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Electronic Companion—“Supply Chain Competition with Multiple Manufacturers and Retailers” by Elodie Adida and Victor DeMiguel, *Operations Research*, doi 10.1287/opre.1100.0863.

Material for on-line appendix

Figure 1 Supply chain efficiency as a function of the degree of retailer differentiation

This figure depicts the efficiency of the decentralized chain as a function of the degree of retailer differentiation. We consider a case with a single product, at least two (differentiated) retailers, and $H = 1$. The vertical axis gives the supply chain efficiency. The horizontal axis gives the value of the degree of retailer differentiation $\nu = H - G = 1 - G$ between 0 and 1. The different lines give the efficiency for a variety of cases with different number of retailers (N) and manufacturers (M).

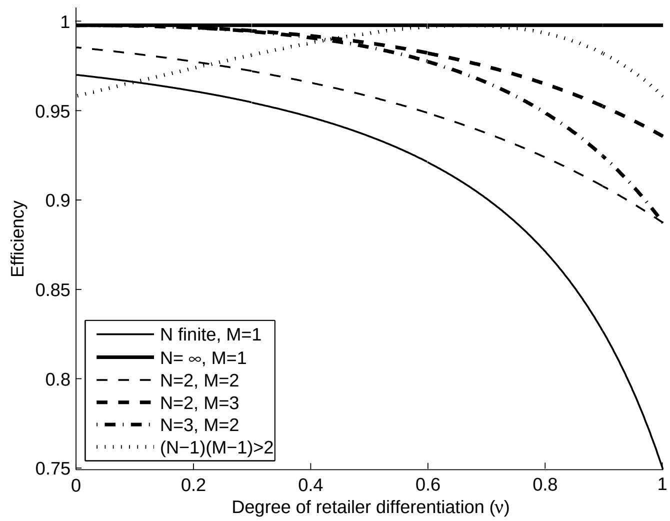


Figure 2 Efficiency with asymmetric manufacturers and retailers

The plot on the left depicts the supply chain efficiency for a supply chain with two products, two symmetric nondifferentiated retailers, and four asymmetric manufacturers—the first three manufacturers' production cost is 0.1 for every product, whereas the fourth manufacturer's production cost for every product varies from 0 to 0.5. The horizontal axis gives the cost of the fourth manufacturer and the vertical axis gives the efficiency. A vertical dashed line indicates the cost of the three symmetric manufacturers. The plot on the right depicts the supply chain efficiency for a supply chain with two products, two symmetric manufacturers, and four asymmetric retailers—the first three retailers' parameter η is 0.9, whereas the fourth retailer's parameter η_4 varies between 0.5 and 1. The horizontal axis gives the fourth retailer's parameter η_4 and the vertical axis gives the efficiency. A vertical dashed line indicates the parameter of the three symmetric retailers.

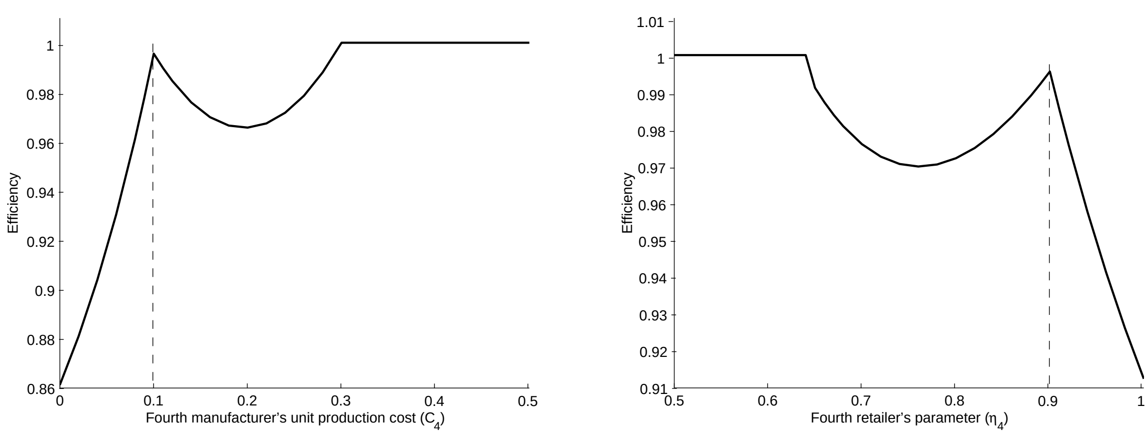


Figure 3 Effect of asymmetric product assortments

The figure depicts the aggregate supply quantity for each product, the manufacturer profit, the expected retailer utility, the expected supply chain utility, the expected retail price, the wholesale price, and the efficiency for a supply chain with two products, three symmetric manufacturers, two nondifferentiated retailers for the cases with asymmetric and symmetric product assortments—in the case with asymmetric assortment the first retailer carries product 1 only, and the second retailer carries product 2 only, while in the case with symmetric assortment both retailers may carry both products.

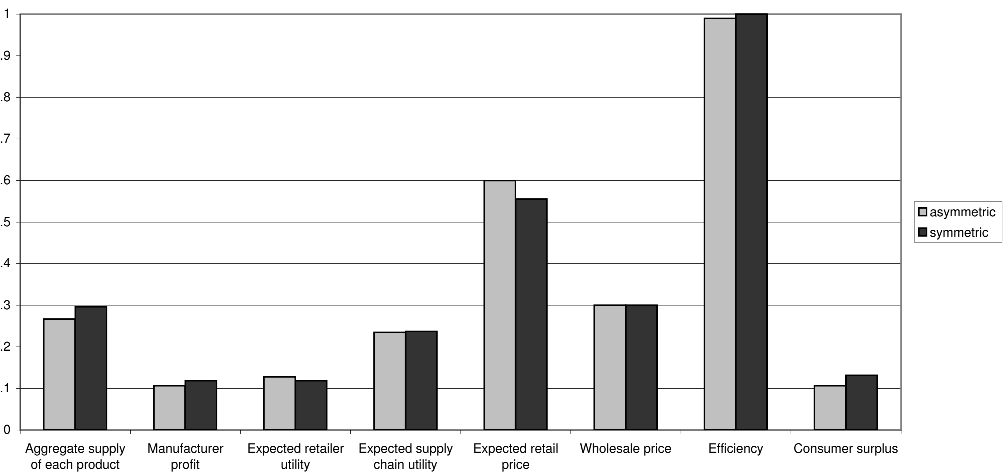
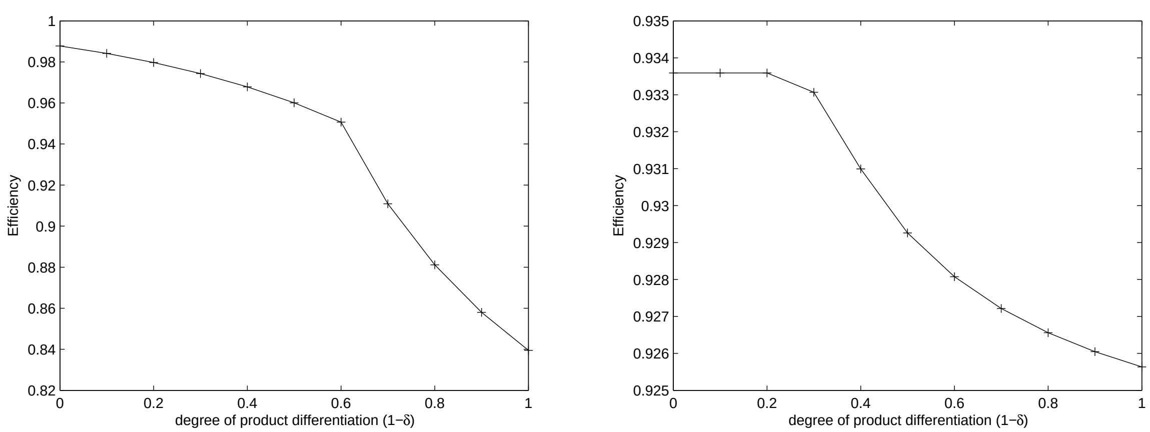


