Revenue Management with Correlated Demand Forecasting

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1 Introduction

Many airlines are struggling to survive in today's economy. A crucial contribution to the success of an airline is an efficient management of its revenues. The problem underlying revenue management is for the airline to decide whether to accept customer requests for air travel on a real time basis in order to maximize its expected revenue. Typically, an airline offers several products (itinerary and fare class combinations) and operates a network of many resources (class–cabin in a given flight).

Demand forecasting is a crucial component of airline revenue management. Most of the papers in this area consider univariate demand which is then modelled with a statistical distribution, the normal and gamma distributions being the most popular (McGill and van Ryzin 1999; Lee 1990). However, in practice the available capacity is allocated dynamically between different products and thus is makes sense to take into account the potential correlations among product demand. At the same time, demand for any single product is recorded over the entire booking horizon at fixed time points (snapshots). In this framework it is natural to investigate whether there is correlation among realizations of demand for any individual product in different booking periods. Inter–product and inter–temporal correlations of demand have been documented empirically (McGill 1995). For efficiency reasons, it is therefore important to take them into account in a demand forecasting model.

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In this paper we propose a novel methodology for demand forecasting in the revenue management framework. Firstly, we develop a multivariate demand model that takes into account both the correlation between demand for different products offered by the airline and the correlation between demand during different time periods over the entire booking horizon. Secondly, we describe how this model can be estimated from historical data using the EM algorithm.

The paper is structured as follows: Section 2 describes the demand model and investigates the correlation structure. The estimation procedure is developed in Section 3, and Section 4 presents the results of simulation experiments designed to test the performance of the estimation algorithm. Several comments and directions for future research are outlined in the concluding Section 5.

2 The Forecasting Model

We consider an airline that offers n products. Time is discrete, indexed by $t = 1, \ldots, T$, where T is the number of periods between the time when a product is opened for booking and the time of service. Let $D(t) = (D_1(t), \ldots, D_n(t))'$ denote the random vector of demand in period t, where $D_i(t)$ is the demand for product i.

We assume that the random demand can be described through the following linear mixed effects model:

$$D(t) = a(t) + W(t) \cdot v + \varepsilon(t), \quad t = 1, \ldots T. \qquad (1)$$

Here $a(t) \in \Re^n$ is a vector of fixed effects representing the mean demand in period t. The random effects are modelled through $v \in \Re^n$, the realization of a common shock which influences demand in all periods. We assume that v has a multivariate normal distribution with covariance matrix $\mathbf{\Sigma}_v$ and zero mean, $v \sim \mathrm{N}_n(\mathbf{0},\ \mathbf{\Sigma}_v)$. The influence of the common random shock on the demand for each product in period t is weighted by the $n \times n$ diagonal matrix $W(t)$. For example, the random shock will affect the demand for economy fare classes to a larger extent at the beginning of the booking period than at the end. Similarly, the shock will have a larger influence on the business demand towards the time of service, rather than at the start of the booking period. The weighting matrices $W(t)$ are thus determined by the product definitions and we assume that they are known by the forecasters. The error terms $\varepsilon(t)$ are normally distributed $\varepsilon(t) \sim \mathrm{N}_n(\mathbf{0},\ \mathbf{I}_n\sigma_e^2)$ where $\mathbf{I}_n$ is the identity matrix of order n, so that in fact the components of the error vectors are independent. We also assume that $\varepsilon(t)$ are independent across time periods $t = 1, \ldots, T$ and independent of v.

With this specification, the random shock v induces correlation of demand both across different time periods and across different products within the same period. Indeed, conditionally on v the demand $D(t)$ has the $\mathrm{N}_n(a(t) + W(t)v,\ \mathbf{I}_n\sigma_e^2)$ distribution. Unconditionally, $D(t)$ is distributed as $\mathrm{N}_n(a(t),\ W(t)\mathbf{\Sigma}_v W(t) + \mathbf{I}_n\sigma_e^2)$, hence the components of demand for different products are correlated. Also, for any $t \neq s$ we have

$$\begin{aligned}\mathrm{Cov}(D(t), D(s)) &= \mathrm{Cov}(W(t)v,\ W(s)v) \\ &= W(t)\mathrm{E}[vv']W(s) \\ &= W(t)\mathbf{\Sigma}_v W(s). \qquad (2)\end{aligned}$$

Thus the demand vectors $D(t)$ and $D(s)$ for different time periods are correlated because they share the influence of the common random shock v. (To derive (2), recall that $W(t)$ is diagonal and hence $W(t) = W'(t)$ for all t. Note also that $\mathrm{E}[vv'] = \mathrm{Cov}(v, v) = \mathbf{\Sigma}_v$ since $\mathrm{E}[v] = 0$.)

