A Dynamic Model with Import Quota Constraints *

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Abstract

A collapse in international trade following the 2007-08 crisis has underscored many dangers of globalization and renewed interest in trade protectionism, one form of which is import quotas. The analysis of import quotas is predominantly based on a static model, which is unable to capture the fact that a quota is imposed over a period of time. This article develops a continuous-time model that incorporates a more realistic dynamic quota constraint into the workhorse model and argues many traditional results to no longer be valid. In particular, a country may choose to refrain from trade in a quota-protected commodity even when its world price is below the domestic price and the quota is not fully exhausted. Distinct economic behavior prevails depending on whether the country is importing the protected good, exporting it or refraining from trade in it. The domestic price of the protected good exceeds the world price in import and no-trade regions, even when the quota is underutilized − in contrast, the workhorse quota model predicts no economic effects of a quota unless it is binding. Additional factors underlying the quota-protected economy, the quota utilization rate to date and the time remaining till the quota horizon, are identified. Various extensions of the baseline analysis support the robustness of our main conclusions.

JEL Classifications: D51, F13, F30, F40, G12

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1. Introduction

The 2007-08 financial crisis brought about a major collapse in international trade, with trade in goods and services falling at a pace unseen since the Great Depression. According to the October 2009 edition of the International Monetary Funds World Economic Outlook, international trade as measured by total exports of goods and services is projected to decline 11.9 percent in 2009. These dramatic developments have been blamed on globalization, raising fears that a new wave of protectionism is just about to arrive, just like that following the Great Depression (the Smoot-Hawley Tariff Act of 1930 in the US, followed by retaliatory measures by US trade partners). In his CRS Report for the US Congress, Ahearn (2009) entertains various scenarios the potential new wave of protectionism may take, including severe violations of WTO rules and a major trade conflict.

Renewed interest in protectionism has brought back into focus the debate about the merits of commonly-used trade barriers. One form of such a barrier to international trade is an import quota. An import quota is simply a quantitative restriction on imports *over a period of time*. That is, a country lets in an amount of a particular good up to the quota during the period of time, typically a year, and then prohibits any further quantity from entering. A sizable body of theoretical and empirical work exists examining the effects of quotas. This work, however, is predominantly based on a static model which, as argued by empiricists and policymakers, offers some undesirable implications and may misrepresent the economic significance of quotas. To our knowledge, the model we develop in this paper is the first to incorporate a more realistic, dynamic quota constraint into the workhorse model and focus on its economic effects. We will demonstrate that within our model, the bulk of traditional results is no longer valid, and argue that the alternative modeling approach we advocate might be a more appropriate benchmark for policy and empirical work.

Our primary goal in this paper is to capture the economic effects of an import quota during the period of time over which it is imposed.[1] Since this question certainly cannot be addressed in a static setup, we depart from the traditional approach to modeling quotas by considering a dynamic, finite horizon, continuous-time economy. To facilitate comparisons with the workhorse quota model, we keep the remaining elements as standard as possible. In particular, we consider a pure-exchange economy with two goods. The first good represents the output of the quota-protected industry, while the second good represents the remainder of the commodities produced in the economy. In our setting, the quota constraint manifests itself as an integral constraint.

[1] Quotas on exports are relatively rare, and typically only present in developing countries (Remark 2).

To highlight the novel effects of our dynamic quota-type constraint, we compare with a country facing instead a period-by-period constraint, analogous to that in the workhorse model.

Under general price uncertainty and state-independent preferences, our model of a small open economy shows that the optimal pre-quota-horizon consumption of the protected commodity falls into three regions, where a quota-constrained country exhibits distinctly different economic behavior. At high enough world price of the commodity or its current endowment, the country exports and behaves just like an unconstrained country. However, at low endowment or world price, it imports the protected commodity and here behaves as if facing an additional "cost" to importing over and above the unconstrained country. This extra cost depends on the expected severity of hitting the quota constraint and is stochastic, mathematically being given by the conditional expectation of the constraint's Lagrange multiplier. The consequence is to reduce the country's level of imports in those states in which it would have imported the most. This extra cost acts much like a state-dependent tariff: the country has a flexibility to increase its imports in response to favorable world prices, but may do so only at the additional cost. Furthermore, the effective nonlinearity in this cost function (since no such cost is applied to exporting) yields an extended region over which there is insufficient incentive to import or export the quota-protected commodity and so the country does not trade in it. The domestic price of the protected commodity responds accordingly: when the quota-constrained country is refraining from trade in or importing the protected commodity, its domestic price exceeds its world price; otherwise when exporting, the domestic price equals to that prevailing in the world market.

The consumption behavior above is considerably different from that of a period-by-period constrained country, which (as in the workhorse model) exhibits only two distinct types of economic behavior, one when binding on the constraint for low price of the protected commodity or its current endowment, and one when not binding. In contrast to the quota-constrained country, for the period-by-period or unconstrained, no-trade only arises as a knife-edge condition (on the state price and endowment). To evaluate the conditional expectation of the severity of the quota constraint, we solve the model numerically. Explicit evaluation yields a natural convexity in this extra "cost" function and reveals two forms of unusual behavior in the import region's consumption policy: local concavity as a function of the state price density and, in spite of an Inada condition on preferences, bounded consumption even when the cost of consumption decreases to zero. As the direct cost of consumption is decreased, the probability of hitting the import quota constraint increases causing the extra cost to go up and so reduce imports.

A valuable implication of our work is that within a plausible framework, a quota need not be fully utilized for it to have economic effects. This is at odds with the standard conclusion of the workhorse and period-by-period models, as well as their applications, that non-binding quotas do not introduce any distortion in the economy and hence are harmless to consumers. Moreover, we identify additional economic factors underlying the quota-constrained economy: the quota utilization rate to date (or, equivalently, cumulative imports to date) and the time remaining till the quota horizon. The higher the current utilization rate, the more likely is the constraint to bind, and so the more costly is further importing. The model also predicts that, all else equal, the farther away the quota horizon, the more restrictive the quota is. The remaining driving economic factors in the quota-constrained economy, the current economic conditions, world price of the protected commodity and the country's endowment of the commodity, are as in the period-by-period constrained economy. We empirically investigate the relevance of the new driving factors and demonstrate that both the utilization rate and time remaining turn out to be economically and statistically significant, and the signs of the effects are as predicted by our analysis.

Finally, we consider several natural extensions of our baseline analysis of a quota-protected small open economy: a general equilibrium model accounting for the impact of quotas on the world economy and two generalizations of the model accounting for spillovers from the protected industry to other sectors in the economy. We show that our main conclusions continue to hold in these settings, and the additional implications we obtain are in line with earlier work. For example, within a general equilibrium world economy consisting of one quota-constrained and one unconstrained country, under logarithmic preferences, the constrained country becomes wealthier at the expense of the unconstrained. Moreover, the stock price of the protected industry increases in the quota-constrained and decreases in the unconstrained country.

The subject of import quotas is, of course, prevalent in the literature on international trade. The workhorse model, which serves as our benchmark, is widely adopted in modern textbooks, e.g., Krugman and Obstfeld (2002), Bhagwati, Panagariya and Srinivasan (1998), as well as in traditional texts, e.g., Helpman and Krugman (1989), Dixit and Norman (1980), and is now even a part of the introductory microeconomics curriculum, e.g., Parkin (2002). This literature, however, restricts attention to static models under certainty, ignoring the dynamic nature of quota restrictions. The standard predictions are that quotas reduce imports and increase the domestic price of the commodity above the world price when the quota is binding, and have no economic effects otherwise. Another issue widely investigated in the existing literature is the comparison of quotas to other trade policy instruments, such as tariffs, export subsidies, etc., and the resulting implications on welfare. This is not the focus of our analysis. Helpman and Razin (1980), Young and Anderson (1982) and subsequent developments extend the basic theory to uncertainty and derive nonlinearity in the price of the imported good depending on whether the quota is binding or not. These works fall into the same framework as our period-by-period model, where the economic implications are driven by the separation of the state space into the quota-binding and quota-not-binding regions, but differ substantially from our dynamically-constrained economy.

A large literature exists on measuring the economic effects of quotas and other nontariff barriers (see Deardorff and Stern (1999), Feenstra (1995), Hufbauer and Elliot (1994), ITC (2002), Linkins and Arce (2002) and references therein). This measurement necessarily relies on a theoretical model, and the textbook model still continues to be the main workhorse. An unfortunate implication of the model for empirical work and policymakers is that products with unfilled quotas should be omitted from a sample. In a recent study, ITC (2002) contests this implication, concerned that doing so may misrepresent the economic significance of quotas, and advocates considering 85% utilized quotas to be binding. Other studies have used an 80% or a 90% threshold; the right threshold remains a subject of an ongoing debate.[2] Our model offers an alternative view: all quotas potentially have an economic impact, which may be quantified by examining the factors discussed earlier. Within duopoly settings, Reitzes' (1991) analysis of firms' strategic R&D and output behavior, and Reitzes and Grawe's (1994) study of market-share quotas reveal that non-binding quotas may have economic effects. Also related to our work, is the strand of literature on quota license pricing under uncertainty in a secondary market for such licenses (Anderson (1987), Eldor and Marcus (1988), and Krishna and Tan (1996)). These authors recognize and characterize the option value of a quota license in effect over a period of time, however, there are some fundamental differences between their modeling approaches and ours (Appendix C).