Figure 4 Effect of product differentiation with asymmetric retailers or manufacturers

The plot on the left depicts the supply chain efficiency for an asymmetric supply chain with two products, two asymmetric manufacturers, and two non differentiated symmetric retailers. The plot on the right depicts the supply chain efficiency for an asymmetric supply chain with two products, two symmetric manufacturers, and two non differentiated asymmetric retailers. In both plots, the horizontal axis gives the degree of product differentiation and the vertical axis gives the supply chain efficiency. The value $1-\delta=0$ corresponds to non differentiated products.



Appendix A: A premier on complementarity problems

The *complementarity problem* (CP) consists in finding a vector $x \in \mathbb{R}^n$ that satisfies the following system of equalities and inequalities:

$$\begin{aligned} x^T F(x) &= 0, \\ x &\geq 0, \\ F(x) &\geq 0, \end{aligned}$$

where $F(x) : \mathbb{R}^n \to \mathbb{R}^n$ is a vector-valued function. Note that the fact that both x and $F(x)$ must be nonnegative, and their scalar product $x^T F(x)$ must be zero, implies that for each component $j = 1, 2, \ldots, n$, we must have that either x_j or $(F(x))_j$ must be zero. The CP is often denoted as

$$0 \leq x \perp F(x) \geq 0.$$

A *linear complementarity problem* (LCP) is a CP where $F(.)$ is a linear function: $F(x) = Mx + q$, where $M \in \mathbb{R}^{n\times n}$ and $q \in \mathbb{R}^n$. The LCP can thus be denoted as

$$0 \leq x \perp Mx + q \geq 0. \tag{15}$$

It is well-known that under certain conditions linear and quadratic programs can be formulated as LCPs; see Cottle et al. [2009]. Moreover, the first-order conditions for Nash games can often be rewritten as LCPs. For instance, we show that the retailers equilibrium conditions can be rewritten as an LCP.

The following proposition gives a sufficient condition for the existence of a unique solution to the LCP.

Lemma A.1. If M is positive definite, then the LCP has a unique solution.

Proof. It is easy to show that if the matrix M is positive definite, then x is a solution to the LCP if an only if it is a minimizer to the following quadratic program:

$$\begin{aligned} \min_x \quad & x^T(Mx + q), \\ \text{s.t.} \quad & Mx + q \geq 0, \\ & x \geq 0. \end{aligned}$$

Moreover, the positive definiteness of M implies that the feasible region of this quadratic program is nonempty. Furthermore, given that the feasible region is nonempty, the positive definiteness of M implies that there exists a unique solution to the quadratic program, and hence to the LCP. □

A *mathematical program with equilibrium constraints* (MPEC) is an optimization problem that includes as part of its constraints a complementarity problem; see Luo et al. [1996] for an in-depth treatment of MPECs. An MPEC can be written as follows:

$$\begin{aligned} \min_x \quad & r(x) \\ \text{subject to} \quad & c_i(x) \geq 0 \\ & c_e(x) = 0 \\ & 0 \leq x \perp F(x) \geq 0. \end{aligned}$$

MPECs can be used to model Stackelberg games, where a leader makes a decision anticipating the equilibrium reached by a set of followers, which compete with each other. We use the MPEC to model the decision of each manufacturer, who anticipates the equilibrium reached by the retailers, who are the followers.

Finally, an *equilibrium problem with equilibrium constraints* (EPEC) is a mathematical program whose solution is the equilibrium point that simultaneously solves a set of MPECs. For instance, an EPEC can be used to represent the manufacturer-retailer equilibrium in our paper, as each manufacturer decision problem can be written as an MPEC, and their simultaneous equilibrium is an EPEC. Equilibrium problems with equilibrium constraints can be solved using numerical methods as the one proposed by Hu [2002].

Appendix B: Auxiliary results

This appendix contains the statements and proofs of some auxiliary results used in the paper.

LEMMA B.1. Let the matrix $\hat{T} \in \mathbb{R}^{NP \times NP}$ be

$$\hat{T} = \begin{pmatrix} 2T & T & \dots & T \\ T & 2T & \dots & T \\ \vdots & \vdots & \ddots & \vdots \\ T & T & \dots & 2T \end{pmatrix},$$

where $T \in \mathbb{R}^{P \times P}$. The matrix $\hat{T}$ defined is positive definite if and only if matrix T is positive definite.

Proof: A matrix is positive definite if and only if its symmetric part is positive definite. Hence without loss of generality, we assume that $\hat{T}$ and T are symmetric. Assume that $\hat{T}$ is positive definite. By definition $\hat{x}^T \hat{T} \hat{x} > 0 \ \ \forall \hat{x} \neq 0 \in \mathbb{R}^{NP}$. Let $x \neq 0 \in \mathbb{R}^P$, and let $\hat{x} = (x, 0, \dots, 0) \in \mathbb{R}^{NP}$.

$$\hat{x}^T \hat{T} \hat{x} = 2x^T T x > 0,$$

where the last strict inequality follows from the fact that $\hat{T}$ is positive definite. Therefore, T is positive definite.

