TECHNICAL REPORT SOL 01-2
Management Science and Engineering
Stanford University

A CLASS OF QUADRATIC PROGRAMMING TEST PROBLEMS WITH GLOBAL VARIABLES

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Abstract

A feature common to many optimization problems is a weak connectivity between component systems. One type of connectivity occurs when only a few of the variables, known as global variables, are relevant to all systems, while the remainder are local to a single component. We term these problems Optimization Problems with Global Variables (OPGVs). In this paper we introduce a new quadratic programming OPGV test problem set. The user can control problem characteristics such as dimension, convexity, degeneracy, and degree of coupling among systems. All local and global minimizers to the test problems are known a priori.

1 Introduction

Many optimization problems arising in business and engineering combine objective and constraint functions belonging to a set of weakly connected systems. In this paper we focus on problems for which only a few of the variables (known as global variables) affect the behaviour of all systems, while the remainder (known as local variables) are needed only within one of the systems. We term these problems Optimization Problems with Global Variables (OPGVs), namely,

$$\begin{array}{lrcl} \min\limits_{x,y_i} & \sum\limits_{i=1}^{N} F_i(x,y_i) & & \\ s.t. & c_i(x,y_i) & \geq & 0, \quad i = 1{:}N, \end{array} \tag{1}$$

where $x \in \mathcal{R}^n$ are the *global* variables, $y_i \in \mathcal{R}^{n_i}$ are the ith system *local* variables, $c_i(x,y_i) : \mathcal{R}^{n+n_i} \rightarrow \mathcal{R}^{m_i}$ are the ith system constraints, and $F_i(x,y_i) : \mathcal{R}^{n+n_i} \rightarrow \mathcal{R}$ is the objective function term corresponding to the ith system.

The structure of the OPGV suggests there might be strong computational and organizational advantages in the use of decomposition algorithms to solve it. Generalized Benders decomposition [Ben62, Geo72] is widely used to solve

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OPGVs whose objective and constraint functions are convex in the local variables. Several decomposition algorithms have been proposed for the nonconvex OPGV [Tam87, Bra96, dM01]. However, the development of these and other decomposition algorithms for the nonconvex OPGV has been hindered by the absence of a suitable suitable test-problem set. Although several OPGV test-problem sets [BDG+87], [Inf94, p. 47] have been developed in the context of the stochastic programming problem [BL97, Inf94], most of them correspond to linear or convex problems.

A more general test-problem set is the multidisciplinary design optimization test suite [PAG96]. For each test problem, a problem description, a benchmark solution method, sample input and output files, as well as source codes are available from the NASA Langley Research Center internet site. Test problems range from simple synthetic problems to some real engineering design problems. Unfortunately, the user has no control over important problem characteristics such as convexity and degree of degeneracy. Moreover, different test problems are given in different formats, and the implementation requires the modification of complicated FORTRAN source codes.

Easy-to-use nonconvex test-problem sets are available for several types of optimization problems related to the OPGV. Calamai and Vicente [CV94] developed a FORTRAN code to generate quadratic bilevel programs. The user can choose the test-problem size and the number of local and global minimizers. Moreover, all local and global minimizers are known a priori. Jiang and Ralph [JR99] developed a MATLAB code to generate mathematical programs with equilibrium constraints. Their test problems are more general than Calamai and Vicente's (which can be generated as a particular case) and the user can choose test-problem characteristics such as size, convexity, degeneracy, and ill conditioning. A disadvantage is that the minimizers of the test problems are not known in general.

In this paper, we modify Calamai and Vicente's test problems to create a quadratic programming OPGV test-problem set. The test problem objective and constraint functions can be evaluated using two MATLAB M-files available upon request. The user can choose the test-problem size, convexity, degeneracy, and degree of coupling. We calculate all local and global minimizers of the test-problem set and study their degree of degeneracy. Finally, the quadratic programming character of these test problems is not the limitation it may appear at first, because the master problem resulting from decomposition of a quadratic programming test problem is not, in general, a quadratic program.