For convenience, we state the model for the demand over all time periods as

$$D = a + \mathbf{W}v + \varepsilon, \qquad (3)$$

where $D, a, \varepsilon \in \Re^{nT}$ are obtained by concatenating the corresponding vectors from periods $t = 1, \ldots, T$, and $\mathbf{W}$ is the $nT \times n$ matrix with rows given by $W(1), \ldots, W(T)$. Note that $\varepsilon \sim \mathrm{N}_{nT}(0, \mathbf{I}_{nT}\sigma_e^2)$ and hence $D \sim \mathrm{N}_{nT}(a,\ \mathbf{W}\mathbf{\Sigma}_v\mathbf{W}' + \mathbf{I}_{nT}\sigma_e^2)$.

3 Model Estimation

Consider a random sample $D_1, \ldots, D_K$ of K independent realizations of the total demand vector. Each realization corresponds to a set of flights leaving on a particular day. Booking limits and aircraft capacity constraints cause censoring of the demand data (van Ryzin and McGill 2000), so that in practice only sales data are available. Denote by $S_1, \ldots, S_K$ the corresponding observed sales; these are censored realizations of demand, so that $D_{ki} \geq S_{ki}$ for all $k = 1, \ldots, K$ and $i = 1, \ldots, n$. Let δ_k be the vector of censoring indicators for D_k, defined as $\delta_{ki} = 1$ if $D_{ki} = S_{ki}$ and $\delta_{ki} = 0$ otherwise, for all $i = 1, \ldots, nT$.

Based on sales and censoring data, we are interested in estimating the fixed effects vector a, the covariance matrix $\mathbf{\Sigma}_v$ of the random shock v, and the error variance σ_e^2. Since v is unobservable and the observations $D_1, \ldots, D_K$ are potentially censored, the EM algorithm (Dempster, Laird and Rubin 1977) can be used to accommodate the missing data. This approach is similar to that of Smith and Helms (1995).

The EM algorithm is the classic tool for obtaining maximum likelihood estimates from incomplete or missing data. The complete data for model (3) consists of the realized values of the random shock $v_1, \ldots, v_k$ and the uncensored demand variables $D_1, \ldots, D_K$. The observed but incomplete data consists in the sales variables $S_1, \ldots, S_K$ and the censoring indicators $\delta_1, \ldots, \delta_K$. We assume that censoring is observable, i.e. the booking limits are known for each product and time period. The EM algorithm iterates between two steps: the expectation (E) step computes expected values of the sufficient statistics for the complete data conditional on the observed data and current values of the parameters. In the maximization (M) step, new estimates of the unknown parameters are obtained by maximizing the likelihood computed with the expected values of the sufficient statistics from the previous E–step.

The joint distribution of the uncensored demand and the random shock is given by

$$\begin{bmatrix} D_k - \mathbf{a} \\ v_k \end{bmatrix} \sim \mathrm{N}_{(n+1)T}(0, \mathbf{\Sigma}),$$

where $\mathbf{\Sigma} = \begin{bmatrix} \mathbf{W}\mathbf{\Sigma}_v\mathbf{W}' + \sigma_e^2\mathbf{I}_{nT} & \mathbf{W}\mathbf{\Sigma}_v \\ \mathbf{\Sigma}_v\mathbf{W}' & \mathbf{\Sigma}_v \end{bmatrix}$.