For the reverse, assume that T is positive definite; then all its eigenvalues are strictly positive. Any eigenvalue λ of $\hat{T}$ associated with eigenvector $\hat{x} \neq 0 \in \mathbb{R}^{NP}$ such that $\hat{x} = (x_1, \dots, x_N)$ satisfies

$$2Tx_i + \sum_{j \neq i} Tx_j = \lambda x_i, \quad i = 1, \dots, N.$$

Summing these equalities over $i = 1, \dots, N$ yields $(N+1)T \sum_i x_i = \lambda \sum_i x_i$. Thus if $\sum_i x_i \neq 0$, then $\lambda \frac{1}{N+1}$ is an eigenvalue of T, which implies $\lambda > 0$. Otherwise, consider i such that $x_i \neq 0$ (there must exist at least such i since $\hat{x} \neq 0$). We have $\sum_{j \neq i} x_j = -x_i$ and thus $2Tx_i + -Tx_i = \lambda x_i$ which imply that λ is an eigenvalue of T associated with eigenvector x_i. Thus $\lambda > 0$. It follows that $\hat{T}$ is positive definite. □

The following proposition states a useful consequence of Assumption 3.2.

PROPOSITION B.2. Let Assumption 3.2 hold, then the matrix D is positive definite.

Let $\hat{x} = (x, x, \ldots, x)$. Then $\hat{x}^T \hat{B} \hat{x} = N x^T (H + (N-1)G) x$. Moreover, by Assumption 3.2 we know that $\hat{B}$ is positive definite. Thus for any $x \neq 0$, we have that $x^T (H + (N-1)G) x > 0$. □

The following proposition states some useful implications of Assumption 3.7.

Proposition B.3. Let Assumptions 3.1 and 3.7 hold, then

1. the matrix H is positive definite,
2. the matrix $B = (2H + (N-1)G)/(N+1)$ is positive definite.

Part 1. Let $\hat{x} = (x, 0, \ldots, 0)$. Then $\hat{x}^T (\hat{B} + \hat{H}) \hat{x} = 2x^T H x$. Because by Assumption 3.7, $\hat{B} + \hat{H}$ is positive definite, for any $x \neq 0$, we must have $2x^T H x > 0$.

Part 2. Let $\hat{x} = (x, x, \ldots, x)$. Then, $\hat{x}^T (\hat{B} + \hat{H}) \hat{x} = N x^T (2H + (N-1)G) x$. Because by Assumption 3.7 we know that $\hat{B} + \hat{H}$ is positive definite, this implies that for any $x \neq 0$, we have that $x^T (2H + (N-1)G) x > 0$. □

Proposition B.4. Let Assumption 5.1 hold, then the matrices H_1, H_2, …, H_N are positive definite.

Let $\hat{x} = (0, \ldots, 0, x, 0, \ldots, 0)$. Then $\hat{x}^T (\hat{B} + \hat{H}) \hat{x} = 2x^T H_i x$. Because by Assumption 5.1, $\hat{B} + \hat{H}$ is positive definite, for any $x \neq 0$, we must have $2x^T H_i x > 0$, which implies that H_i is positive definite. □

Appendix C: Proof for the results in the paper

Proof of Theorem 3.4

Part 1 and 2. The decision problem in a centralized supply chain with a risk averse central planner is

$$\max_{\hat{x} \geq 0} \eta \hat{x}^T \left(\hat{a} - \hat{B} \hat{x} \right) - \hat{c}^T \hat{x}, \tag{16}$$

where $\hat{c} = (c, c, \ldots, c)$. By Assumption 3.2, matrix $\hat{B}$ is positive definite, and therefore there exists a unique solution to the central planner decision problem, which is the solution to the following LCP:

$$0 \leq -\hat{a} + 2\hat{B}\hat{x} + \frac{\hat{c}}{\eta} \perp \hat{x} \geq 0. \tag{17}$$

Because the assumptions of Proposition B.2 in Appendix B hold, the matrix D is positive definite and thus invertible. Moreover, because Assumption 3.3 holds, we have that all the components of the vector $D^{-1}(a - c/\eta)$ are strictly positive. Then it is easy to see that the unique solution to the central planner problem is given by $\hat{x} = (x, x, \ldots, x)$, where

$$x = \frac{1}{2N} D^{-1} \left(a - \frac{c}{\eta} \right).$$

Part 3. The expected retail price at any retailer is given by the linear inverse demand function $E[p] = a - NDx$. Using the result in Part 1 to substitute x yields the result.

Part 4. The central planner expected utility can be rewritten as $CTU = \eta N x^T (a - NDx - c)$. Using the result in Part 1 to substitute x yields

$$CTU = \eta N \frac{1}{2N} \left(a - \frac{c}{\eta} \right)^T D^{-T} \left(a - c - ND \frac{1}{2N} D^{-1} \left(a - \frac{c}{\eta} \right) \right).$$

The result is obtained by straightforward simplification.

Part 5. From [Vives 1999, Exercise 6.9], the consumer utility that corresponds to the linear demand considered is

$$U(\hat{x}) = \hat{a}^T \hat{x} - \frac{1}{2} \hat{x}^T \hat{B} \hat{x}$$

therefore, the expected consumer surplus is

$$CS = U(\hat{x}) - \hat{x}^T E[\hat{p}] = \hat{a}^T \hat{x} - \frac{1}{2} \hat{x}^T \hat{B} \hat{x} - \hat{x}^T (\hat{a} - \hat{B}\hat{x}) = \frac{1}{2} \hat{x}^T \hat{B} \hat{x}$$

In the symmetric supply chain the expected consumer surplus simplifies to $N^2 x^T D x/2$. The result follows by replacing x with its expression given in Part 1. □

Proof of Proposition 3.6

Part 1. We have assumed that H is positive definite. Therefore the matrix $\hat{B}$ is positive semidefinite. As a result, the central planner decision problem in (16) is convex and a supply schedule is a global maximizer to problem (16) if and only if it satisfies the first-order optimality conditions in (17). For the case without retailer differentiation we have that $H = G$. Moreover, by Assumption 3.1 the supply chain is symmetric, and thus the LCP in (17) is equivalent to the following set of LCPs:

$$0 \leq -a + 2H \sum_{k=1}^{N} x_k + \frac{c}{\eta} \perp x_j \geq 0 \quad \forall j. \tag{18}$$

Note that Assumption 3.3 implies that $H^{-1}(a - c/\eta) > 0$, and therefore it is easy to see that the supply schedule

$$x_j = x = \frac{1}{2N} H^{-1} \left(a - \frac{c}{\eta} \right) \quad \forall j,$$

satisfies (18) and hence is a global maximizer of the central planner problem. Moreover, it is easy to see that there are other asymmetric supply schedules that will lead to identical total supply and identical central planner expected utility.

Parts 2, 3, and 4. These parts follow trivially by using the result in Part 1 to substitute x in the expressions given in Parts 3, 4, and 5 of Theorem 3.4.