Our additional cost to imports, reflecting the expected severity of hitting the quota constraint, resembles (Appendix C) processes arising in the contexts of exchange rate target zones (Krugman (1991), Cadenillas and Zapatero (1999)) or shipping costs (Dumas (1992)). Our integral constraint on imports is similar to the liquidity constraint of Detemple and Serrat (2003) – an integral constraint on net consumption path. Their integral constraint, however, is imposed in expectation, while ours has to be satisfied path-by-path. Consequently, unlike in their economy, the value of the multiplier associated with our integral constraint is not revealed till the expiration of the quota restriction. To our knowledge, the only other work studying an integral-type constraint such as ours is the continuous-time liquidity model of Wang (2001).[3] He introduces a cash-in-advance constraint for purchases over a finite period of time, as opposed to at a single point in time as in the existing literature.

[2]See Linkins and Arce (2002) for more on this debate and for references to other studies.

[3]A somewhat related constraint is considered in a discrete-time model by Bertsimas and Lo (1998). They study the optimal policies of trading a large block of securities within a fixed period in a market microstructure framework.

The rest of the article is organized as follows. Section 2 describes the economy, including a dynamic quota constraint. Section 3 solves for domestic equilibrium in a quota-protected small open economy, and Section 4 extends our analysis to general equilibrium, as well as to the case of the quota-protected industry constituting a large fraction of the domestic economy and to the case of non-separable preferences. Section 5 concludes, Appendix A provides all proofs, Appendix B the procedure employed in the numerical analysis, and Appendix C some empirical analysis supporting our implications.

2. The Economic Setting

2.1. The Economy

We consider a finite-horizon, $[0, T]$, pure-exchange economy with two goods. The first good represents the output of the quota-protected industry, while the second good represents the remaining commodities in the economy and serves as the numeraire. A symbol with an asterisk (*) denotes a quantity related to the second good, henceforth the composite good. There are two countries comprising the world economy: an import quota-constrained country Q and a normal unconstrained country N. We first treat country Q as a small open economy so as to concentrate on the domestic effects of the quota protection policy (Section 3), and then study general equilibrium effects of quotas by assuming that country Q is large enough to affect world prices (Section 4).

Uncertainty in the economy is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, on which is defined a four-dimensional Brownian motion $w(t) = (w_1(t), \ldots, w_4(t))^\top$, $t \in [0, T]$. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, the augmented filtration generated by w. All stated (in)equalities involving random variables hold P-almost surely. In what follows, given our focus is on characterization, we assume all processes introduced to be well-defined, without explicitly stating the regularity conditions ensuring this.

In each country, goods are produced by two representative firms: one producing the quota-protected good and the other the composite good. The production process of each industry is modeled as a Lucas (1978) tree, where the strictly positive flows of output $(\varepsilon_i, \varepsilon_i^*)$, with support $(0, \infty)$, are specified exogenously, satisfying the dynamics:

$$\begin{aligned} d\varepsilon_i(t) &= \varepsilon_i(t)[\mu_{\varepsilon_i}(t)\, dt + \sigma_{\varepsilon_i}(t)\, dw(t)], \qquad i \in \{N,\ Q\}\,, \quad \text{(quota-protected good)} \\ d\varepsilon_i^*(t) &= \varepsilon_i^*(t)[\mu_{\varepsilon_i}^*(t)\, dt + \sigma_{\varepsilon_i}^*(t)\, dw(t)], \qquad i \in \{N,\ Q\}\,, \quad \text{(composite good)} \end{aligned}$$

where $(\mu_{\varepsilon_i},\ \mu_{\varepsilon_i}^*)$, $\sigma_{\varepsilon_i} \equiv (\sigma_{\varepsilon_{i_1}}, \ldots, \sigma_{\varepsilon_{i_4}})^\top$, $\sigma_{\varepsilon_i}^* \equiv (\sigma_{\varepsilon_{i_1}^*}, \ldots, \sigma_{\varepsilon_{i_4}^*})^\top$ are adapted stochastic processes. The price of the quota-protected good (in term of the composite good), p, prevailing in markets with no restrictions will be shown in equilibrium to have dynamics

$$dp(t) = p(t)[\mu_p(t)\,dt + \sigma_p(t)\,dw(t)]\,,$$

where μ_p and σ_p are possibly path-dependent, while that faced by country Q consumers, p_Q (which may differ from p due to import restrictions), to have dynamics

$$dp_Q(t) = p_Q(t)[\mu_{p_Q}(t)\,dt + \sigma_{p_Q}(t)\,dw(t)]\,.$$

Investment opportunities are represented by five securities; an instantaneously riskless bond, B, in zero net supply, and two risky stocks, S_i and S_i^*, of the representative firms located in each country N and Q, in unit net supply. Stocks S_N and S_Q are claims to the output of the quota-protected industry in countries N and Q, ε_N and ε_Q, respectively. S_N^* and S_Q^* are claims to ε_N^* and ε_Q^*, respectively. The bond price and stock prices are posited to follow

$$\begin{aligned}
dB(t) &= B(t)r(t)dt\,, \\
dS_N(t) + p(t)\varepsilon_N(t)dt &= S_N(t)[\mu_N(t)dt + \sigma_N(t)dw(t)]\,, && (1)\\
dS_Q(t) + p_Q(t)\varepsilon_Q(t)dt &= S_Q(t)[\mu_Q(t)dt + \sigma_Q(t)dw(t)]\,, && (2)\\
dS_i^*(t) + \varepsilon_i^*(t)dt &= S_i^*(t)[\mu_i^*(t)dt + \sigma_i^*(t)dw(t)], \quad i \in \{N,\ Q\}\,,
\end{aligned}$$

where the interest rate r, drift coefficients $\mu \equiv (\mu_N,\ \mu_Q,\ \mu_N^*,\ \mu_Q^*)^\top$ and volatilities $\sigma \equiv \{\sigma_{ij}, \sigma_{ij}^*;\ i \in \{N,\ Q\},\ j = 1, \ldots, 4\}$, are possibly path-dependent. Note that due to imperfections in the market for the quota-protected commodity, the stock price dynamics of protected industries are location-specific: output in country Q is valued at the country-Q specific price p_Q (equation (1)), while that in country N valued at price p (equation (2)). The posited dynamic market completeness implies the existence of a unique state price density process, ξ^*, consistent with no-arbitrage, given by

$$d\xi^*(t) = -\xi^*(t)[r(t)dt + \theta(t)^\top dw(t)]\ ,$$

where $\theta(t) \equiv \sigma(t)^{-1}(\mu(t) - r(t)\bar{1})$ is the market price of risk (or the Sharpe ratio) process, and $\bar{1} \equiv (1, \ldots, 1)^\top$. The quantity $\xi^*(t, \omega)$ is interpreted as the Arrow-Debreu price per unit probability P of one unit of the composite good in state $\omega \in \Omega$ at time t. The state-price densities associated with the quota-protected good would simply be given by $\xi_Q(t) = \xi^*(t)\,p_Q(t)$ (prevailing in country Q) and $\xi(t) = \xi^*(t)\,p(t)$ (prevailing outside country Q).

The representative consumer-investor of each country i is endowed at time 0 with the total supply of the stock market located in country i, providing him with the initial financial wealth

$W_i(0) = S_i(0) + S_i^*(0)$. The stocks' payoffs can then be interpreted as the consumer's endowment stream consisting of ε_i units of the quota-protected and ε_i^* units of the composite goods. Additionally, the consumer in country Q receives a quota revenue transfer process δ_Q from his government (described in Section 3.1).[4] Since financial markets are complete, this transfer process is spanned by traded securities. (Since our focus until Section 4 is on the small open economy Q, we drop for now the subscript i.) Each consumer chooses a nonnegative consumption process c, and a portfolio process π, where $\pi(t) = (\pi_N(t),\ \pi_Q(t),\ \pi_N^*(t),\ \pi_Q^*(t))^\top$ denotes the vector of amounts invested in each stock. The consumer's financial wealth process W then follows

$$dW(t) = W(t)\,r(t)\,dt - (p_Q(t)\,c(t) + c^*(t) - \delta_Q(t))dt + \pi(t)^\top\left[\mu(t) - r(t)\bar{1}\right]dt + \pi(t)^\top\sigma(t)\,dw(t)\,,$$

with $W(T) \geq 0$. To isolate the effects of the quota protection of a particular industry, and abstract away from the spillover effects through the consumer demand, we assume that consumer preferences are separable across goods. (The case of nonseparable preferences is considered in Section 4.3.) In particular, we assume that each individual derives time-additive, state-independent utility $u(c) + v(c^*)$ from intertemporal consumption in $[0, T]$. The functions $u(\cdot)$ and $v(\cdot)$ are assumed twice continuously differentiable, strictly increasing, strictly concave, and to satisfy $\lim_{x\to 0} u'(x) = \infty$, $\lim_{x\to 0} v'(x) = \infty$, $\lim_{x\to\infty} u'(x) = 0$ and $\lim_{x\to\infty} v'(x) = 0$.

2.2. Modeling Import Quotas

A country's import of a commodity at time t is defined as any excess consumption over its current endowment of that commodity, $(c(t) - \varepsilon(t))^+$, where $x^+ = \max\{0, x\}$ denotes the positive part of x. (Analogously, exports would be defined as $(\varepsilon(t) - c(t))^+$.)