2 A Convex Separable Test Problem

We propose the following convex quadratic programming OPGV:

$$
\begin{array}{lllll}
\min\limits_{x,y_1,y_2} & \multicolumn{4}{l}{\frac{1}{2}k_1\|x-a\|^2 + \frac{1}{2}k_2\|y_{11}-x\|^2 + \frac{1}{2}\|y_{12}\|^2 +} \\
 & \multicolumn{4}{l}{\frac{1}{2}k_1\|x-a\|^2 + \frac{1}{2}k_2\|y_{21}+x\|^2 + \frac{1}{2}\|y_{22}\|^2} \\
s.t. & e \;\le & x+y_{11} & \le & 2e, \\
 & & x-y_{11} & \le & e, \\
 & e \;\le & -x+y_{21} & \le & 2e, \\
 & & -x-y_{21} & \le & e,
\end{array}
\tag{2}
$$

where $x \in \mathcal{R}^n$ are the global variables, $y_i = (y_{i1}, y_{i2}) \in \mathcal{R}^{n_i}$ are the ith system local variables with $y_{i1} \in \mathcal{R}^n$, $y_{i2} \in \mathcal{R}^{n_i-n}$, $k_1, k_2 \in \mathcal{R}$, and $e \in \mathcal{R}^n$ is the vector whose components are all ones.

For $k_1, k_2 > 0$ the objective function Hessian corresponding to (2) is positive definite and therefore the quadratic program is strictly convex. By changing n, n_1, and n_2, we can choose the size of the test problem. Likewise, by changing the ratios n_1/n and n_2/n, the user can control the degree of coupling among the two systems that compose (2). Finally, different degrees of degeneracy can be obtained by careful choice of a.

Note that Problem 2 can be separated into $n+2$ independent problems. Each of the first n problems is formed by the objective function terms and the constraints that depend on the rth component of the vectors x, y_{11}, and y_{21}, which we denote as x_r, y_{11r}, and y_{21r}. We term these n problems *three-variable convex problems*, namely,

$$
\begin{array}{lllll}
\min\limits_{x_r,y_{11r},y_{21r}} & \multicolumn{4}{l}{\frac{1}{2}k_1(x_r-a)^2 + \frac{1}{2}k_2(y_{11r}-x_r)^2 +} \\
 & \multicolumn{4}{l}{\frac{1}{2}k_1(x_r-a)^2 + \frac{1}{2}k_2(y_{21r}+x_r)^2} \\
s.t. & 1 \;\le & x_r+y_{11r} & \le & 2, \\
 & & x_r-y_{11r} & \le & 1, \\
 & 1 \;\le & -x_r+y_{21r} & \le & 2, \\
 & & -x_r-y_{21r} & \le & 1.
\end{array}
\tag{3}
$$

The last two problems that compose (2) are unconstrained quadratic programs formed by the objective function terms that depend only on y_{12} or y_{22}, namely,

$$
\min_{y_{r2}} \frac{1}{2}\|y_{r2}\|^2, \qquad r = 1, 2. \tag{4}
$$

Although these unconstrained problems may seem trivial at first glance, in Section 4 we explain how a change of variables can be used to intertwine Problems 3 and 4 into a nonseparable test problem. Moreover, these unconstrained problems allow us to control the degree of coupling among systems by changing the dimension of y_{12} and y_{22}.

2.1 Minimizers

To find the minimizer of the convex test problem it suffices to find the minimizer of the $n+2$ problems that compose it. Since the minimizers of the two unconstrained problems (4) are obviously $y_{12}^* = 0$ and $y_{22}^* = 0$, it only remains to calculate the minimizers of the three-variable convex problem.