Proof of Proposition 3.8

Part 1. For the symmetric supply chain, the jth retailer decision given in equation (2) can be rewritten as:

$$\max_{x_j \geq 0} \quad -x_j^T v + \eta x_j^T (a - H x_j - G \sum_{l \neq j} x_l). \tag{19}$$

Note that because Assumption 3.7 holds, by Proposition B.3 in Appendix B we have that the matrix H is positive definite, and thus for any wholesale price (v) and for any order quantities from the other retailers (x_l for $l \neq j$), there exists a unique maximizer to problem (19). Moreover, from Lemma A.1 we know that the unique maximizer coincides with the unique solution to the following LCP:

$$0 \leq \frac{v}{\eta} - (a - 2H x_j - G \sum_{l \neq j} x_l) \perp x_j \geq 0.$$

In other words, for any v and x_l for $l \neq j$, there exists a unique jth retailer best response.

Hence, for given v, the retailer order vector $\hat{x}=(x_1,x_2,\ldots,x_N)$ is a retailers equilibrium if and only if it solves the following LCP, which is obtained by concatenating the LCPs characterizing the best response of the N retailers:

$$0\leq \hat{q}+(\hat{B}+\hat{H})\hat{x}\perp\hat{x}\geq 0, \tag{20}$$

where $\hat{q}=(v/\eta-a, v/\eta-a,\ldots,v/\eta-a)$. Note that by Assumption 3.7, $\hat{B}+\hat{H}$ is positive definite and thus the LCP in (20) has a unique solution (see Appendix A), which is the unique retailers equilibrium.

Part 2. Because the retailers equilibrium is unique and the game is symmetric with respect to all retailers, the equilibrium must be symmetric. Indeed, if the equilibrium was not symmetric, because the game is symmetric with respect to all retailers, it would be possible to permute the strategies among the retailers and obtain a different equilibrium, hereby contradicting the uniqueness of the equilibrium.

Part 3. Because the assumptions of Proposition B.3 in Appendix B hold, we know that the matrix $2H+(N-1)G$ is positive definite and thus the LCP in (6) has a unique solution. Moreover, it is easy to see that the vector $\hat{x}=(x,x,\ldots,x)$, where x is the unique solution to the LCP in (6), solves the LCP in (20) and thus is the unique retailers equilibrium. □

Proof of Proposition 3.10
Given that $B^{-1}\Big(a-v/\eta\Big)>0$, it is easy to see that x given by (7) is the unique solution to the LCP in (6). Then, the result follows from Proposition 3.8. □

Statement of Theorem C.1

THEOREM C.1. Let Assumptions 3.1, 3.7, and 3.9 hold, then

1. there exists a unique manufacturer retailer equilibrium,
2. at equilibrium every manufacturer supplies the same quantity y, every retailer orders the same quantity x, and the total supply is

$$\text{Total supply}=My=Nx=\frac{MN}{(M+1)(N+1)}B^{-1}\Big(a-\frac{c}{\eta}\Big),$$

3. the wholesale market price is

$$v=\frac{\eta}{M+1}\Big(a+\frac{Mc}{\eta}\Big),$$

4. the expected retail price at every retailer is

$$E[p]=a-\frac{MN}{(M+1)(N+1)}DB^{-1}\Big(a-\frac{c}{\eta}\Big),$$

5. the decentralized manufacturers profit is

$$DMP=\eta\frac{MN}{(M+1)^2(N+1)}\Big(a-\frac{c}{\eta}\Big)^TB^{-1}\Big(a-\frac{c}{\eta}\Big),$$

6. the decentralized retailers utility is

$$DRU=\eta\frac{M^2N}{(M+1)^2(N+1)}\Big(a-\frac{c}{\eta}\Big)^T\Big(I-\frac{N}{N+1}B^{-T}D\Big)B^{-1}\Big(a-\frac{c}{\eta}\Big),$$

7. the decentralized supply chain total expected utility is

$$DTU = \eta\frac{MN}{(M+1)(N+1)}\Big(a-\frac{c}{\eta}\Big)^T\Big(I-\frac{MN}{(M+1)(N+1)}B^{-T}D\Big)B^{-1}\Big(a-\frac{c}{\eta}\Big).$$

8. the decentralized expected consumer surplus is

$$CS = \frac{1}{2}\frac{M^2N^2}{(M+1)^2(N+1)^2}\Big(a-\frac{c}{\eta}\Big)^T B^{-T}DB^{-1}\Big(a-\frac{c}{\eta}\Big).$$

Proof of Theorems 3.11 and C.1

Part 1. Under Assumption 3.9, we have from Proposition 3.10 that the ith manufacturer decision problem can be rewritten as

$$\max_{y_i\geq 0,\ v} \quad v^T y_i - c^T y_i \tag{21}$$

$$\text{s.t.} \quad \sum_{k=1}^{M} y_k = \frac{N}{N+1}B^{-1}\Big(a-\frac{v}{\eta}\Big). \tag{22}$$

From the equality constraint we know that

$$v = \eta\left[a-\frac{N+1}{N}B\sum_{k=1}^{M}y_k\right], \tag{23}$$

and hence the ith manufacturer decision is

$$\max_{y_i\geq 0} \quad \eta\left[a-\frac{N+1}{N}B\sum_{k=1}^{M}y_k\right]^T y_i - c^T y_i.$$

Because the assumptions in Proposition B.3 hold, we have that B is positive definite and thus the ith manufacturer decision has a unique maximizer that is given by the first-order optimality conditions:

$$0 \leq -\eta\left[a-\frac{N+1}{N}B\sum_{k=1}^{M}y_k\right] + \eta\frac{N+1}{N}By_i + c \perp y_i \geq 0. \tag{24}$$

Concatenating the optimality conditions for all M manufacturers we have that the supply vector $\hat{y} = (y_1, y_2, \ldots, y_M)$ is an equilibrium if and only if it is the solution to the following LCP:

$$0 \leq \hat{r} + \hat{F}\hat{y} \perp \hat{y} \geq 0, \tag{25}$$

where $\hat{r} = (\frac{c}{\eta}-a, \frac{c}{\eta}-a, \ldots, \frac{c}{\eta}-a)$, and

$$\hat{F} = \frac{N+1}{N}\begin{pmatrix} 2B & B & \ldots & B \\ B & 2B & \ldots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \ldots & 2B \end{pmatrix}.$$

Moreover, since B is positive definite, Lemma B.1 implies that $\hat{F}$ is positive definite. Thus there exists a unique manufacturer retailer equilibrium.