Our concern here is to model a quota constraint, where a country's cumulative imports of the first good during some period of time are restricted not to exceed an amount K. Specifically, we focus on the following constraint

$$\int_0^T (c(t) - \varepsilon(t))^+ dt \leq K\,, \tag{3}$$

which must hold almost surely along each path. So in our continuous-time setting a quota-type constraint manifests itself as an integral constraint. We assume the constraint is applied on imports rather than net imports, meaning past or future exports are not allowed to offset current

[4] Consistent with the literature (e.g., Helpman and Razin (1980)), we assume that the governing body of the quota constrained country auctions off rights to import and distributes the proceeds to the consumer. Alternatively, one can assume that the governing body assigns rights to export to a number of foreign firms – then the quota rents are captured by the exporting firms.

imports. Note that, since the integrand in (3) is nonnegative, this constraint automatically implies

$$\int_0^t (c(s) - \varepsilon(s))^+ ds \leq K \,, \qquad \forall t \in [0, T] \,.$$

Hence, one can alternatively think of an agent checking continuously that he has not yet exceeded his quota constraint for the full quota period $[0, T]$. For comparison, and to highlight the properties unique to a quota-type constraint, we will also consider a country P with a period-by-period import constraint, along the lines of the workhorse model, i.e.,

$$(c(t) - \varepsilon(t))^+ \leq k, \qquad t \in [0, T] \,. \tag{4}$$

For $k = K/T$, this constraint implies the quota constraint (3), but not vice versa. Hence, the quota constraint can be thought of as a "softer" version of import constraint. The constraint is faced by the country as a whole; domestic consumers do not explicitly account for it in their optimization. Through the effect of the constraint on domestic prices, however, consumers' choices will be indirectly affected by the quota.

3. Small Open Economy with Import Quota Constraint

In this section, we treat the import-constrained country as a small open economy. We characterize the optimal consumption and domestic prices in the quota-constrained country Q and compare with the period-by-period constrained country P and the unconstrained country N.

3.1. Country's and Domestic Consumer's Problems

The open economy Q (or P) has unrestricted access to world capital markets, however it is too small for its investment decisions to affect valuations in the world market, ξ^*, or for its choice of consumption to affect the world prices of the quota-protected good, ξ or p. Hence, throughout this section we treat ξ^* and ξ (or p) as exogenously specified state variables. The role of import quotas is to shield domestic owners/producers of the protected good from low world prices by restricting the supply of the good, thereby increasing its domestic price p_Q (and hence ξ_Q) above p (ξ) in some states of the world. To solve for domestic prices and allocations prevailing in the domestic equilibrium of a quota-constrained economy, we follow the usual steps in the international trade literature (e.g., Dixit and Norman (1980), Young and Anderson (1982)). We first consider the choice of a benevolent agent acting in the interest of the whole country, henceforth referred to as the country. The agent determines optimal levels of consumption in the country, taking into account the import quota constraint, and collects and distributes any revenues raised by auctioning off rights to import the protected good. We then use the optimal consumption allocations to identify the domestic prices faced by consumers in the decentralized domestic economy.

The country auctions off rights to firms for importing the quota-protected good. With domestic prices higher than world prices, the firms benefit from importing the good to country Q. Their revenue per unit imported is $p_Q(t) - p(t) \geq 0$. This market is competitive with free entry and firms make zero profits in each time and state (t, ω), hence the price of a right to import a unit of the quota-protected good over the next instant is $p_Q(t) - p(t)$. The revenue raised by sales of these rights is then given by

$$\delta_Q(t) = (p_Q(t) - p(t))(c(t) - \varepsilon(t)) \,. \tag{5}$$

This revenue is positive only in those states of the world where the domestic price of the quota-protected good is higher than the world price and the domestic demand for imports is positive; in the remainder of the states the revenue is zero.

The country's total wealth at time 0 is equal to the initial financial wealth of the consumers plus the revenue due to the rights sales. Under dynamic market completeness, we can then restate the dynamic budget constraint faced by the country in its static Arrow-Debreu form: the present value of the country's consumption expenditures cannot exceed its total wealth

$$E\left[\int_0^T [\xi_Q(t)\, c(t) + \xi^*(t) c^*(t)] dt\right] \leq W(0) + E\left[\int_0^T \xi^*(t)\,(p_Q(t) - p(t))(c(t) - \varepsilon(t)) dt\right] \,, \tag{6}$$

where we have substituted $\xi_Q(t) = \xi^*(t)\, p_Q(t)$. The second term on the right-hand side is the present value of license revenues (5). In addition to the budget constraint, the country has to take into account the quota constraint in its optimization. The problem of the country, using the martingale representation approach (Cox and Huang (1989), Karatzas, Lehoczky and Shreve (1987)), can be restated as the following static variational one:

$$\max_{c,\, c^*} \; E\left[\int_0^T [u(c(t)) + v(c^*(t))]\, dt\right] \tag{7}$$

subject to (3) and (6) for the quota-constrained country Q,

or (4) and (6) for the period-by-period constrained country P.

The solutions to these problems are reported in Sections 3.2–3.3.

The final ingredient of the model is the domestic consumer. By solving for prices that he faces in the domestic equilibrium with transfers, we complete the analysis of the small open economy.

The problem of the consumer in country Q can be cast as the following static variational one:

$$\max_{c,\, c^*} \; E\left[\int_0^T [u(c(t)) + v(c^*(t))]\, dt\right]$$

$$\text{subject to } E\left[\int_0^T [\xi_Q(t)\, c(t) + \xi^*(t) c^*(t)]\, dt\right] \leq W(0) + E\left[\int_0^T \xi^*(t)\, \delta_Q(t)\, dt\right] ,$$

where we have substituted $\xi_Q(t) = \xi^*(t)\, p_Q(t)$. The second term on the right-hand side is the present value of (lump-sum) transfers to the consumer. The solution to this problem is given by

$$c_Q(t) = I(y_Q \xi_Q(t)) \,, \quad c^*_Q(t) = I^*(y_Q \xi^*(t)) \,, \tag{8}$$

where $I(\cdot)$, $I^*(\cdot)$ are the inverse functions of $u'(\cdot)$, $v'(\cdot)$, respectively, and $y_Q > 0$ solves

$$E\left[\int_0^T [\xi_Q(t)\, c_Q(t;\, y_Q) + \xi^*(t) c^*_Q(t;\, y_Q)] dt\right] = W(0) + E\left[\int_0^T \xi^*(t)\, \delta_Q(t) dt\right] . \tag{9}$$

The optimal policies of the consumer in the period-by-period constrained country P are given by the same expressions up to the obvious replacement of the subscript Q by P.

3.2. Period-By-Period Constrained Country's Equilibrium

Proposition 1 characterizes the optimal solution to the period-by-period constrained country's problem, assuming it exists.

Proposition 1. *The optimal consumption of the period-by-period constrained country, for* $t \in [0, T]$, *is*

$$c_P(t) \;=\; \begin{cases} \varepsilon(t) + k & \textit{if} \;\; \xi(t) < \frac{1}{y}\, u'(\varepsilon(t) + k) \quad (\textit{binding}) \,, \\ I(y\, \xi(t)) & \textit{if} \;\; \frac{1}{y}\, u'(\varepsilon(t) + k) \leq \xi(t) \quad (\textit{not binding}) \,, \end{cases} \tag{10}$$

$$c^*_P(t) \;=\; I^*(y \xi^*(t)) \,, \tag{11}$$

where $y > 0$ *solves*

$$E\left[\int_0^T [\xi(t)\, c_P(t;\, y) + \xi^*(t)\, c^*_P(t;\, y)]\, dt\right] = E\left[\int_0^T [\xi(t)\, \varepsilon(t) + \xi^*(t)\, \varepsilon^*(t)]\, dt\right] . \tag{12}$$

Moreover, $y \leq y_N$*, where* y_N *is the analogous constant for an unconstrained country with the same endowment process* $(\varepsilon,\, \varepsilon^*)$*.*

At each time t, there are two regions of country P's consumption behavior: a region where it is unconstrained, and a region where it binds on the constraint and so imports the maximum amount of the quota-protected good allowable. While an unconstrained country's optimal consumption, $I(y_N\, \xi(t))$, depends on the world state prices associated with the quota-protected good ξ alone, country P's consumption is driven both by the state prices and its current endowment $\varepsilon(t)$. Figure 1 depicts the country's optimal consumption of the protected good. For low enough quota-protected commodity price or endowment, the country is binding on the import constraint; for high enough state price density or endowment, it is unconstrained.

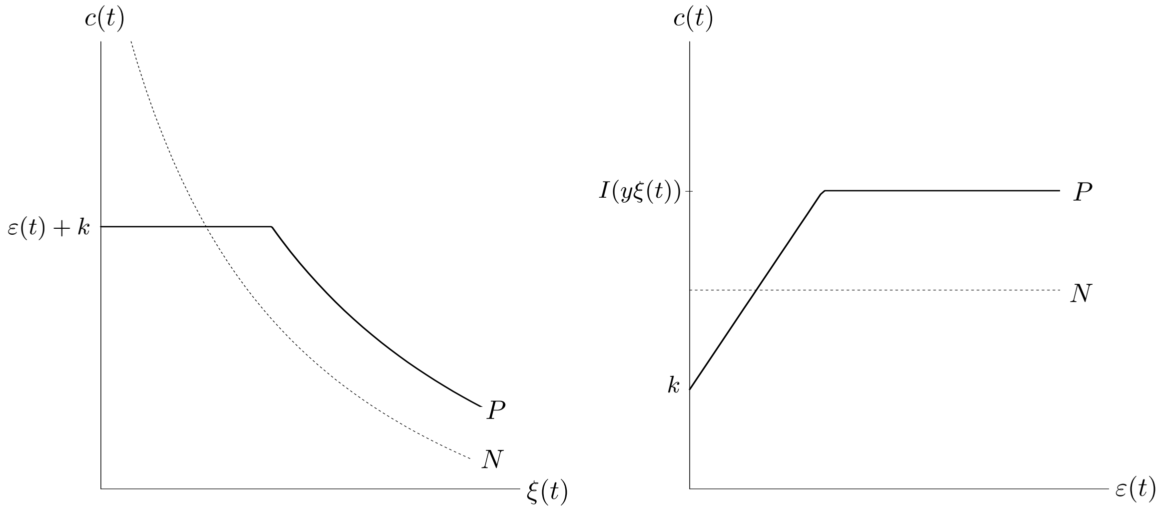


Figure 1: Country P's optimal consumption of the quota-protected commodity versus concurrent world state price density associated with the quota-protected good, ξ, and endowment of the quota-protected good, ε. The dotted plot is for the unconstrained normal country N.