Provided $k_1, k_2 > 0$, (3) is a strictly convex quadratic program. Moreover, its objective function is nonnegative and hence bounded below. Therefore, for each a, there exists a unique minimizer of (3). This unique minimizer can be found by solving the KKT conditions. Here, we give the minimizer for $a \geq 0$. Because of the symmetry of the problem, the minimizer for $a < 0$ is just $(-x_r^*, y_{11r}^*, y_{21r}^*)$, where $(x_r^*, y_{11r}^*, y_{21r}^*)$ is the minimizer corresponding to $|a|$. We distinguish four cases:

Case 1 ($0 \leq a \leq 1/2 + 2k_2/k_1$): The active set is formed by the constraints $x_r + y_{11r} = 1$ and $-x_r + y_{21r} = 1$. The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \frac{k_1}{k_1+4k_2}a \\ 1 - x_r^* \\ 1 + x_r^* \end{pmatrix}.$$

Case 2 ($1/2 + 2k_2/k_1 \leq a \leq 1 + 3k_2/k_1$): The active set is formed by the constraint $-x_r + y_{21r} = 1$. The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \frac{k_1 a - k_2}{k_1+2k_2} \\ x_r^* \\ 1 + x_r^* \end{pmatrix}.$$

Case 3 ($1 + 3k_2/k_1 \leq a \leq 3/2 + 5k_2/k_1$): The active set is formed by the constraints $x_r + y_{11r} = 2$ and $-x_r + y_{21r} = 1$. The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \frac{k_1 a + k_2}{k_1+4k_2} \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}.$$

Case 4 ($3/2 + 5k_2/k_1 \leq a$): The active set is formed by the constraints $x_r + y_{11r} = 2$, $x_r - y_{11r} = 1$ and $-x_r + y_{21r} = 1$. The minimizer is

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0.5 \\ 2.5 \end{pmatrix}.$$

The set of minimizers of the three-variable convex problem corresponding to $a \in (-\infty, \infty)$ is depicted in Figure 1. The graph at the top represents y_{11r}^* as a function of x_r^* and the graph at the bottom represents y_{21r}^* as a function of x_r^*.

2.2 Degeneracy

The degree of degeneracy of the minimizer of (2) depends on the value of a. Provided $n \geq 1$ and $k_1, k_2 > 0$, the following propositions give the set of values of a for which the LICQ, SCSC, and SOSC hold at the minimizer.

Proposition 2.1 *The LICQ and SOSC hold at the unique minimizer of (2) for all a.*

