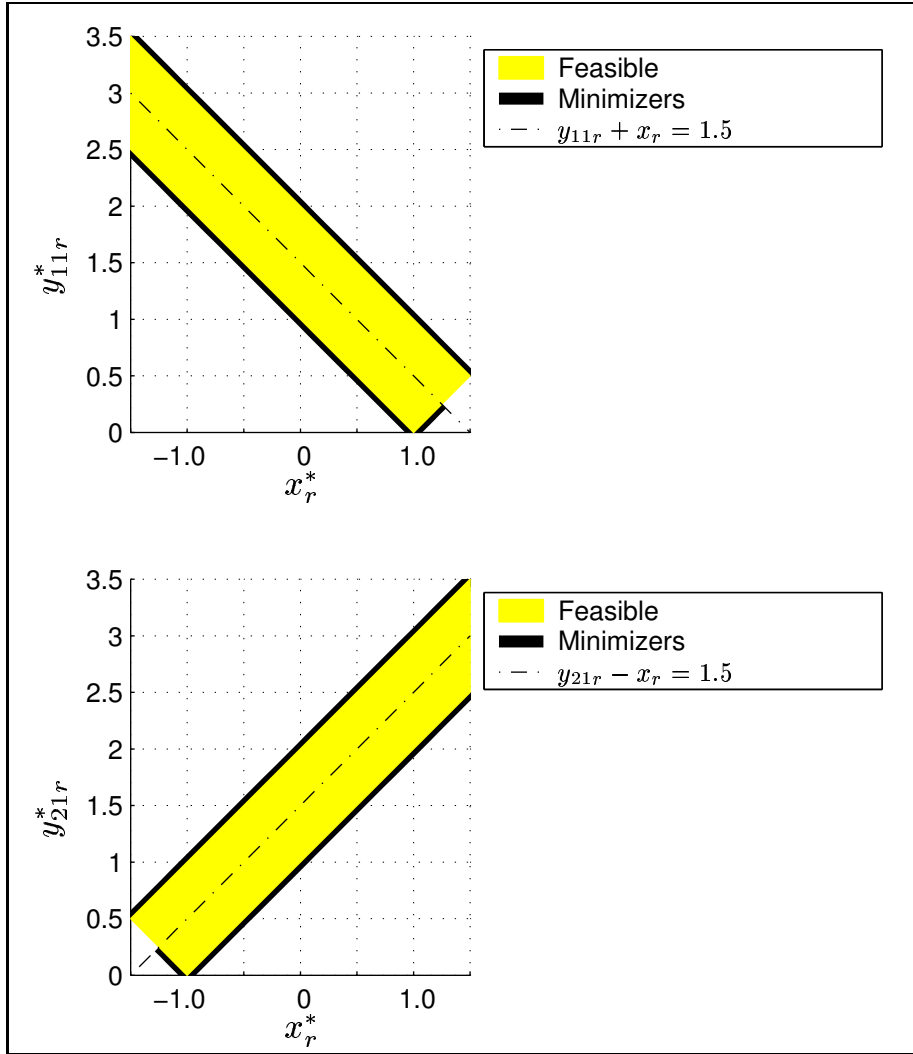


Figure 2: Minimizers to the three-variable nonconvex quadratic program for  $a \in (-\infty, \infty)$ .

where  $x \in \mathbb{R}^n$  are the global variables,  $y_i = (y_{i1}, y_{i2}) \in \mathbb{R}^{n_i}$  are the  $i$ th system local variables with  $y_{i1} \in \mathbb{R}^n$ ,  $y_{i2} \in \mathbb{R}^{n_i - n}$ ,  $k_1, k_2 \in \mathbb{R}$ , and  $e \in \mathbb{R}^n$  is the vector whose components are all ones. Note that the objective function of the nonlinear programming test problem proposed is quadratic. Moreover, its Hessian matrix is identical to that of the convex quadratic programming test problem proposed in Section 2.1. However, in the nonlinear program proposed, the linear constraints of the convex quadratic programming test problem are substituted by nonconvex nonlinear constraints.