Given the optimal allocations determined at the country level in Proposition 1, we can infer the price of the quota-protected commodity faced by the P-country's consumers, consistent with domestic market clearing.

Proposition 2. *The state-price density associated with the quota-protected good in the period-by-period constrained country prevailing in the domestic market equilibrium, at time $t \in [0,T]$, is*

$$\xi_P(t) \quad = \quad \begin{cases} \frac{1}{y} u'(\varepsilon(t)+k) & \text{if} \quad \xi(t) < \frac{1}{y} u'(\varepsilon(t)+k) \qquad (\textit{binding}), \\ \xi(t) & \text{if} \quad \frac{1}{y} u'(\varepsilon(t)+k) \leq \xi(t) \qquad (\textit{not binding}), \end{cases} \tag{13}$$

where $y > 0$ solves (12). Consequently, $\xi_P(t) > \xi(t)$ in the binding region, $\xi_P(t) = \xi(t)$ otherwise.

Consistent with the existing literature on import quotas, when the period-by-period constraint is binding, the domestic state price density of the quota-protected commodity (and hence its price $p_P(t)$) is higher than its world price. The domestic price has to exceed the world price because otherwise there would be excess demand for the quota-protected commodity in the domestic market and hence the small open economy P cannot be in equilibrium. In the states of the world where the constraint is not binding, there is no effect of quotas on domestic prices in country P.

3.3. Quota Constrained Country's Equilibrium

Before characterizing the optimal time-t consumption of the quota-constrained country Q, we consider its consumption at time T in order to highlight similarities with the period-by-period constrained case. Under further regularity conditions on the optimal consumption process,[5] we may derive the optimal time-T consumption as:

$$c_Q(T) = \begin{cases} \varepsilon(T) + \dot{X}(T) & \text{if} \quad \xi(T) < \frac{1}{y}\, u'\Big(\varepsilon(T) + \dot{X}(T)\Big) \quad \text{(binding)}, \\ I(y\,\xi(T)) & \text{if} \quad \frac{1}{y}\, u'\Big(\varepsilon(T) + \dot{X}(T)\Big) \leq \xi(T) \quad \text{(not binding)}, \end{cases} \tag{14}$$

$$c_Q^*(T) = I^*(y\xi^*(T)),$$

where $X(t) \equiv \int_0^t (c(s) - \varepsilon(s))^+ ds$ denotes the past cumulative imports of the quota-protected good at time t and $\dot{X}(T)$ denotes the time derivative of $X(T)$ given by $\lim_{t \to T}(K - X(t))/(T - t)$ in the quota binding region. The terminal consumption of the quota-protected country behaves similarly to that of the period-by-period constrained.

Figure 2 depicts the optimal terminal consumption of the quota-protected country, seen to resemble the plots in Figure 1. For high enough state price density or terminal endowment it is unconstrained and behaves like a normal country N. At a lower state price density or endowment the country hits its constraint and imports the remaining amount allowed. Whereas in country P the maximal amount allowed is the constant k, in country Q it is equal to the quantity $\dot{X}(T)$, which can be approximated by $(K - X(t))/(T - t)$. Hence, unlike that of country P, the terminal consumption of the quota-constrained country depends not only on the terminal state price density and endowment, but also on an extra variable, the past cumulative imports, $X(t)$, lending a path-dependence to the terminal consumption.

[5] In particular, we require that c belongs to the class of left-continuous processes on $[0, T]$. Then the first order condition with respect to $c(t)$, holding a.s.-for-Lebesque-almost-every $t \in [0, T]$, becomes valid in a stronger sense, a.s. for all t, including $t = T$. This regularity condition is natural and not restrictive. It amounts to replacing the process c solving (7) by its continuous modification, which can be done in our context thanks to Theorem 5.16 in Liptser and Shiryayev (1977). If the original solution to (7) is optimal, so is its continuous modification, as it disagrees with the original solution only on a set of $\mathcal{P} \times \lambda$-measure 0. This regularity condition would, of course, not be necessary in a discrete-time version of our model, where each period has a positive mass.

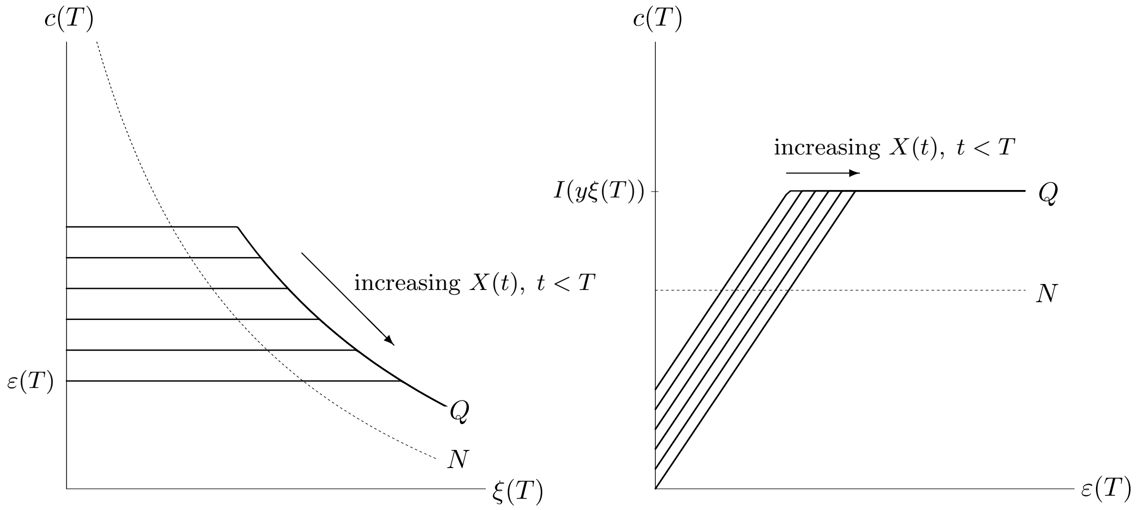


Figure 2: Country Q's optimal terminal consumption of the quota-protected commodity versus terminal world state price density associated with the quota-protected good, $\xi(T)$, and endowment of the quota-protected good, $\varepsilon(T)$. The plots are for varying levels of past cumulative imports of the quota-protected good, $X(t) \equiv \int_0^t (c_Q(s) - \varepsilon(s))^+ ds$. The dotted plot is for the unconstrained country N.

We now turn to the optimal consumption of the country for $t < T$. The solution to (7), assuming it exists, is characterized in Proposition 3.

Proposition 3. *The optimal time-t, $t \in [0, T)$, consumption of the quota constrained country is:*

$$c_Q(t) = \begin{cases} I(y\,\xi(t) + E_t[\lambda(T)]) & \text{if} \quad \xi(t) < \frac{1}{y}\left(u'(\varepsilon(t)) - E_t[\lambda(T)]\right), & \textit{(import)} \\ \varepsilon(t) & \text{if} \quad \frac{1}{y}\left(u'(\varepsilon(t)) - E_t[\lambda(T)]\right) \le \xi(t) < \frac{1}{y}\,u'(\varepsilon(t)), & \textit{(no trade)} \\ I(y\,\xi(t)) & \text{if} \quad \frac{1}{y}\,u'(\varepsilon(t)) \le \xi(t), & \textit{(export)} \end{cases} \tag{15}$$

$$c_Q^*(t) = I^*(y\,\xi^*(t)), \tag{16}$$

where $E_t[\cdot]$ is shorthand for $E[\cdot|\mathcal{F}_t]$, and the constant $y > 0$ and $\mathcal{F}_T$-measurable random variable $\lambda(T) \ge 0$ solve

$$\begin{cases} E\left[\int_0^T [\xi(t)\,c_Q(t;\,y,\,E_t[\lambda(T)]) + \xi^*(t) c_Q^*(t;\,y)]dt\right] = E\left[\int_0^T [\xi(t)\,\varepsilon(t) + \xi^*(t)\,\varepsilon^*(t)]dt\right], \\ \int_0^T (c_Q(t;\,y,\,E_t[\lambda(T)]) - \varepsilon(t))^+ dt = K \quad \textit{or} \quad \lambda(T) = 0. \end{cases} \tag{17}$$

Moreover, (i) the quota never gets exhausted before time T, (ii) $y \le y_N$, where y_N is the analogous constant for the unconstrained country with the same endowment process $(\varepsilon, \varepsilon^*)$, and (iii) $c_Q(t) \ge c_N(t)$ in the export region and $c_Q^*(t) \ge c_N^*(t)$ across all regions, where $c_N(t)$ and $c_N^*(t)$ are the consumption of the quota-protected and composite goods, respectively, of the unconstrained country with the same endowment process $(\varepsilon, \varepsilon^*)$.