The logarithm of the complete data likelihood function has the following component of interest:

$$\begin{aligned} -K \log \mid \mathbf{\Sigma} \mid &- \sum_{k=1}^{K} \begin{bmatrix} D_k - a \\ v_k \end{bmatrix}' \mathbf{\Sigma}^{-1} \begin{bmatrix} D_k - a \\ v_k \end{bmatrix} \\ &= -KnT \log(\sigma_e^2) - K \log \mid \mathbf{\Sigma}_v \mid \\ &- \sum_{k=1}^{K} [(D_k - a - \mathbf{W}v_k)'(D_k - a - \mathbf{W}v_k)\frac{1}{\sigma_e^2} \\ &\quad + v_k'\mathbf{\Sigma}_v^{-1}v_k], \end{aligned} \tag{4}$$

where $\mid \mathbf{\Sigma} \mid = (\sigma_e^2)^{nT} \cdot \mid \mathbf{\Sigma}_v \mid$ and

$$\mathbf{\Sigma}^{-1} = \begin{bmatrix} \frac{1}{\sigma_e^2}\mathbf{I}_{nT} & -\frac{1}{\sigma_e^2}\mathbf{W} \\ -\frac{1}{\sigma_e^2}\mathbf{W}' & \mathbf{\Sigma}_v^{-1} + \frac{1}{\sigma_e^2}\mathbf{W}'\mathbf{W} \end{bmatrix}.$$

Taking partial derivatives of (4) and setting them equal to zero, it follows that the estimates of a, $\mathbf{\Sigma}_v$, σ_e^2 and $\{v_k\}$ which maximize the likelihood function satisfy the following equations:

$$\widehat{a} = \frac{1}{K} \sum_{k=1}^{K} (D_k - \mathbf{W}\widehat{v}_k) \tag{5}$$

$$\widehat{\mathbf{\Sigma}}_v = \frac{1}{K} \sum_{k=1}^{K} \widehat{v}_k \widehat{v}_k' \tag{6}$$

$$\widehat{\sigma}_e^2 = \frac{1}{KnT} \sum_{k=1}^{K} (D_k - \widehat{a} - \mathbf{W}\widehat{v}_k)'(D_k - \widehat{a} - \mathbf{W}\widehat{v}_k) \tag{7}$$

$$\widehat{v}_k = (\mathbf{W}'\mathbf{W} + \widehat{\sigma}_e^2 \widehat{\mathbf{\Sigma}}_v^{-1})^{-1}\mathbf{W}'(D_k - \widehat{a}), \tag{8}$$

for $k = 1, \ldots, K$. These are the underlying equations for the EM algorithm.

The algorithm starts with a set of initial estimates $a^{(0)}$, $\mathbf{\Sigma}_v^{(0)}$, $\sigma_e^{2(0)}$ and $\{v_k^{(0)}\}$, which may be obtained by using mixed model techniques and treating censored observations as ignorably missing. At the rth iteration, the E–step computes the expected values of the sufficient statistics for the complete data, conditional on the observed data $\{\mathbf{S}_k, \delta_k\}$ and on the estimated values of the parameters from the $(r-1)$st iteration. From (5)–(8) it follows that a set of sufficient statistics for $\mathbf{a}$, σ_e^2 and $\{v_k\}$ are given by $\sum_{k=1}^{K}(D_k - \mathbf{W}\widehat{v}_k)$, $\sum_{k=1}^{K}(D_k - \widehat{a} - \mathbf{W}\widehat{v}_k)'(D_k - \widehat{a} - \mathbf{W}\widehat{v}_k)$, and $\{\mathbf{W}'(D_k - \widehat{a})\}_k$, respectively. Let $\boldsymbol{\theta} = \{a, \mathbf{\Sigma}_v, \sigma_e^2, \{v_k\}\}$ and let $\boldsymbol{\theta}^{(r)}$ be the value of $\boldsymbol{\theta}$ after the rth iteration. We denote $\widetilde{D}_k^{(r)} = E[D_k \mid S_k, \delta_k, \boldsymbol{\theta}^{(r)}]$. Then the expected values of the sufficient statistics for $\mathbf{a}$, $\{v_k\}$, and σ_e^2 at the rth iteration are determined by $t_1^{(r)}$, $\{t_{2k}^{(r)}\}$ and $t_3^{(r)}$, respectively, where

$$\begin{aligned} t_1^{(r)} &= \mathrm{E}[\sum_{k=1}^{K}(D_k - \mathbf{W}\widehat{v}_k) \mid S_k, \delta_k, \boldsymbol{\theta}^{(r-1)}] \\ &= \sum_{k=1}^{K}(\widetilde{D}_k^{(r-1)} - \mathbf{W}v_k^{(r-1)}), \end{aligned} \tag{9}$$