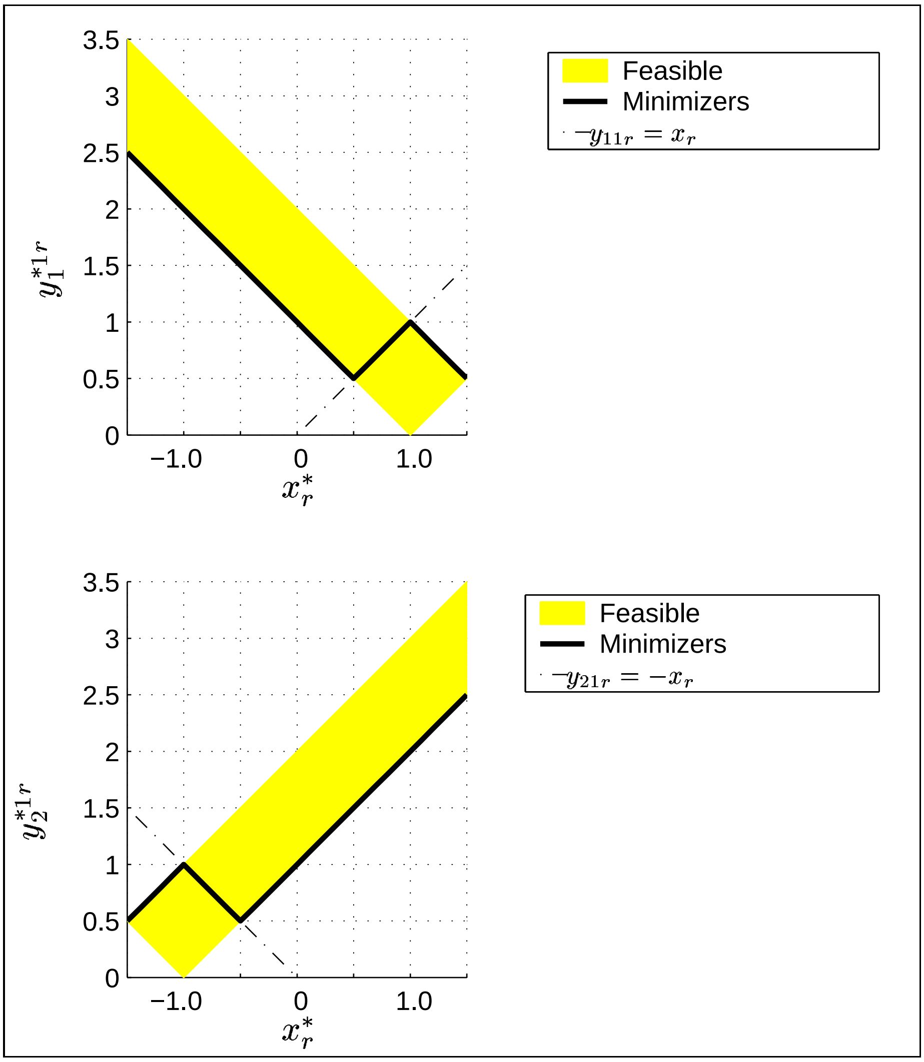

Figure 1: Minimizers to the three variable convex problem for $a \in (-\infty, \infty)$.

Proof: It is easy to show from the structure of the active set at the minimizer of (3) that the LICQ holds. Also, if $k_1, k_2 > 0$, then the Hessian of the Lagrangian for Problem 2 is positive definite for all a and therefore the SOSC hold. ■

Proposition 2.2 *The SCSC holds at the minimizer of (2) iff for $i = 1{:}n$, a_i is not in the set $\{1/2 + 2k_2/k_1, 1 + 3k_2/k_1, 3/2 + 5k_2/k_1\}$.*

Proof: This follows immediately from the KKT conditions of (3). ■

Proposition 2.3 *The SLICQ holds at the minimizer of (2) iff for $i = 1{:}n$, $a_i < 3/2 + 5k_2/k_1$.*

Proof: This is obvious from the active set at the minimizer of (3). ■

3 A Nonconvex Separable Test Problem

We propose the following nonconvex quadratic programming OPGV:

$$\begin{array}{llcccl} \min\limits_{x,y_1,y_2} & \multicolumn{5}{l}{\frac{1}{2}k_1\|x-a\|^2 - \frac{1}{2}k_2\|y_{11} - (-x+be)\|^2 + \frac{1}{2}\|y_{12}\|^2 +} \\ & \multicolumn{5}{l}{\frac{1}{2}k_1\|x-a\|^2 - \frac{1}{2}k_2\|y_{21} - (x+be)\|^2 + \frac{1}{2}\|y_{22}\|^2} \\ s.t. & e & \le & x + y_{11} & \le & 2e, \\ & & & x - y_{11} & \le & e, \\ & e & \le & -x + y_{21} & \le & 2e, \\ & & & -x - y_{21} & \le & e. \end{array} \tag{5}$$

Note that the feasible regions of the convex and nonconvex test problems are identical. The nonconvex test problem is obtained from the convex test problem by replacing the objective function terms $\frac{1}{2}k_2\|y_{11} - x\|^2 + \frac{1}{2}k_2\|y_{21} + x\|^2$ by the terms $-\frac{1}{2}k_2\|y_{11} - (-x+be)\|^2 - \frac{1}{2}k_2\|y_{21} - (x+be)\|^2$.

As in the convex case, the nonconvex test problem can be separated into $n+2$ independent problems. The first n problems are termed *three-variable nonconvex problems* and can be written as follows:

$$\begin{array}{llcccl} \min\limits_{x_r,y_{11r},y_{21r}} & \multicolumn{5}{l}{\frac{1}{2}k_1(x_r-a)^2 + \frac{1}{2}k_2(y_{11r} - (-x_r+b))^2 +} \\ & \multicolumn{5}{l}{\frac{1}{2}k_1(x_r-a)^2 + \frac{1}{2}k_2(y_{21r} - (x_r+b))^2} \\ s.t. & 1 & \le & x_r + y_{11r} & \le & 2, \\ & & & x_r - y_{11r} & \le & 1, \\ & 1 & \le & -x_r + y_{21r} & \le & 2, \\ & & & -x_r - y_{21r} & \le & 1. \end{array} \tag{6}$$