Although (15) is proven in the Appendix by a convex duality technique, for intuition it is helpful to think of $\lambda(T)$ as the Lagrange multiplier associated with the quota constraint, capturing how tightly the constraint binds at time T. For high enough terminal state price density or endowment, or low enough cumulative past imports, the constraint does not bind, so that $\lambda(T) = 0$. As the quota gets depleted, with decreasing state price density or endowment, or increasing past imports, the constraint binds and $\lambda(T)$ begins to increase above zero. At earlier times, this multiplier appears in expectation because the constraint must hold almost surely along all paths.[6] The conditional expectation $E_t[\lambda(T)]$ then captures the expectation, at earlier times, of how tightly the constraint will bind at time T.[7] The quantity $E_t[\lambda(T)]$ can be mapped into the price of a quota license, as discussed in Appendix C.

Pre-horizon consumption of the quota-protected commodity is considerably different from that of country P. There are now three regions of distinct economic behavior, depending on whether the country is importing, exporting or neither. For high enough state price density associated with the quota-protected good or current endowment of the quota-protected good, the country exports and behaves like an unconstrained one. For low enough state price density or current endowment, it imports and here it behaves as if facing an additional "cost" $E_t[\lambda(T)]$ to consumption over and above the normal country. In this region, consuming (which amounts to importing) is "costly" to the country for two reasons: via the usual cost $y\xi(t)$ because its budget constraint prevents it from consuming as much in other states and times, and via an additional cost $E_t[\lambda(T)]$ because any import it makes contributes towards its hitting its import constraint at the horizon. Finally, there is an extended intermediate region in which the quota-constrained country does not trade in the quota-protected commodity and so consumes exactly its concurrent endowment $\varepsilon(t)$. This contrasts sharply with an unconstrained country or country P for whom the no-trade region is just a knife-edge. This extended no-trade region is analogous to behavior exhibited by agents facing other types of nonlinearity in their cost/price structure.[8] The nonlinear cost structure implicitly arises here because exports do not contribute to the quota allowance, while imports

[6]Wang (2001), who also has an integral-type constraint in his model, deduces a similar expectation term in an optimal policy, but unlike our case this expectation term arises over the whole region.

[7]Under further regularity on the consumption process (see footnote 5), the multiplier $\lambda(T)$ is given by $\lambda(T) = (u'(\varepsilon(T)+\dot{X}(T))-y\,\xi(T))^+$. This structure resembles the payoff of an exchange option, giving the right to exchange the quantity $u'(\varepsilon(T)+\dot{X}(T))$ for $y\,\xi(T)$. The former quantity is the country's marginal utility when binding on its constraint, and the latter the unconstrained country's marginal utility. The "cost" to the country of the quota constraint at $t < T$ can then be linked to the price of this exchange option.

[8]Examples include: an agent facing a different interest rate for borrowing versus lending who exhibits an extended region over which he neither borrows nor lends (Cvitanić and Karatzas (1992)); an agent facing a nonlinear taxation on portfolio holdings (a higher tax rate for long positions than short positions) who exhibits an extended region of no-holdings (Basak and Croitoru (2001)); an investor facing a securities market with proportional transaction costs who exhibits an extended region where he does not rebalance his portfolio (Davis and Norman (1990)); countries facing shipping costs who exhibit an extended no-trade region in equilibrium (Dumas (1992), Uppal (1993)).

do. At any time prior to the horizon ($t < T$), as long as there is some probability of hitting the constraint (so that $E_t[\lambda(T)] \neq 0$), the additional cost to importing is non-zero. Once the horizon is reached, however, the three regions collapse to only two regions. All uncertainty about $\lambda(T)$ has been resolved, with $\lambda(T)$ only being non-zero once the quota constraint binds, and zero for the full non-binding region, including the no-trade point. This explains how the three regions prior to the horizon collapse to only two regions at the horizon.

The consequence of the additional cost in the import region is to make the country consume less of the protected good (and hence import less) than it would have done if facing $y\xi(t)$ alone. How much less it consumes depends on his current expectation of the severity of the quota constraint $\lambda(T)$ at the horizon. Hence this expectation $E_t[\lambda(T)]$ is an extra driving factor in consumption in this region. Consequently, the structure of the optimal consumption is considerably more complex than that for the period-by-period constrained country. Not only is it driven by the concurrent state price density and endowment, but also (via $E_t[\lambda(T)]$) is driven by the past cumulative imports $X(t) = \int_0^t (c_Q(s) - \varepsilon(s))^+ ds$, and the conditional distribution of the future cumulative imports $X(T) - X(t) = \int_t^T (c_Q(s) - \varepsilon(s))^+ ds$. To solve explicitly for the optimal consumption $c_Q(t)$ involves a forward-backward solution due to the multiplier $\lambda(T)$ in Proposition 3, which does not appear possible in our general setting, hence we merely treat the past cumulative imports as an extra variable and characterize optimal consumption behavior as a function thereof.

Since the endowment process ε can get arbitrarily close to zero, (such as geometric Brownian motion), the quota can never be exhausted before the horizon. This is because if the quota binds prematurely, the country's consumption cannot exceed its endowment for the remainder of the time horizon. This policy is shown to be suboptimal, as it forces the country to forgo opportunities to import when its endowment or state prices become very low. However, for the case of endowment and the state prices being bounded away from zero, we have worked out an example in discrete time where the constraint may bind before time T. In practice, quotas are not likely to be fully utilized prior to their expiration. For example, in the data set we employ in our empirical analysis in Appendix C, out of the total 362 listed observations for all quota-protected categories from 1999 to 2002, only one quota was fully utilized by the end of October. If we use a milder, 85% threshold to connote "binding", 0% of quotas were binding by the end of May, and only 12.68% were binding by the end of October.

Finally, Proposition 3 reveals that, like country P, the quota-constrained country exhibits an unambiguously lower multiplier associated with its budget constraint (or indirect marginal utility of initial wealth) than an identically endowed normal country. This is because either constrained country is forced to consume less once its import constraint kicks in, and so can afford to consume more in the unconstrained states. Hence, as seen in Figures 1, 2, and 5, either constrained country consumes more of the protected good than an identically endowed normal country in states with high state price or endowment (the export region). Additionally, the quota-constrained country shifts consumption across goods; its composite good consumption is higher than that of the normal country.

Given the optimal allocations determined in Proposition 3, we can infer the price of the protected commodity faced by the country-Q consumers, consistent with domestic market clearing.

Proposition 4. *The state-price density associated with the quota-protected good in the quota-constrained country prevailing in the domestic market equilibrium, at time $t \in [0,T]$, is given by*

$$\xi_Q(t) = \begin{cases} \xi(t) + \frac{1}{y} E_t[\lambda(T)] & \textit{if} \quad \xi(t) < \frac{1}{y}\left(u'(\varepsilon(t)) - E_t[\lambda(T)]\right), & \textit{(import)} \\ \frac{1}{y} u'(\varepsilon(t)) & \textit{if} \quad \frac{1}{y}\left(u'(\varepsilon(t)) - E_t[\lambda(T)]\right) \leq \xi(t) < \frac{1}{y}\, u'(\varepsilon(t)), & \textit{(no trade)} \\ \xi(t) & \textit{if} \quad \frac{1}{y}\, u'(\varepsilon(t)) \leq \xi(t), & \textit{(export)} \end{cases} \quad (18)$$

where $y > 0$ solves (17). Consequently, $\xi_Q(t) > \xi(t)$ in import and no trade regions, $\xi_Q(t) = \xi(t)$ in export region.

Similarly to country P and the workhorse quota model, import quota protection gives rise to higher domestic prices; here, the domestic state price density $\xi_Q(t)$ (and hence $p_Q(t)$) is higher than the world state price density $\xi(t)$ when the country is importing or not trading. In sharp contrast to the workhorse model, which predicts quotas to have no economic effects unless they are binding, an import quota does not have to be fully exhausted for it to have an effect on prices in our economy. In fact, the quota constraint never binds in our model for $t \in [0,T)$, and yet prices go up on the expectation that the constraint may bind in the future. This result is especially important since in practice quotas are rarely fully exhausted (see Appendix C for empirical evidence).

Quantitative restrictions are often blamed for being a too rigid form of protection. To contest this, we present Figure 3, which sketches the domestic state price density for the quota-constrained country (plot (b)) and the domestic state price density prevailing in country P for comparison (plot (a)). As seen from Figure 3a, indeed, unlike under a tariff protection, the import policy of a period-by-period constrained country binding on its quota constraint becomes completely insensitive to economic conditions (both the domestic price and the quantity imported are independent of world prices). The behavior of the quota-constrained country however is quite different. Figure 3b demonstrates that under a quota-type constraint, the country's policy may also become insensitive to economic conditions – but only for a range of prices for which the country refrains from trade. In that region, the country does not import the protected commodity even when its world price is below the domestic price and the quota is not fully exhausted − in contrast to the prediction of the workhorse model. Once a critically low level of prices is reached, the country transitions from the no-trade region to the import region, where domestic policies again become sensitive to economic conditions. The lower the world price, the more the country trades and hence the lower the domestic price drops. The dynamic nature of the quota constraint therefore ensures that, albeit at an additional cost, the country may increase its imports in the constrained states if it wishes to do so.