Part 2. Because the manufacturers equilibrium is unique and the game is symmetric with respect to all manufacturers, the equilibrium must be symmetric. Indeed, if the equilibrium was not symmetric, because the game is symmetric with respect to all manufacturers, it would be possible to permute the strategies among the manufacturers and obtain a different equilibrium, hereby contradicting the uniqueness of the equilibrium. Therefore at the equilibrium $\hat{y}=(y,y,\dots,y)$. Constraint (22) thus implies

$$My = \frac{N}{N+1}B^{-1}\Big(a - \frac{v}{\eta}\Big).$$

Therefore, under Assumption 3.9, $My > 0$ and hence $y > 0$. It follows that the nonnegativity constraint in problem (21) is redundant and can be ignored. Therefore y is obtained by solving the first order optimality conditions of the manufacturers decision problem

$$-\eta\left(a - \frac{N+1}{N}MBy\right) + \eta\frac{N+1}{N}By + c = 0.$$

It follows that

$$y = \frac{N}{(M+1)(N+1)}B^{-1}\Big(a - \frac{c}{\eta}\Big) \tag{26}$$

and thus $\hat{y}=(y,y,\dots,y)$ is the unique solution to the LCP in (25). By adding the supply quantities of all manufacturers we get that

$$\text{Total supply} = My = \frac{MN}{(M+1)(N+1)}B^{-1}\Big(a - \frac{c}{\eta}\Big). \tag{27}$$

Moreover, we also know by Proposition 3.10 that at the unique equilibrium all retailers order the same quantity x. Because the wholesale market clearing conditions must hold at the unique equilibrium, we must then have that

$$Nx = \frac{MN}{(M+1)(N+1)}B^{-1}\Big(a - \frac{c}{\eta}\Big). \tag{28}$$

Part 3. Substituting (26) into (23), we have that

$$v = \eta\left[a - \frac{N+1}{N}B\frac{MN}{(M+1)(N+1)}B^{-1}\Big(a - \frac{c}{\eta}\Big)\right].$$

Simplifying we have

$$v = \eta\left[a - \frac{M}{M+1}\Big(a - \frac{c}{\eta}\Big)\right].$$

The result follows by simple algebra.

Part 4. The linear inverse demand function implies that at every retailer the expected retail price is $E[p] = a - (H + (N-1)G)x$. Using the result in Part 2, we have that

$$E[p] = a - (H + (N-1)G)\frac{M}{(M+1)(N+1)}B^{-1}\Big(a - \frac{c}{\eta}\Big).$$

The result follows because by definition $D = (H + (N-1)G)/N$.

Part 5. The decentralized manufacturers profit is

$$\begin{aligned} DMP &= M(v^T y - c^T y) = \frac{M}{M+1}(\eta a - c)^T \frac{N}{(M+1)(N+1)} B^{-1}\Big(a - \frac{c}{\eta}\Big) \\ &= \eta \frac{MN}{(M+1)^2(N+1)} \Big(a - \frac{c}{\eta}\Big)^T B^{-1}\Big(a - \frac{c}{\eta}\Big) \end{aligned}$$

Part 6. The decentralized retailers utility is

$$\begin{aligned} DRU &= -Nv^T x + N\eta x^T (a - Hx - (N-1)Gx) \\ &= N\eta\Big(a - \frac{v}{\eta}\Big)^T x - N^2 \eta x^T D x \\ &= \eta \frac{MN}{M+1}\Big(a - \frac{c}{\eta}\Big)^T x - N^2 \eta x^T D x \\ &= \eta \frac{M^2 N}{(M+1)^2(N+1)}\Big(a - \frac{c}{\eta}\Big)^T B^{-1}\Big(a - \frac{c}{\eta}\Big) - \eta \frac{M^2 N^2}{(M+1)^2(N+1)^2}\Big(a - \frac{c}{\eta}\Big)^T B^{-T} D B^{-1}\Big(a - \frac{c}{\eta}\Big) \\ &= \eta \frac{M^2 N}{(M+1)^2(N+1)}\Big(a - \frac{c}{\eta}\Big)^T \Big(I - \frac{N}{N+1} B^{-T} D\Big) B^{-1}\Big(a - \frac{c}{\eta}\Big) \end{aligned}$$

Part 7. The result follows by adding the expressions for the decentralized manufacturers profit and the decentralized retailers utility provided in the two previous parts.

Part 8. From [Vives 1999, Exercise 6.9], it is easy to show that for the linear demand considered and in the symmetric supply chain the expected consumer surplus is $N^2 x^T D x/2$. The result follows by replacing x with its expression given in Part 2. □

Proof of Proposition 3.12

The result follows trivially from Theorem 3.11 because for the case without retailer differentiation we have that the three matrices B, D, and H are identical. □

Proof of Proposition 3.13

The expression of the supply chain efficiency given in (8) follows trivially from Propositions 3.6 and 3.12.

We now prove the monotonicity results shown in Table 2. The **total supply** is equal to

$$\frac{MN}{(M+1)(N+1)} H^{-1}\Big(a - \frac{c}{\eta}\Big).$$

Because $H^{-1}(a - c/\eta) > 0$, and since $M/(M+1) = 1 - 1/(M+1)$ is increasing in M and $N/(N+1) = 1 - 1/(N+1)$ is increasing in N, it follows that the total supply is strictly increasing in the number of manufacturers M and in the number of retailers N.

The **wholesale market price** is

$$v = \frac{\eta}{M+1}\Big(a + \frac{Mc}{\eta}\Big) = \frac{\eta a - c}{M+1} + c,$$

thus it is independent of the number of retailers N. Because we have assumed that $a - c/\eta > 0$, the wholesale price is strictly decreasing in the number of manufacturers M.

The **expected retail price** at every retailer is

$$E[p] = a - \frac{MN}{(M+1)(N+1)}\Big(a - \frac{c}{\eta}\Big).$$

Because $M/(M+1)$ is increasing in M, $N/(N+1)$ is increasing in N, and $a-c/\eta>0$, it follows that the expected retail price is strictly decreasing in the number of manufacturers M and in the number of retailers N.

The **manufacturers profit** is

$$DMP = \eta\frac{MN}{(M+1)^2(N+1)}\theta.$$

We know that $N/(N+1)$ is increasing in N, $\eta>0$, and by Assumption 3.7 and Proposition B.3 H is positive definite and thus $\theta>0$. Moreover, it is easy to derive

$$\frac{\partial}{\partial M}\frac{M}{(M+1)^2} = \frac{1-M}{(M+1)^3} \le 0,$$

which is zero only for M=1, and hence the manufacturer profit is strictly decreasing in the number of manufacturers M and strictly increasing in the number of retailers N.

The **expected retailers utility** is

$$DRU = \eta\frac{M^2N}{(M+1)^2(N+1)^2}\theta.$$

Because $\eta>0$, $\theta>0$, $M^2/(M+1)^2$ is increasing in M and $N/(N+1)^2$ is decreasing in N, it follows that the expected retailer utility is strictly increasing in the number of manufacturers M and strictly decreasing in the number of retailers N.

The **expected consumer surplus** is

$$CS = \frac{1}{2}\frac{M^2N^2}{(M+1)^2(N+1)^2}\theta.$$

The result follows by noting that $\eta>0$, $\theta>0$, $M^2/(M+1)^2$ is increasing in M, and $N^2/(N+1)^2$ is increasing in N.