The last two problems that compose the nonconvex test problem are the following unconstrained optimization problems:

$$\min_{y_{r2}} \frac{1}{2}\|y_{r2}\|^2, \qquad r = 1,2. \tag{7}$$

3.1 Minimizers

Since the minimizers of the two unconstrained problems (4) are obviously $y_{12}^* = 0$ and $y_{22}^* = 0$, we only need to calculate the minimizers of the three-variable nonconvex problem. Because of the symmetry of the problem it suffices to compute the local minimizers for $a > 0$. Here, we give the local minimizers for $k_1 > 2k_2 > 0$ and $b = 1.5$. We distinguish five cases:

Case 1 ($0 \le a \le 1$): There exist four local minimizers that are also global:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} a \\ 1 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 1 - x_r^* \\ 2 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 2 + x_r^* \end{pmatrix}.$$

Case 2 ($1 < a \le 1 + (b-1)k_2/k_1$): There exist two global minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} a \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 2 + x_r^* \end{pmatrix},$$

and two local minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix}.$$

Case 3 ($1 + (b-1)k_2/k_1 \le a < 1.25$): There exist two global minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} a \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 2 + x_r^* \end{pmatrix},$$

and two local minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} \frac{k_1 a - (1+b)k_2}{k_1 - 2k_2} \\ x_r^* - 1 \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} \frac{k_1 a - (1+b)k_2}{k_1 - 2k_2} \\ x_r^* - 1 \\ 2 + x_r^* \end{pmatrix}.$$

Case 4 ($1.25 \le a \le 1.5$): There exist two global minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} a \\ 2 - x_r^* \\ 1 + x_r^* \end{pmatrix}, \begin{pmatrix} a \\ 2 - x_r^* \\ 2 + x_r^* \end{pmatrix}.$$

Case 5 ($1.5 \le a$): There exist two global minimizers:

$$\begin{pmatrix} x_r^* \\ y_{11r}^* \\ y_{21r}^* \end{pmatrix} = \begin{pmatrix} 1.5 \\ 0.5 \\ 2.5 \end{pmatrix}, \begin{pmatrix} 1.5 \\ 0.5 \\ 3.5 \end{pmatrix}.$$

The set of minimizers of the three-variable nonconvex problem for $k_1 > 2k_2 > 0$ and $b = 1.5$ is depicted in Figure 2 for $a \in (-\infty, \infty)$. The graph at the top represents y_{11r}^* as a function of x_r^* and the graph at the bottom represents y_{21r}^* as a function of x_r^*.