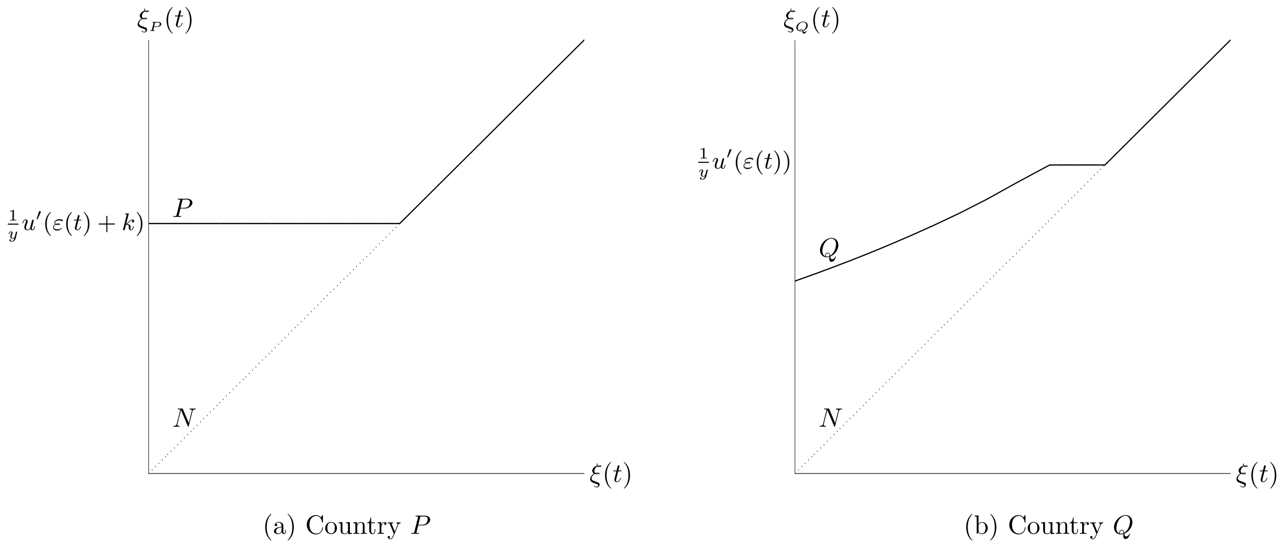


Figure 3: Domestic state price density associated with the quota-protected good in the period-by-period P (plot (a)) and quota-constrained Q (plot (b)) countries versus world state price density $\xi(t)$. The dotted plots are for the unconstrained country N. The quantity $E_t[\lambda(T)]$ used in the construction of plot (b) is from Figure 4a (computed numerically in Section 3.4).

Remark 1 (Certainty Case). It is of interest to compare with the certainty case since the bulk of the existing literature covers this arena. The certainty case is a special case of our model, where now ε is a deterministic function of time, with no ω and σ processes existing. Here, whether the quota binds ($\lambda > 0$) or not ($\lambda = 0$) will be known with certainty at $t = 0$. If not binding, the quota will have no economic effects. (This case never arises in our uncertainty model, except at T, since there is always some probability of the quota binding in the future.) If the quota binds, there will still be three regions for optimal consumption, except now the Lagrange multiplier, or additional cost term λ, will be a constant, so none of the discussed state-dependency of this term will arise.

Remark 2 (Export Quota Constraint). The analysis of the export quota constraint,

$$\int_0^T (\varepsilon(t) - c(t))^+ dt \leq K,$$

follows from the import constraint symmetrically. It is now the export region, prior to the horizon, which is directly impacted by the constraint. The Lagrange multiplier $\lambda(T)$, or $E_t[\lambda(T)]$, appears as a negative cost, encouraging the country to consume more, or export less, in the export region:

$$c_Q(t) = I(y\,\xi(t) - E_t[\lambda(T)]).$$

Remark 3 (Repeated Quota Constraints). The analysis of a series of repeated import quota constraints,

$$\int_{mT}^{(m+1)T} (c(t) - \varepsilon(t))^+ dt \leq K, \qquad m = 1, \ldots, M,$$

follows along similar lines to the single constraint. The optimal consumption inherits an expression identical to Proposition 3, but the cost term $E_t[\lambda(T)]$ is replaced by $E_t[\lambda(mT)]$, where mT is the next closest quota horizon. All other quantities in the import region are analogously impacted by this and only this cost term associated with the next horizon; the only difference is that at each horizon the cost term expires and then is reset to the next horizon's expected value.

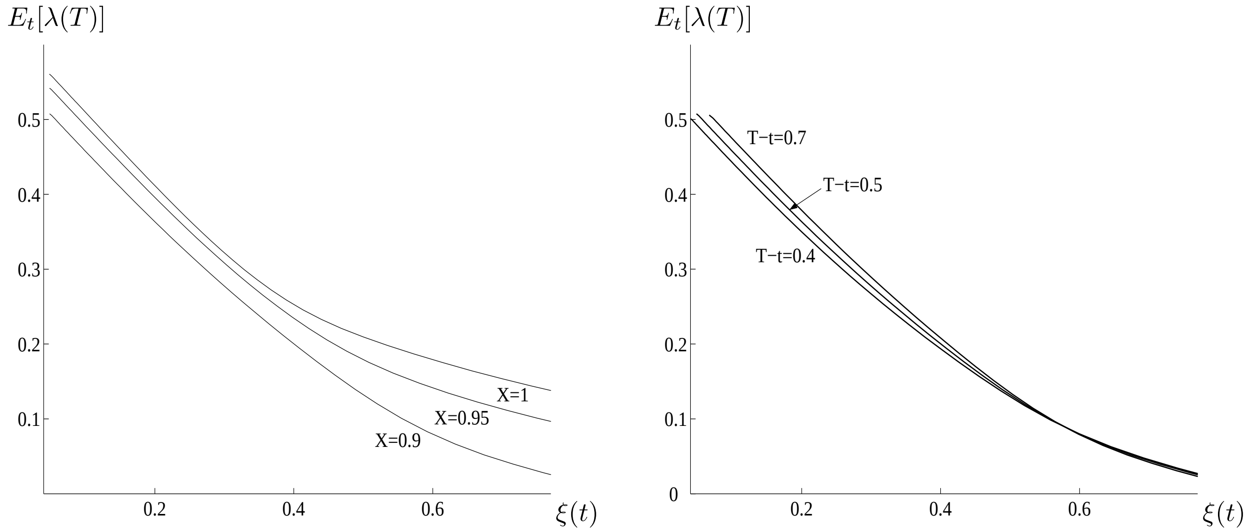
3.4. Further Properties of the Quota-Constrained Strategy

To further analyze the optimal behavior under a quota constraint, we need a more explicit representation of the martingale $E_t[\lambda(T)]$ as a function of concurrent state variables. This requires either numerical analysis or simplifying distributional assumptions coupled with approximation. A ballpark formula can be obtained along the lines of the earlier version of this paper.[9] In the interest of rigor however we chose to present the numerical analysis. We derive the Hamilton-Jacobi-Bellman equation for the quota-constrained country's problem and solve it numerically. The details are in Appendix B.

The preceding analysis in Section 3.3 identifies the driving state variables of the quota-constrained economy. Some driving factors, the current economic conditions, the world price

[9] In particular, we recognize that $\lambda(T) = (u'(\varepsilon(T) + \dot{X}(T)) - y\xi(T))^+$ has the form of the payoff of an option to exchange one asset for another. To evaluate its conditional expectation, one may make distributional assumptions on the primitives to be consistent with the Black-Scholes framework, and use option-pricing techniques. We may approximate $\varepsilon(t) + \dot{X}(t)$ as a geometric Brownian motion with parameters $\mu_{\varepsilon+\dot{X}}$ and $\sigma_{\varepsilon+\dot{X}}$, the mean growth rate and volatility of output plus the rate of imports, respectively. For CRRA preferences, this will ensure that $u'(\varepsilon(t) + \dot{X}(t))$ follows a geometric Brownian motion. Additional inputs into the Black-Scholes-type formula are market parameters such as the price of the protected commodity, which also follows a geometric Brownian motion, with constant mean growth rate and volatility.

of the protected commodity (or the world state price density associated with the protected commodity) and the country's endowment of the commodity are the same as in the period-by-period constrained economy. However, the analysis identifies two additional economic factors: the time-remaining till the quota horizon and the past cumulative imports or, equivalently, quota utilization rate to date, as typically reported in practice. Figure 4 plots $E_t[\lambda(T)]$ against the concurrent world state price density associated with the quota-protected good for varying levels of (a) past cumulative imports $X(t)$ and (b) time-remaining $T-t$.