We now prove the **supply chain efficiency** properties.
From (8), the supply chain efficiency is given by

$$\text{efficiency} = \frac{4MN(M+N+1)}{(M+1)^2(N+1)^2}.$$

Note that the efficiency is given by a function that is symmetric with respect to M and N. Therefore it suffices to characterize the monotonicity properties of the efficiency as a function of N for fixed M, and we know that the same monotonicity properties hold for the efficiency as a function of M for N fixed.

We derive

$$\frac{\partial}{\partial N}\frac{N(M+N+1)}{(N+1)^2} = \frac{M+1-N(M-1)}{(N+1)^3},$$

which is positive if $M=1$ or if $N<(M+1)/(M-1)$, and is negative otherwise. It follows that:

- If $M=1$, the supply chain efficiency is strictly increasing in the number of retailers N. Moreover, if $M=1$ and $N=\infty$, then $DRU=0$ and $DMP=DTU=\eta\theta/4=CTU$ so the efficiency converges to one.

- If $M = 2$, the supply chain efficiency is strictly increasing for $N < (M+1)/(M-1) = 3$ and strictly decreasing otherwise, therefore it is strictly concave in N and reaches a maximum for $N = 3$. We find that the efficiency equals 1 at the maximum.
- If $M = 3$, the supply chain efficiency is strictly increasing for $N < (M+1)/(M-1) = 2$ and strictly decreasing otherwise, therefore it is strictly concave in N and reaches a maximum for $N = 2$. We find that the efficiency equals one at the maximum.

- If $M \geq 4$, the supply chain efficiency is strictly increasing for $N < (M+1)/(M-1)$ and strictly decreasing otherwise. Note that $(M+1)/(M-1) \in (1,2)$ is not an integer. Thus we must compare the value of the efficiency at $N = 1$ and $N = 2$. Some easy calculations lead to

$$\text{efficiency}(M, N=1) - \text{efficiency}(M, N=2) = \frac{M(M+2)}{(M+1)^2} - \frac{8M(3+M)}{9(M+1)^2} = \frac{M(M-6)}{9(M+1)^2}.$$

Therefore, if $M = 4$ or $M = 5$, efficiency$(M, N=1) <$ efficiency$(M, N=2)$ so the supply chain efficiency is strictly concave in N and reaches a maximum for $N = 2$. If $M = 4$ and $N = 2$, we obtain that the efficiency equals $224/225 < 1$. If $M = 5$ and $N = 2$ (or $N = 5$ and $M = 2$), we obtain that the efficiency equals $80/81 < 1$.

- If $M \geq 6$, as seen above, the supply chain efficiency is strictly decreasing for $N > (M+1)/(M-1) > 1$. If $M = 6$, efficiency$(M, N=1) =$ efficiency$(M, N=2)$, so the supply chain efficiency is decreasing in N. If $M \geq 7$, efficiency$(M, N=1) >$ efficiency$(M, N=2)$, so the supply chain efficiency is strictly decreasing in N. Therefore if $M \geq 6$ the efficiency reaches its highest value is for $N = 1$, where it equals $M(M+2)/(M+1)^2 = ((M+1)^2 - 1)/(M+1)^2 < 1$.

As discussed above, due to the symmetry of the efficiency as a function of M and N, the same results hold for the monotonicity properties of the efficiency as a function of M for fixed N.

Proof of Proposition 4.1

Because $G = H$ we know from the linear inverse demand function that $p = (a - HX)\epsilon$, where X is the total supply. Then we can rewrite this expression as $X = a - H^{-1}p/\epsilon$. This implies that the aggregate demand function depends only on the retail price p, and not on the number of retailers in the market. $\square$

Proof of Proposition 4.2

In the case with one product ($P = 1$) and $H = 1$, we can simplify the expression of the efficiency, in order to study how it varies with respect to the value of G. We have

$$\text{Efficiency} = \frac{DTU}{CTU} = \frac{\eta\frac{MN}{(M+1)(N+1)}\left(a - \frac{c}{\eta}\right)\left(1 - \frac{MN}{(M+1)(N+1)}\frac{D}{B}\right)\frac{1}{B}\left(a - \frac{c}{\eta}\right)}{\frac{\eta}{4}\left(a - \frac{c}{\eta}\right)\frac{1}{D}\left(a - \frac{c}{\eta}\right)}$$

After some straightforward calculations, we obtain

$$\text{Efficiency} = \left(\frac{4M}{M+1}\right)\left(\frac{1+(N-1)G}{2+(N-1)G}\right)\left(1 - \frac{M}{M+1}\frac{1+(N-1)G}{2+(N-1)G}\right).$$

Let $r \equiv M/(M+1)$ and $\varphi(G) \equiv 1 + (N-1)G$. Then the efficiency can be rewritten as

$$\text{Efficiency} = 4r\frac{\varphi(G)}{1+\varphi(G)}\left(1 - r\frac{\varphi(G)}{1+\varphi(G)}\right).$$

We have

$$\frac{d(\varphi(G)/(1+\varphi(G)))}{dG} = \frac{\varphi'(G)}{(1+\varphi(G))^2} = \frac{N-1}{(1+\varphi(G))^2},$$

therefore the partial derivative of the efficiency with respect to G is:

$$\frac{\partial \text{ Efficiency}}{\partial G} = 4r\frac{N-1}{(1+\varphi(G))^2}\Big(1-2r\frac{\varphi(G)}{1+\varphi(G)}\Big).$$

Since we assume $N \geq 2$, it is easy to derive

$$\frac{\partial \text{ Efficiency}}{\partial G} \geq 0 \quad \text{if and only if} \quad G(N-1)(M-1) \leq 2,$$

which is trivially true when $M=1$. Moreover, if $M \geq 2$,

$$\frac{\partial \text{ Efficiency}}{\partial G} \geq 0 \quad \text{if and only if} \quad G \leq \frac{2}{(N-1)(M-1)}$$

which implies that the efficiency is a unimodal function of G. It attains its maximum for $G = 2/((N-1)(M-1))$ (as long as this value is within the allowed range $[0,1]$ for G), where $\varphi(G)/(1+\varphi(G)) = 1/2r$, that is Efficiency $= (4r/2r)(1-r/2r) = 1$.

Part 1. If $M=1$ and $N<\infty$, then $r=1/2$ and the efficiency is increasing with G, so it is decreasing with ν. Therefore its maximum value is reached for $G=1$ (or $\nu=0$) and is equal to

$$\frac{2N}{N+1}\Big(1-\frac{N}{2(N+1)}\Big) = 1-\Big(1-\frac{N}{N+1}\Big)^2$$

which is strictly less than 1.

If $M=1$ and $N=\infty$, then $r=1/2$ and $\varphi(G)=\infty$ so Efficiency $=4r(1-r)=1$ $\forall \nu$.