$$\begin{aligned} t_{2k}^{(r)} &= \mathrm{E}[\mathbf{W}'(D_k - \widehat{a}) \mid S_k, \delta_k, \boldsymbol{\theta}^{(r-1)}] \\ &= \mathbf{W}'(\widetilde{D}_k^{(r-1)} - a^{(r-1)}), \quad k = 1, \ldots, K \end{aligned} \tag{10}$$

and

$$\begin{aligned} t_3^{(r)} &= \mathrm{E}[\sum_{k=1}^{K}(\widehat{ER}_k' \cdot \widehat{ER}_k \mid S_k, \delta_k, \boldsymbol{\theta}^{(r-1)}] \qquad (11) \\ &= \sum_{k=1}^{K}\{\mathrm{E}[D_k'D_k \mid S_k, \delta_k, \boldsymbol{\theta}^{(r-1)}] \\ &\quad -2(a^{(r-1)} + \mathbf{W}v_k^{(r-1)})'\widetilde{D}_k^{(r-1)} \\ &\quad +(a^{(r-1)} + \mathbf{W}v_k^{(r-1)})'(a^{(r-1)} + \mathbf{W}v_k^{(r-1)})\}, \end{aligned}$$

where we have denoted $\widehat{ER}_k = D_k - \widehat{a} - \mathbf{W}\widehat{v}_k$ for $k = 1, \ldots, K$. In order to compute these expectations we need to evaluate $\widetilde{D}_k^{(r-1)} = E[D_k \mid S_k, \delta_k, \boldsymbol{\theta}^{(r)}]$ and

$\mathrm{E}[D_k' D_k \mid S_k, \delta_k, \boldsymbol{\theta}^{(r-1)}]$. Conditional on $\boldsymbol{\theta}^{(r-1)}$, D_k has the $\mathrm{N}_{nT}(a^{(r-1)} + \mathbf{W}v_k^{(r-1)},\ \sigma_e^{2(r-1)}\mathbf{I}_{nT})$ distribution, thus the components of D_k are independent and $D_{kj} \mid \boldsymbol{\theta}^{(r-1)} \sim N(a_j^{(r-1)} + (\mathbf{W}v_k^{(r-1)})_j,\ \sigma_e^{2(r-1)})$, for $j = 1, \ldots, nT$.

If $\delta_{kj} = 1$ then D_{kj} is observed and $D_{kj} = S_{kj}$, thus $\widetilde{D}_{kj}^{(r)} = E[D_{kj} \mid S_{kj}, \theta^{(r)}] = S_{kj}$ and $\mathrm{E}[D_{kj}^2 \mid S_{kj}, \theta^{(r)}] = S_{kj}^2$. If $\delta_{kj} = 0$ then D_{kj} is censored and $D_{kj} > S_{kj}$, thus

$$\begin{aligned}\widetilde{D}_{kj}^{(r)} &= E[D_{kj} \mid D_{kj} > S_{kj}, \theta^{(r)}] \qquad (12)\\ &= a_j^{(r)} + (\mathbf{W}v_k^{(r)})_j + \sigma_e^{(r)} \cdot \frac{\phi(z_{kj}^{(r)})}{1 - \Phi(z_{kj}^{(r)})},\end{aligned}$$

where $\phi(\cdot)$ and $\Phi(\cdot)$ are the standard normal density and cumulative density functions, and

$$z_{kj}^{(r)} = \frac{S_{kj} - (a_j^{(r)} + (\mathbf{W}v_j^{(r)})_j)}{\sigma_e^{(r)}}.$$

This follows from standard results for the truncated normal distribution (Johnson et al. (1994), p.156–162). Note also that

$$\mathrm{E}[D_k' D_k \mid S_k, \delta_k, \boldsymbol{\theta}^{(r-1)}] = \sum_{j=1}^{nT} \mathrm{E}[D_{kj}^2 \mid S_k, \delta_k, \boldsymbol{\theta}^{(r-1)}] \qquad (13)$$

and

$$\begin{aligned}\mathrm{E}[D_{kj}^2 \mid D_{kj} > S_{kj}, \theta^{(r)}] &= (a_j^{(r)} + (\mathbf{W}v_k^{(r)})_j)^2 + \sigma_e^{2(r)}\\ &+ \sigma_e^{(r)} \cdot \frac{\phi(z_{kj}^{(r)})}{1 - \Phi(z_{kj}^{(r)})} \cdot (S_{kj} + a_j^{(r)} + (\mathbf{W}v_k^{(r)})_j).\end{aligned}$$