(a) The effect of $X(t)$ (b) The effect of $T-t$

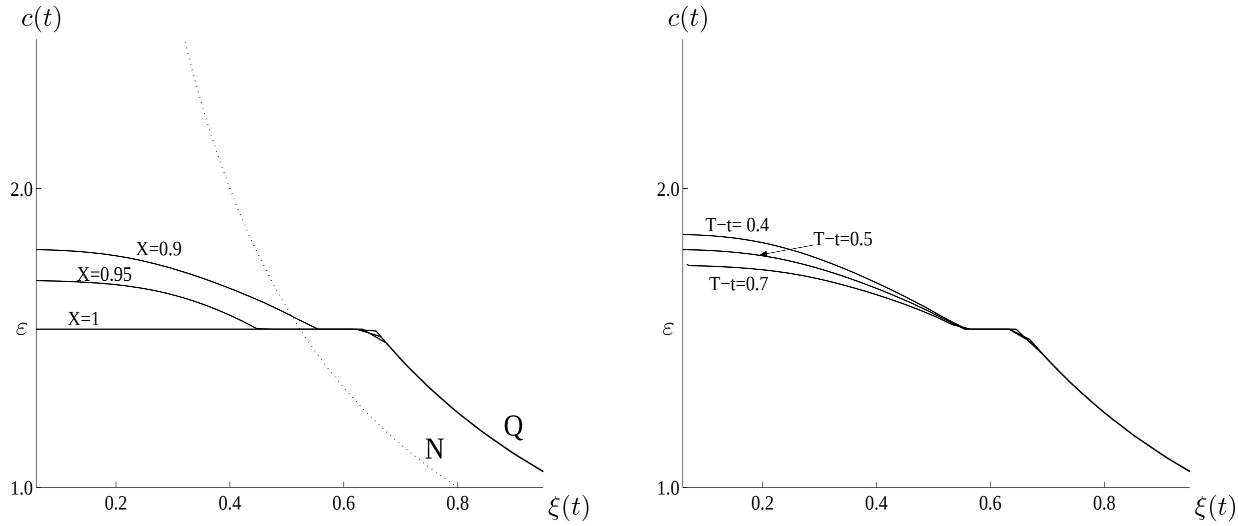
Figure 4: Conditional expectation of the quota constraint multiplier $E_t[\lambda(T)]$ versus concurrent world state price density associated with the quota-protected good, $\xi(t)$. The plots assume $u(c) = \log(c)$, $v(c^*) = \log(c^*)$ and are for varying levels of : (a) past cumulative imports $X(t) \in \{0.9, 0.95, 1\}$ and (b) time-remaining till the quota horizon $T-t \in \{0.4, 0.5, 0.7\}$. The fixed parameter values are: $K = 1$, $p = 1.5$, $r = 0.05$, $\mu_S = 0.1$, $\mu_e = 0.1$, $\sigma_S = 0.2$, $\sigma_e = 0.04$, $\rho = 0.5$. $T-t = 0.5$ (plot (a)), $X = 0.9$ (plot (b)), and $\varepsilon = 1.52$.[10]

Recall that $E_t[\lambda(T)]$ captures the time-t expectation of how tightly the constraint will bind at T. As we might anticipate, $E_t[\lambda(T)]$ is a smoothed version of $\lambda(T) = (u'(\varepsilon(T)+\dot{X}(T)) - y\xi(T))^+$, an option-like payoff (see footnote 7). As an option price would be, the process $E_t[\lambda(T)]$ is convex in the underlying processes $\xi(t)$ and $\varepsilon(t)$. $E_t[\lambda(T)]$ is decreasing in the current state price density and endowment; a higher current $\xi(t)$ or $\varepsilon(t)$ imply a higher terminal state price or endowment so that the constraint is less likely to bind. $E_t[\lambda(T)]$ is increasing in the past cumulative imports $X(t)$ (quota utilization); the more the country has already imported, the more likely the constraint is to bind at time T. Figure 4(b) shows that at low levels of current state prices, $E_t[\lambda(T)]$ decreases over time. This is because, for the same level of past cumulative imports of the protected commodity,

[10]The definitions of the parameters μ_S, μ_e, σ_S, σ_e and ρ are provided in Appendix B.

the longer the time-remaining till the quota horizon, the more restrictive the quota is. $E_t[\lambda(T)]$ must then be higher in the import (low ξ) states to discourage imports.

Figure 5 plots the quota-constrained country's pre-horizon consumption of the protected commodity against the concurrent world state price density for varying levels of (a) past cumulative imports and (b) time-remaining. The graphs demonstrate the behavior, in the three regions of the quota-constrained country's pre-horizon consumption as described in Section 3.3. As for an unconstrained country, the consumption is decreasing in the concurrent cost of consumption. The consumption is decreasing in past cumulative imports (or quota utilization); the more the country has already imported the more likely its constraint is to bind and so the more costly is a current import. A country facing a longer time till the quota expiration has lower consumption in the import region, due to increased cost of consumption $E_t[\lambda(T)]$ (Figure 3b).



(a) The effect of $X(t)$ (b) The effect of $T-t$

Figure 5: Country Q's optimal time-t consumption of the quota-protected commodity versus concurrent world state price density associated with the quota-protected good. The plots assume $u(c) = \log(c)$, $v(c^*) = \log(c^*)$ and are for varying levels of : (a) past cumulative imports $X(t) \in \{0.9, 0.95, 1\}$ and (b) time-remaining till the quota horizon $T-t \in \{0.4, 0.5, 0.7\}$. The dotted plot is for the unconstrained normal country N. The fixed parameter values are: $K = 1$, $p = 1.5$, $r = 0.05$, $\mu_S = 0.1$, $\mu_e = 0.1$, $\sigma_S = 0.2$, $\sigma_e = 0.04$, $\rho = 0.5$. $T - t = 0.5$ (plot (a)), $X = 0.9$ (plot (b)) and $\varepsilon = 1.52$.

An interesting feature of the quota constrained country's consumption choice over that of an unconstrained country is that as the direct cost of consumption of the protected commodity, $\xi(t)$, tends to zero, its consumption does not rise without bound. This is because any import also has the indirect cost of contributing to cumulative imports, which are constrained. Hence the net cost does not tend to zero and the consumption is bounded. A second curious feature in Figure 5(a) is the concavity of $c(t)$ in $\xi(t)$. The consumption of an unconstrained (decreasing absolute risk aversion) country is convex in the state price density. For the quota-constrained country, this may break down because the additional cost, $E_t[\lambda(T)]$, is convex in the state price density yielding a concave component to the consumption behavior (since a decreasing function of a convex one is concave). One can see how this concavity is required in order to collapse to the option-type payoff at the horizon, infinitely concave at the kink.[11]

Figure 5 also allows us to compare the consumption (and import/export) levels of the quota-constrained country to those of an unconstrained country. At low enough state price density and endowment, the quota-constrained country consumes less of the protected commodity than the unconstrained country because it is importing and this has positive additional cost of contributing towards the cumulative imports. At high enough state price density and endowment, the quota-constrained country can, then, afford to consume more than the unconstrained. As we mentioned in Section 3.3, Figure 4 shows the export region to shrink and exports to be reduced. The import region may shrink or grow, and at low enough state price density or endowment imports are reduced while at high enough they may be increased.

4. Extensions and Ramifications

The primary object of our investigation has been a country whose production of a protected commodity is marginal relative to that produced in the world or the country's total output, or has no spillover effects on other sectors in the economy. Here, we explore several important extensions of our small open economy (Section 3). First, we consider a country that is large enough to affect world prices. Second, we examine a country whose quota-protected industry constitutes a significant fraction of the country's economy. Third, we study a country populated by consumers who exhibit non-separable preferences across the two goods in the economy, allowing for the spillover from the quota-protected sector to the rest of the economy through the consumers' demands.

4.1. World Equilibrium under Import Quota Constraints

In this Section, we examine the equilibrium world prices in a heterogeneous economy, consisting of one quota-constrained and one unconstrained country, and discuss the effects of an import

[11]For brevity, we omitted the graphs illustrating the dependence of $E_t[\lambda(T)]$ and consumption on the country's endowment, however the intuition is clear. Since $E_t[\lambda(T)]$ is decreasing in the underlying process $\varepsilon(t)$, optimal consumption is an increasing function of the country's endowment in the import region, in addition to the no trade region. As the country's endowment increases, the expectation of terminal endowment increases so the constraint is less likely to bind and so the country treats an import as less "costly" in this sense.

constraint in the case where the quota-constrained country is large enough to affect world prices.

Definition. *An equilibrium is a collection of price parameters* $(r,\ \mu,\ \sigma,\ \mu_p,\ \sigma_p)$ *and associated optimal policies* $(c_N,\ c_N^*,\ c_Q,\ \pi_N,\ \pi_Q)$ *such that the good, stock and bond markets clear:*

$$\begin{aligned} c_N(t) + c_Q(t) &= \varepsilon_N(t) + \varepsilon_Q(t)\,, \quad c_N^*(t) + c_Q^*(t) = \varepsilon_N^*(t) + \varepsilon_Q^*(t)\,, \\ \pi_{Ni}(t) + \pi_{Qi}(t) &= S_i(t)\,, \quad \pi_{Ni}^*(t) + \pi_{Qi}^*(t) = S_i^*(t)\,, \qquad i \in \{N,\, Q\}\,, \\ W_N(t) + W_Q(t) &= S_N(t) + S_Q(t) + S_N^*(t) + S_Q^*(t)\,. \end{aligned} \tag{19}$$

Recall, the multipliers $(y_N,\, y_Q)$ solve each country's budget constraint holding with equality at the optimum. However, good market clearing (19) together with one budget constraint implies the other budget constraint, and hence one of $(y_N,\, y_Q)$ is indeterminate. Without loss of generality, we can set $y_N = 1$ and define $y \equiv y_Q$. Then, for convenience, we sometimes make use of a "world representative agent" with utilities over the quota-protected and composite good defined by

$$U(c;\, y) \equiv \max_{c_N + c_Q = c} u_N(c_N) + \frac{1}{y} u_Q(c_Q)\,, \qquad V(c^*;\, y) \equiv \max_{c_N^* + c_Q^* = c^*} v_N(c_N^*) + \frac{1}{y} v_Q(c_Q^*)\,.$$

In the benchmark unconstrained economy, the constant $1/y$ would fully represent the relative weight of country Q as compared with country N.