If $N=M=2$, then $2/((N-1)(M-1))=2$ therefore the efficiency is an increasing function of G on $[0,H]=[0,1]$, so it is a decreasing function of ν on $[0,1]$. It thus attains its maximum for $G=1$ (or $\nu=0$) where it takes a value lower than 1 (the efficiency is actually 80/81).

Part 2. if $(N,M)=(2,3)$ or $(3,2)$, then $2/((N-1)(M-1))=1$ therefore the efficiency is an increasing function of G on $[0,1]$, so it is a decreasing function of ν on $[0,1]$, and it reaches a maximum for $G=1$ (or $\nu=0$) where the efficiency is equal to 1.

Part 3. If $(N-1)(M-1)>2$, then $2/((N-1)(M-1))<1$ therefore the efficiency is a strictly concave function of G on $[0,1]$, so it is a strictly concave function of ν on $[0,1]$, and it reaches its unique maximum equal to 1 for $G=2/((N-1)(M-1))$. □

Proof of Proposition 4.3

This follows in a straightforward manner from the results in Propositions 3.12 and 3.6. □

Proof of Proposition 4.4

Part 1. The aggregate supply, wholesale price, and expected retail price in a supply chain with risk-averse retailers depend on the risk aversion only via the term c/η, therefore risk neutral retailers with a cost of $\tilde{c}=c/\eta$ would yield the same results.

Part 2. The manufacturer profit, expected retailer utility, and aggregate expected utility in a supply chain with risk-averse retailers depend on the risk aversion via the term c/η and because of a factor η. □

Proof of Corollary 4.5

We first obtain a detailed expression for $H^{-1}(a-c/\eta)$ and θ as a function of η to study how the expected retail price, efficiency, aggregate supply, manufacturer profit, expected retailer utility, and expected aggregate supply chain utility vary with η.

It is clear that each component $a^k - c^k/\eta$ of the vector $a-c/\eta$ is increasing with η. Moreover, if H is diagonal with diagonal coefficients h^k, from Proposition B.3 H is positive definite so $h^k > 0$ for all k. Then the kth component of vector $H^{-1}(a-c/\eta)$ is $(1/h^k)(a^k - c^k/\eta)$ and is thus an increasing function of η. By Assumption 3.3, $H^{-1}(a-c/\eta) > 0$ therefore $a - c/\eta > 0$. Hence

$$\theta = \sum_{k=1}^{P} \frac{1}{h^k}\left(a^k - \frac{c^k}{\eta}\right)^2$$

is an increasing function of η. We finally recall that risk neutral retailers correspond to $\eta = 1$, and a lower value of η represents a higher risk aversion.

Part 1. Based on Propositions 3.12 and 3.6, the expected decentralized (resp. centralized) retail price in the supply chain with risk-averse retailers are given by

$$E[p] = a - \frac{MN}{(M+1)(N+1)}\left(a - \frac{c}{\eta}\right) \quad \text{(resp.} \quad E[p] = \frac{1}{2}\left(a + \frac{c}{\eta}\right) \text{)},$$

which are clearly decreasing with η, hence both prices are increasing with the risk aversion.

Part 2. Based on Proposition 4.3, the efficiency of the decentralized chain with risk-averse retailers is $4MN(M+N+1)/((M+1)^2(N+1)^2)$, which is independent of η.

Part 3. Based on Propositions 3.12 and 3.6, the decentralized (resp. centralized) aggregate supply in the chain with risk-averse retailers are given by

$$\frac{MN}{(M+1)(N+1)} H^{-1}\left(a - \frac{c}{\eta}\right) \quad \text{(resp.} \quad \frac{1}{2} H^{-1}\left(a - \frac{c}{\eta}\right) \text{)},$$

which are clearly increasing with η when H is diagonal, hence decreasing with the risk aversion.

Part 4. Based on Propositions 3.12 and 3.6, the decentralized manufacturer profit and expected retailer utility are given by

$$\eta \frac{MN}{(M+1)^2(N+1)}\theta \quad \text{and} \quad \eta \frac{M^2 N}{(M+1)^2(N+1)^2}\theta,$$

which are clearly increasing with η when H is diagonal, hence decreasing with the risk aversion. Moreover, the decentralized (resp. centralized) expected aggregate supply chain utility are given by

$$\eta \frac{MN(M+N+1)}{(M+1)^2(N+1)^2}\theta \quad \text{(resp.} \quad \frac{\eta}{4}\theta \text{)},$$

which are clearly increasing with η when H is diagonal, hence decreasing with the risk aversion. □

Proof of Proposition 5.2

The jth retailer decision can be rewritten as:

$$\max_{x_j \geq 0} \quad -x_j^T v + \eta_j x_j^T (a_j - H_j x_j - \sum_{l \neq j} G_{jl} x_l).$$

Note that because Assumption 5.1 hold, we know by Proposition B.4 in Appendix B, that the matrix H_j is positive definite and thus for any v and x_l for $l \neq j$ there exists a unique maximizer to the jth retailer decision, which is the unique solution to the following LCP:

$$0 \leq \frac{v}{\eta_j} - (a_j - 2H_j x_j - \sum_{l \neq j} G_{jl} x_l) \perp x_j \geq 0.$$

In other words, for any v and x_l for $l \neq j$, there exists a unique jth retailer best response.

Hence, for a given v, the order quantity vector $\hat{x} = (x_1, x_2, \ldots, x_N)$ is a retailers equilibrium if and only if it solves the LCP in (9), which is simply the concatenation of the LCPs characterizing the best response of the different players. Note also that by Assumption 5.1, $\hat{B} + \hat{H}$ is positive definite and thus the LCP in (9) has a unique solution. □

Proof of Proposition D.1

Under the revenue-sharing contracts described, the jth retailer objective is

$$\max_{x_j \geq 0} \quad -\phi c^T x_j - \phi \eta x_j^T G \sum_{l \neq j} x_l + \phi \eta x_j^T (a - H x_j - G \sum_{l \neq j} x_l),$$

where the first term is the wholesale cost, the second term is the externality payment, and the third term is the expected utility from sales in the wholesale market.

Because Assumption 3.7 holds, by Proposition B.3 we have that matrix H is positive definite, and hence there exists a unique maximizer to the jth retailer decision problem, which is the solution to the following LCP:

$$0 \leq -a + 2Hx_j + 2G \sum_{l \neq j} x_l + \frac{c}{\eta} \perp x_j \geq 0. \tag{29}$$

Therefore $\hat{x}$ is a retailers equilibrium under the revenue-sharing contracts if and only if it solves the following LCP:

$$0 \leq -\hat{a} + 2\hat{B}\hat{x} + \frac{\hat{c}}{\eta} \perp \hat{x} \geq 0.$$

Because Assumption 3.3 holds, it is easy to see that the unique solution to this LCP is given by $\hat{x} = (x, x, \ldots, x)$, where

$$x = \frac{1}{2N} D^{-1} \Big(a - \frac{c}{\eta} \Big), \tag{30}$$

and the order quantity given by (30) coincides with that in the centralized chain.