Replacing the expressions for $\widetilde{D}_{kj}^{(r)}$ and $\mathrm{E}[D_{kj}^2 \mid S_{kj}, \theta^{(r)}]$ in (9)–(11), we obtain the expected values of the sufficient statistics for $\mathbf{a}$, $\{v_k\}$, and σ_e^2.

At the rth iteration of the M–step, new estimates $a^{(r)}$, $\boldsymbol{\Sigma}_v^{(r)}$, $\sigma_e^{2(r)}$, and $\{v_k^{(r)}\}$ are computed from (5) – (8) using the values of $t_1^{(r)}$, $\{t_{2k}^{(r)}\}$ and $t_3^{(r)}$ derived in the previous iteration of the E–step:

$$\begin{aligned}a^{(r)} &= \frac{1}{K} t_1^{(r)}\\ v_k^{(r)} &= (\mathbf{W}'\mathbf{W} + \sigma_e^{2(r-1)}(\boldsymbol{\Sigma}_v^{(r-1)})^{-1})^{-1} \cdot t_{2k}^{(r)},\\ \boldsymbol{\Sigma}_v^{(r)} &= \frac{1}{K}\sum_{k=1}^{K} v_k^{(r)} v_k^{(r)\prime}\\ \sigma_e^{2(r)} &= \frac{1}{TnK} t_3^{(r)}.\end{aligned}$$

The algorithm then iterates until convergence.

4 Simulation Experiments

The performance of the EM algorithm described in the previous section has been investigated through a series of simulation experiments. The objective was to examine the effects of demand correlation, degree of censorship and length of booking horizon on the bias and consistency of estimators and on the convergence of the EM algorithm.

We considered a subset of $n = 2$ products offered by an airline, corresponding to two fare classes (business and economy) on the same itinerary. The total flight capacity was fixed at 100 seats, with 30 seats allocated to the first fare class and 70 seats allocated to the second.

We simulated the demand for the two products over a time horizon with $T = 4$, 6, and 8 snapshots. The mean demand $a(t) = (a_1(t), a_2(t))$ was uniform across all periods. The demand variances σ_{v1}^2 and σ_{v2}^2 were chosen such that the coefficient of variation is 0.4, and the correlation between demand for the two products was given values of $\rho = 0.2$ and 0.4. These choices for the coefficient of variation and for the correlation coefficient are typical values in airline demand modelling (McGill 1995; McGill and van Ryzin 1999). The error standard deviation was $\sigma_e^2 = 0.8$.

With the parameters specified above we generated $K = 100$ realizations of demand from model (1), corresponding to a hundred different departure dates.

These demand paths were then censored in order to obtain the sales data, in such a manner that the percentage of censorship had values $C = 0\%$, 20% and 40%. The censoring indicators for each demand value were also recorded at this stage. The EM algorithm as described in Section 3 was then used to estimate the parameters of the demand distribution from sales and censoring indicators. The convergence tolerance of the algorithm was set at 0.001, leading to stability to the third significant figure in the log–likelihood.

The results of the simulations are summarized in tables 1 and 2. The values reported in the tables are the empirical bias and mean squared error of the parameter estimates evaluated from 100 iterations; the values for the estimates of a_1 and a_2 are averages over all time periods.

The bias for the estimated parameters is mostly negative, showing that the EM algorithm slightly underestimates the true values. The bias and mean squared error for all parameters are generally increasing with the percentage of censoring C, and decreasing with the number of time periods T, as expected. This simply reflects the fact that more information in the sample naturally leads to better estimates. It should be noted that the bias is larger for the estimates of the random effect variances and of the error variance, partly due to the relatively moderate sample size $K = 100$. By contrast, the estimates of ρ are quite precise.

The EM algorithm has notoriously slow convergence. However, it should be noted that in the simulation scenarios that we considered the algorithm converged rather rapidly — a running time of ten seconds is a loose upper bound for all instances. The actual convergence speed decreased both with increasing percentage of censorship C and with increasing demand correlation ρ.