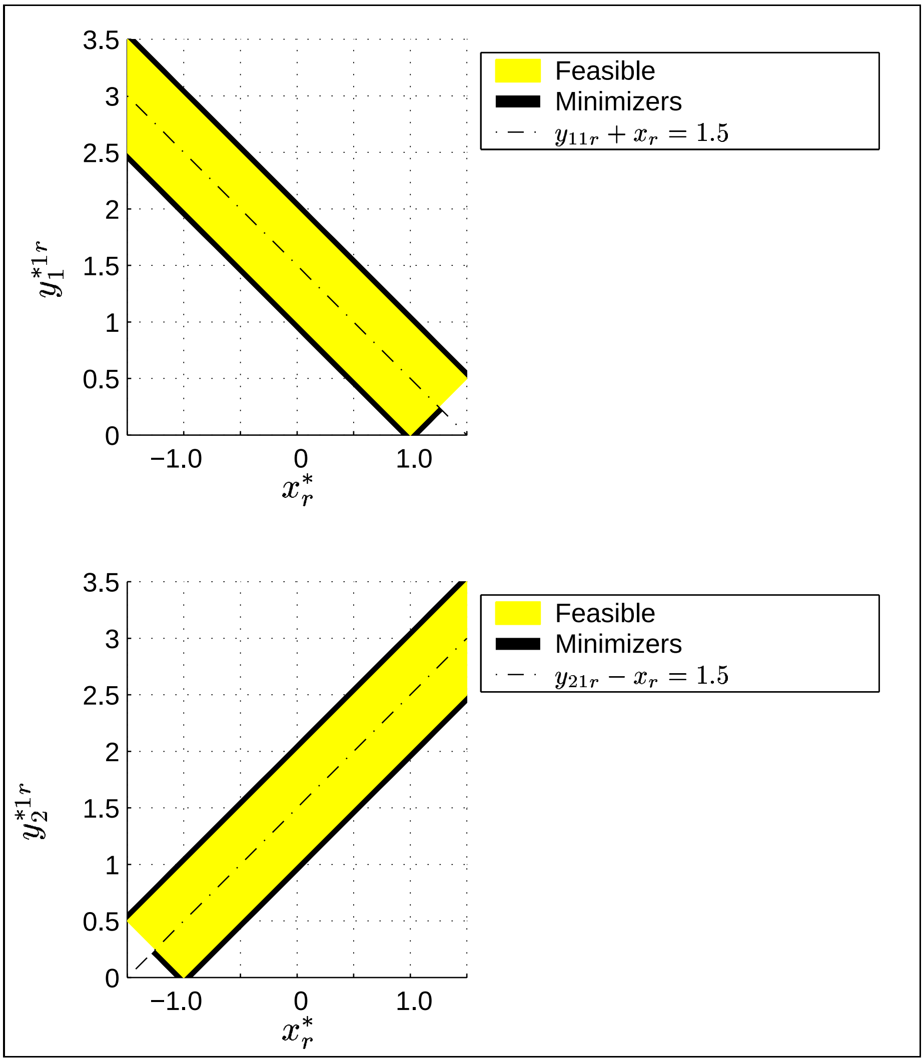

Figure 2: Minimizers to the three variable nonconvex problem for $a \in (-\infty, \infty)$.

3.2 Degeneracy

Provided $n \geq 1$, $k_1 > 2k_2 > 0$ and $b = 1.5$ the following propositions give the particular values of a for which the LICQ, SCSC, and SOSC hold at the minimizer of (5).

Proposition 3.1 *The LICQ and SOSC hold at all local minimizers of (5) for all a.*

Proof: It is easy to show from the structure of the active set at the minimizer of (6) that the LICQ holds. Likewise, if $k_1 > 2k_2 > 0$ and $b = 1.5$ it is easy to show that the SOSC are satisfied at all local minimizers. ■

Proposition 3.2 *The SCSC holds at all local minimizers of (5) iff for $i = 1{:}n$, a_i is not in the set $\{1, 1.5\}$.*

Proof: This follows immediately from the KKT conditions of (6). ■

Proposition 3.3 *The SLICQ holds at the minimizer of (2) iff for $i = 1{:}n$, $a_i < 3/2$.*

Proof: This is obvious from the active set at the minimizer. ■

4 A Nonseparable Test Problem

The convex and nonconvex test problems can be separated into $n + 2$ independent problems. The iterative procedure required to solve these separable test problems is numerically equivalent, for most algorithms, to the one needed to solve the $n + 2$ problems independently. Therefore, to analyze how the performance of a decomposition algorithm depends on problem size, we need to modify our test problems so that they are not separable.

Vicente and Calamai used a transformation matrix to obtain nonseparable test problems from their separable bilevel quadratic test problems. Here, we need to ensure that the test problems maintain the OPGV structure. The transformation we propose is

$$\begin{pmatrix} \hat{x} \\ \hat{y}_1 \\ \hat{y}_2 \end{pmatrix} = \begin{pmatrix} Q_x & & \\ & Q_{y_1} & \\ & & Q_{y_2} \end{pmatrix} \begin{pmatrix} x \\ y_1 \\ y_2 \end{pmatrix},$$

where Q_x, Q_{y_1}, and Q_{y_2} are randomly generated orthogonal matrices. It is easy to show that the test problems in the variables $(\hat{x}, \hat{y}_1, \hat{y}_2)$ are OPGVs. Moreover, the transformed test problems are not separable.

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