Note that each manufacturer knows the contractual conditions and thus knows the expected equilibrium retail price, which is entirely determined by the retailers equilibrium and thus is independent of its own decisions. In particular, each manufacturer knows that the expected retail price is, using expression (30) and the inverse demand function,

$$E[p] = a - \Big(H + (N-1)G \Big) \frac{1}{2N} D^{-1} \Big(a - \frac{c}{\eta} \Big) = \frac{1}{2} \Big(a + \frac{c}{\eta} \Big).$$

We can then write the manufacturer decision problem as

$$\max_{y_i \geq 0} \quad (1-\phi)\eta y_i^T \frac{1}{2}(a + \frac{c}{\eta}) - (1-\phi)c^T y_i - (1-\phi)\eta \frac{M}{M+1} y_i^T D \sum_{k=1}^{M} y_k,$$

where the first term is the expected utility from the revenue shared by the retailers, the second term is the wholesale market revenues minus the manufacturing cost, and the third term is the externality payment. Note that we have not included the wholesale market clearing conditions in the manufacturer's decision problem because the wholesale market price is set by the revenue-sharing contracts. We are going to see, however, that the externality payments are such that the manufacturers choose production quantities that satisfy the retailers demand and clear the wholesale market.

Because the assumptions in Proposition B.2 in Appendix B hold, we know that matrix D is positive definite, therefore there exists a unique solution to the manufacturer best response problem, which is the solution to the following LCP:

$$0 \leq -\frac{1}{2}(a+\frac{c}{\eta}) + \frac{c}{\eta} + \frac{M}{M+1} D \sum_{k \neq i} y_k + 2\frac{M}{M+1} D y_i \perp y_i \geq 0,$$

which can be rewritten as

$$0 \leq -\frac{1}{2}\frac{M+1}{M}(a-\frac{c}{\eta}) + D(2y_i + \sum_{k \neq i} y_k) \perp y_i \geq 0. \tag{31}$$

Hence a supply schedule $\hat{y} = (y_1, y_2, \ldots, y_M)$ is a manufacturers equilibrium if an only if it solves the following LCP, which is formed as the concatenation of the LCPs (31) for all M manufacturers:

$$0 \leq \hat{r} + \hat{S}\hat{y} \perp \hat{y} \geq 0,$$

where $\hat{r} = ((M+1)/(2M))(c/\eta - a, c/\eta - a, \ldots, c/\eta - a)$, and and

$$\hat{S} = \begin{pmatrix} 2D & D & \ldots & D \\ D & 2D & \ldots & D \\ \vdots & \vdots & \ddots & \vdots \\ D & D & \ldots & 2D \end{pmatrix}. \tag{32}$$

Because the assumptions of Proposition B.2 in Appendix B hold, the matrix D is positive definite, and hence, by Lemma B.1 $\hat{S}$ is positive definite, and hence there exists a unique manufacturers equilibrium under the proposed revenue-sharing contracts.

Because the equilibrium is unique and the problem is symmetric, the equilibrium is symmetric. Therefore, the equilibrium is the solution to the following LCP:

$$0 \leq -\frac{1}{2}(a - \frac{c}{\eta}) + MDy \perp y \geq 0.$$

Since Assumption 3.3 holds, it is easy to see that the solution is

$$y = \frac{1}{2M} D^{-1}\left(a - \frac{c}{\eta}\right)$$

and thus $\sum_{i=1}^{M} y_i = \frac{1}{2} D^{-1}\left(a - \frac{c}{\eta}\right)$; that is, the manufacturing production quantities are identical to those in the centralized supply chain and the wholesale market clears. □

Appendix D: Supply chain coordination

We show that there exists a continuum of theoretical revenue sharing contracts that coordinate the symmetric supply chain and allow for any arbitrary split of utility between manufacturers and retailers. Our contracts are similar to those studied by Cachon and Lariviere [2005], but we extend the analysis to the case with competing manufacturers. We must note, however, that the purpose of our studying the revenue-sharing contracts is of a theoretical nature. Specifically, our purpose is to improve our understanding of the effect of competition in a supply chain with competing manufacturers and retailers, but we do not advocate the practical implementation of these complex contracts.

Our model allows for both vertical competition (manufacturers versus retailers) as well as horizontal competition (manufacturers versus manufacturers, or retailers versus retailers). Consequently, the coordinating contracts mush achieve vertical as well as horizontal coordination. To achieve vertical coordination, the contracts stipulate that the retailers must *share* their revenue from retail sales with the manufacturers. Specifically, every retailer must keep a proportion ϕ of its revenue, where $\phi \in (0,1)$, and pass on to the manufacturer a proportion $(1-\phi)$ of its revenue. Moreover, both the manufacturers and the retailers agree to a wholesale market price $v = \phi c$, where c is the manufacturer cost. As a result, manufacturers and retailers share revenue, costs, and risk, and hence their incentives are perfectly aligned. [23] To achieve horizontal coordination, the contracts require that every retailer pays a quantity equal to the externality that it imposes on its fellow competing retailers as a result of its supplying a particular quantity to the retail market. Likewise, every manufacturer must make an externality payment that depends on the amount they produce for each of the products.[24]

Proposition D.1. Let the assumptions of Theorems 3.4 and 3.11 hold. Assume also that the manufacturers and retailers have accepted the following revenue-sharing contracts:

1. every retailer keeps a proportion ϕ of its revenue, where $\phi \in (0,1)$, and passes on to the manufacturer a proportion $(1-\phi)$ of its revenue,
2. the wholesale market price is $v = \phi c$, where c is the manufacturer cost,
3. the jth retailer has to make an externality payment equal to $\phi \eta x_j^T G(\sum_{l \neq j} x_l)$ for ordering a quantity x_j,
4. the ith manufacturer has to make an externality payment for supplying a quantity y_i equal to $(1-\phi)\eta(M/(M+1)) y_i^T D(\sum_{k=1}^{M} y_k)$.

Then there exists a unique decentralized equilibrium at which the retailer order quantities and manufacturer supply quantities are equal to those in the centralized supply chain given in Theorem 3.4.

[23] Note that in the decentralized supply chain the manufacturer profit is deterministic, and hence the manufacturer risk aversion does not play a role. With the revenue sharing contracts in place, the manufacturers receive a share of the uncertain retail revenue, and hence their risk aversion comes into play. For tractability we assume in this section (and in the context of the symmetric supply chain) that all manufacturers and retailers in the supply chain have the same risk-aversion parameter.

[24] The spirit of the proposed revenue-sharing contracts is that the externality payments of the manufacturers and retailers are collected by a third party, and that this third party will decide how to reallocate these externality payments.