As in the quadratic case, the nonlinear programming test problem can be separated into  $n + 2$  independent problems. The first  $n$  problems are termed *three-variable nonlinear programs* and can be written as follows:

$$\begin{aligned} \min_{x_r, y_{1r}, y_{2r}} \quad & \frac{1}{2}k_1(x_r - a)^2 + \frac{1}{2}k_2(y_{11r} - x_r)^2 + \\ & \frac{1}{2}k_1(x_r - a)^2 + \frac{1}{2}k_2(y_{21r} - (6 - x_r))^2 \\ \text{s.t.} \quad & 1 \leq x_r \cdot y_{11r} \leq 9 \\ & 4 \leq (6 - x_r) \cdot y_{11r} \\ & 1 \leq (6 - x_r) \cdot y_{21r} \leq 9 \\ & 4 \leq x_r \cdot y_{21r} \end{aligned} \quad (9)$$

The last two problems are the following unconstrained quadratic programs:

$$\min_{y_{r2}} \frac{1}{2} \|y_{r2}\|^2, \quad r = 1, 2. \quad (10)$$

### 3.0.3 Minimizers

It is easy to show that for  $k_1, k_2 > 0$ , there exists a unique minimizer to the three-variable nonlinear programming test problem. Here, we give the minimizer for  $a \geq 3$ . Because of the symmetry of the problem, the minimizer for  $a < 3$  is just  $(6 - x_r^*, y_{21r}^*, y_{11r}^*)$ , where  $(x_r^*, y_{11r}^*, y_{21r}^*)$  is the minimizer corresponding to  $6 - a$ . We distinguish three cases:

**Case 1** ( $a = 3$ ): The active set is formed by the constraints  $x \cdot y_{11} = 9$  and  $(6 - x) \cdot y_{21} = 9$ . The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \end{pmatrix}.$$

**Case 2** ( $3 < a \leq \frac{54}{13} + \frac{76415}{50544} \cdot \frac{k_2}{k_1}$ ): The only active constraint is  $x \cdot y_{11} = 9$ . The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \nu(a) \\ 9/x_r^* \\ 6 - x_r^* \end{pmatrix},$$

where

$$\nu(a) : \left[3, \frac{54}{13} + \frac{76415}{50544} \cdot \frac{k_2}{k_1}\right] \rightarrow \mathbb{R}$$

is a strictly increasing function such that  $\nu(3) = 3$  and

$$\nu\left(\frac{54}{13} + \frac{76415}{50544} \cdot \frac{k_2}{k_1}\right) = 54/13. \text{ }^3$$

**Case 3** ( $\frac{54}{13} + \frac{76415}{50544} \cdot \frac{k_2}{k_1} \leq a$ ): The active set is formed by the constraints  $x \cdot y_{11} = 9$  and  $(6 - x) \cdot y_{11} = 4$ . The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} 54/13 \\ 13/6 \\ 24/13 \end{pmatrix}.$$

The set of minimizers of the three-variable nonlinear program corresponding to  $a \in (-\infty, \infty)$  is depicted in Figure 3. The graph at the top represents  $y_{11r}^*$  as a function of  $x_r^*$  and the graph at the bottom represents  $y_{21r}^*$  as a function of  $x_r^*$ .

### 3.0.4 Degeneracy

The degree of degeneracy of the minimizer of (8) depends on the value of  $a$ . Provided  $n \geq 1$  and  $k_1, k_2 > 0$ , the following propositions give the set of values of  $a$  for which the LICQ, SCSC, and SOSC hold at the minimizer.

**Proposition 3.1** *The LICQ and SOSC hold at the unique minimizer of (8) for all  $a$ .*

**Proposition 3.2** *The SCSC holds at the minimizer of (2) iff for  $i = 1:n$ ,  $a_i$  is not in the set*

$$\left\{ \frac{24}{13} - \frac{76415}{50544} \cdot \frac{k_2}{k_1}, 3, \frac{54}{13} + \frac{76415}{50544} \cdot \frac{k_2}{k_1} \right\}.$$

**Proposition 3.3** *The SLICQ holds at the minimizer of (2) iff for  $i = 1:n$ ,*

$$\frac{24}{13} - \frac{76415}{50544} \cdot \frac{k_2}{k_1} < a_i < \frac{54}{13} + \frac{76415}{50544} \cdot \frac{k_2}{k_1}.$$

## 4 Nonseparable Test Problems

All test problems introduced so far can be separated into  $n + 2$  independent problems. The iterative procedure required to solve these separable test problems is numerically equivalent, for most algorithms, to the one needed to solve the  $n + 2$  problems independently. Therefore, to analyze how the performance of a decomposition algorithm depends on problem size, we need to modify our test problems so that they are not separable.