5 Conclusions

We have investigated in this paper a novel multivariate model for airline demand forecasting. The model is flexible enough to allow correlation both between different products offered by the airline and between several time periods over the booking horizon. In practice, historical data consist not of actual demand but of observed sales which represent demand censored by capacity constraints. Any methodology for estimating the model from historical data needs to take into account the censored nature of demand. We have proposed the use of the EM algorithm for estimating the model parameters within the maximum likelihood framework, and showed how this approach allows uncensoring of the demand data, while at the same time naturally accommodating the random effects features of the model.

The model can be extended to take into account overbooking and cancellations over the booking period. At the same time, this forecasting model may be complemented by a multistage programming approach to generating bid prices (Talluri and van Ryzin 1999; Williamson 1992). These and other generalizations are the subject of future research.

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Table 1: Empirical bias and mean squared error (MSE) for EM estimates based on 100 iterations. $K = 100$ samples of demand generated under model (1) with $\rho = 0.4$.

	Censoring		a_1	a_2	σ_{v1}	σ_{v2}	ρ_v	σ_e
$T = 4$	$C = 0\%$	Bias	-0.038	-0.015	-0.623	-0.171	0.018	-0.156
		MSE	0.105	0.029	3.022	0.133	0.009	0.044
	$C = 20\%$	Bias	-0.244	-0.080	-2.115	-0.469	0.044	-0.187
		MSE	0.149	0.035	6.801	0.394	0.031	0.062
	$C = 40\%$	Bias	-0.613	-0.194	-4.442	-0.887	0.090	-0.207
		MSE	0.471	0.067	21.473	0.954	0.065	0.091
$T = 6$	$C = 0\%$	Bias	0.013	-0.013	-0.107	-0.132	0.041	-0.135
		MSE	0.044	0.017	0.808	0.040	0.012	0.019
	$C = 20\%$	Bias	-0.127	-0.055	-0.883	-0.232	0.043	-0.160
		MSE	0.089	0.020	1.170	0.075	0.013	0.027
	$C = 40\%$	Bias	-0.225	-0.085	-1.709	-0.358	0.060	-0.181
		MSE	0.111	0.023	3.277	0.152	0.029	0.039
$T = 8$	$C = 0\%$	Bias	-0.006	0.004	-0.055	-0.120	0.065	-0.101
		MSE	0.037	0.013	0.156	0.028	0.018	0.011
	$C = 20\%$	Bias	-0.048	-0.047	-0.434	-0.183	0.068	-0.121
		MSE	0.044	0.015	0.369	0.046	0.028	0.017
	$C = 40\%$	Bias	-0.152	-0.059	-0.763	-0.228	0.136	-0.131
		MSE	0.057	0.017	0.722	0.064	0.061	0.023

Table 2: Empirical bias and mean squared error (MSE) for EM estimates based on 100 iterations. $K = 100$ samples of demand generated under model (1) with $\rho = 0.2$

	Censoring		a_1	a_2	σ_{v1}	σ_{v2}	ρ_v	σ_e
$T = 4$	$C = 0\%$	Bias	-0.022	0.000	-0.834	-0.245	-0.009	-0.210
		MSE	0.121	0.030	2.300	0.191	0.012	0.045
	$C = 20\%$	Bias	-0.290	-0.054	-2.458	-0.449	0.005	-0.230
		MSE	0.198	0.029	7.686	0.287	0.010	0.054
$T = 6$	$C = 0\%$	Bias	-0.023	-0.019	-0.080	-0.156	0.013	-0.134
		MSE	0.054	0.019	0.612	0.054	0.014	0.019
	$C = 20\%$	Bias	-0.160	-0.039	-0.729	-0.215	0.027	-0.160
		MSE	0.091	0.018	0.956	0.062	0.012	0.027
$T = 8$	$C = 0\%$	Bias	0.014	-0.006	-0.122	-0.105	0.009	-0.100
		MSE	0.044	0.016	0.217	0.018	0.016	0.011
	$C = 20\%$	Bias	-0.097	-0.042	-0.421	-0.142	0.019	-0.130
		MSE	0.051	0.016	0.296	0.031	0.022	0.018

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