<sup>3</sup>An analytical expression of this function can be constructed. However due to its complexity we do not included here. Nevertheless, for each specific value of  $a$ ,  $\nu(a)$  can be computed by solving the KKT conditions of problem (9)

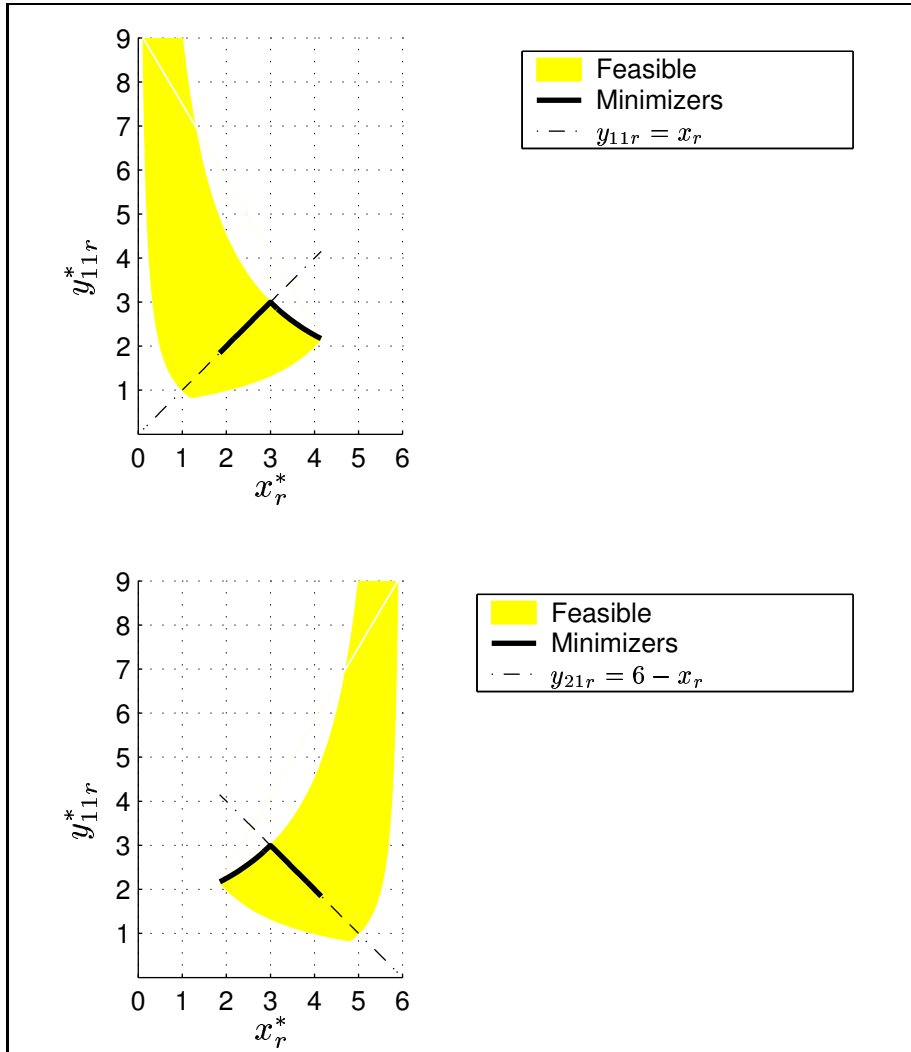


Figure 3: Minimizers to the three-variable nonlinear program for  $a \in (-\infty, \infty)$ .

Vicente and Calamai used a transformation matrix to obtain nonseparable test problems from their separable bilevel quadratic test problems. Here, we need to ensure that the test problems maintain the OPGV structure. The transformation we propose is

$$\begin{pmatrix} \hat{x} \\ \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} P_x & & \\ & P_{y_1} & \\ & & P_{y_2} \end{pmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix},$$

where  $P_x \in \mathbb{R}^{n \times n}$ ,  $P_{y_1} \in \mathbb{R}^{n_1 \times n_1}$ , and  $P_{y_2} \in \mathbb{R}^{n_2 \times n_2}$  are nonsingular. It is easy to show that the test problems in the variables  $(\hat{x}, \hat{y}_1, \hat{y}_2)$  are OPGVs. Moreover, the transformed test problems are not separable.

## 5 Conclusions

To the best of our knowledge, the test problem set introduced is the first to allow the generation of nonconvex nonlinear programming as well as convex and nonconvex quadratic programming OPGV test problems. Moreover, the user has control over crucial problem characteristics such as size, degeneracy, and degree of coupling among subproblems.

These test problems have been effectively used for the comparison of the inexact and exact penalty decomposition algorithms developed by De Miguel and Murray [dM01].

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