For brevity, we omit the analysis of equilibrium under period-by-period import constraints. The small open economy results extend naturally, with the equilibrium consisting of two regions, depending on the relative endowments of the two countries. Domestic price implications of a small open economy (Section 3.2) also carry through: the domestic price of the protected good rises above the world's in the binding region, and equals otherwise. The period-by-period constrained country now affects the world state price density associated with the protected good, pushing it lower than in the unconstrained world.

Analogously to the small open economy, equilibrium at the horizon in the quota-constrained economy resembles that in the period-by-period constrained. Horizon quantities fall into two regions, when the quota-constrained country binds on its quota constraint and when it does not. In order to highlight its novel features, Proposition 5 characterizes the pre-horizon equilibrium in the economy with one quota-constrained and one unconstrained country. (Equations (20)–(26) are also valid at the horizon, but the proof in Appendix A provides simpler expressions for $t = T$ revealing the collapse to two regions.)

Proposition 5. *If equilibrium exists in an economy with one unconstrained country* N *and one quota-constrained country* Q*, the equilibrium world state price densities, country-*Q *domestic state* *price density associated with the quota-protected good and consumption allocations are*

$$\xi(t) = \begin{cases} G(\varepsilon_N(t)+\varepsilon_Q(t);\, y,\, E_t[\lambda(T)]) & \text{if} \quad u'_N(\varepsilon_N(t)) < \frac{1}{y}\Big(u'_Q(\varepsilon_Q(t)) - E_t[\lambda(T)]\Big) \qquad (Q \text{ imports}) \\ u'_N(\varepsilon_N(t)) & \text{if} \quad \frac{1}{y}\Big(u'_Q(\varepsilon_Q(t)) - E_t[\lambda(T)]\Big) \le u'_N(\varepsilon_N(t)) < \frac{1}{y}u'_Q(\varepsilon_Q(t)) \qquad (\text{no trade}) \\ U'(\varepsilon_N(t)+\varepsilon_Q(t);\, y) & \text{if} \quad \frac{1}{y}u'_Q(\varepsilon_Q(t)) \le u'_N(\varepsilon_N(t))\,, \qquad (Q \text{ exports}) \end{cases} \tag{20}$$

$$\xi^*(t) = V'(\varepsilon^*_N(t)+\varepsilon^*_Q(t);\, y)\,, \tag{21}$$

$$\xi_Q(t) = \begin{cases} G(\varepsilon_N(t)+\varepsilon_Q(t);\, y,\, E_t[\lambda(T)]) + \frac{1}{y}E_t[\lambda(T)] & \text{if} \quad Q \text{ imports}\,, \\ \frac{1}{y}u'_Q(\varepsilon_Q(t)) & \text{if} \quad \text{no trade}\,, \\ U'(\varepsilon_N(t)+\varepsilon_Q(t);\, y) & \text{if} \quad Q \text{ exports}\,, \end{cases} \tag{22}$$

$$c_N(t) = \begin{cases} I_N\Big(G(\varepsilon_N(t)+\varepsilon_Q(t);\, y,\, E_t[\lambda(T)])\Big) & \text{if} \quad Q \text{ imports}\,, \\ \varepsilon_N(t) & \text{if} \quad \text{no trade}\,, \\ I_N\Big(U'(\varepsilon_N(t)+\varepsilon_Q(t);\, y)\Big) & \text{if} \quad Q \text{ exports}\,, \end{cases} \tag{23}$$

$$c_Q(t) = \varepsilon_N(t)+\varepsilon_Q(t) - c_N(t)\,, \quad c^*_N(t) = I^*_N(V'(\varepsilon^*_N(t)+\varepsilon^*_Q(t)))\,, \quad c^*_Q(t) = I^*_Q(y\,V'(\varepsilon^*_N(t)+\varepsilon^*_Q(t)))\,, \tag{24}$$

where $G(\,\cdot\,; y, E_t[\lambda(T)])$ *is the inverse aggregate demand function with respect to* $\xi(t)$*, i.e., is the solution to* $c = I_Q(y\,G(c\,;\, y,\, E_t[\lambda(T)]) + E_t[\lambda(T)]) + I_N(G(c\,;\, y,\, E_t[\lambda(T)]))$*, and in particular* $G(\,\cdot\,;\, y,\, 0) = U'(\,\cdot\,;\, y)$*;* y *satisfies one of the countries' budget constraints, i.e.,*

$$E\left[\int_0^T [\xi(t;\, y,\, E_t[\lambda(T)])I_N(\xi(t;\, y,\, E_t[\lambda(T)])) + \xi^*(t;\, y)I^*_N(\xi^*(t;\, y))]\,dt\right] = E\left[\int_0^T [\xi(t;\, y,\, E_t[\lambda(T)])\varepsilon_N(t) + \xi^*(t;\, y)\varepsilon^*_N(t)]\,dt\right]\,, \tag{25}$$

and $\lambda(T) \ge 0$ *solves* $$\int_0^T (c_Q(t;\, y,\, E_t[\lambda(T)]) - \varepsilon_Q(t))^+ dt = K \text{ or } \lambda(T) = 0\,. \tag{26}$$

Conversely, if there exists ξ*,* ξ^**,* ξ_Q*,* c_N*,* c_Q *and* y *satisfying (20)–(26), then the associated optimal policies clear all markets.*

The pre-horizon equilibrium contains three regions: when the quota constrained country is relatively highly endowed with the quota-protected good compared with the unconstrained country, it "exports" and the state price density is determined, as in the benchmark unconstrained economy, by the world representative agent's marginal utility of the protected good. Then there is an extended intermediate "no-trade" region where the countries are relatively evenly endowed in which the costliness of importing to the constrained country is sufficient to prevent trade. In this region, the market for the protected good is effectively segmented, and hence the quota-constrained country does not participate in setting the world price for the good. As the relative endowment of the unconstrained country increases further, eventually there is enough imbalance to overcome the costliness of importing, and the constrained country begins to import from the unconstrained. In the "import" region, the world state price density is priced by both countries, reflecting both marginal utilities of the quota-protected good consumption. However, these marginal utilities are not proportional, so the state prices are not given by the standard representative agent's marginal utility of the quota-protected good. The (stochastic) costliness of importing, $E_t[\lambda(T)]$, to the constrained country, must also be taken into account. Depending on the size of $E_t[\lambda(T)]$, the higher the expectation of how tightly the quota constraint will bind at the horizon, the less "weight" is assigned to the quota-protected country in that state of the world. This causes it to import less than it would otherwise have done. The world state price density ξ^* is determined in the usual way by the world representative agent's marginal utility of the composite good consumption. The behavior of the domestic state price density in country Q retains its small open economy implications (Section 3.3): the domestic price of the protected good is higher than its world price when there is no trade or Q is importing. Further characterizations regarding the dynamics of the equilibrium state price densities (interest rates and market prices of risk) and the consumption allocations (volatility and mean growth rates) may be derived from Proposition 5 – we have omitted them here for brevity (details available upon request).

Proposition 6 reports additional results on relative good prices, stock prices and countries' wealth that are derived as a consequence of Proposition 5.

Proposition 6. *Let the superscript ^{u} refer to the pertinent variable in the unconstrained economy with two unconstrained countries and no import constraints. Then*

$$\frac{1}{y} > \frac{1}{y^u} .$$

Moreover, assume $u_i(c) = log(c)$ and $v_i(c^*) = log(c^*)$, $i \in \{N, Q\}$. Then

(i) $p(t) < p^u(t) < p_Q(t)$ *when Q imports or does not trade, and* $p(t) = p_Q(t) = p^u(t)$ *when Q exports;*

(ii) $S_N(t) < S_N^u(t)$, $S_Q(t) > S_Q^u(t)$;

(iii) $S_i^*(t) = S_i^{*u}(t)$, $i \in \{N, Q\}$;

(iv) Consider the total wealth of each country: $\hat{W}_N(t) = \frac{1}{\xi^*(t)} E_t \left[\int_t^T [\xi(s) c_N(s) + \xi^*(s) c_N(s)] ds \right]$, $\hat{W}_Q(t) = \frac{1}{\xi^*(t)} E_t \left[\int_t^T [\xi_Q(s) c_Q(s) + \xi^*(s) c_Q(s)] ds \right]$. *We have:* $\hat{W}_N(t) + \hat{W}_Q(t) = \hat{W}_N^u(t) + \hat{W}_Q^u(t)$, *while* $\hat{W}_N(t) < \hat{W}_N^u(t)$ *and* $\hat{W}_Q(t) > \hat{W}_Q^u(t)$.

In the "Q exports" region, the quantity $1/y$ captures the relative weightings of the two countries. Since the quota-constrained country Q is restricted to import less of the protected commodity than it would have done if unconstrained, prices must adjust to persuade the unconstrained country to export less, which they do. Then Q can afford to export less when exporting while the unconstrained country cannot afford to import as much. Hence the unconstrained country behaves in this region as if it has less weighting, and the constrained country has higher weighting. Under logarithmic preferences with no wealth effects present, further implications obtain. In the constrained states (Q imports or no-trade), the unconstrained country must be persuaded to export less (i.e., consume more) of the quota-protected commodity, while in unconstrained states to import less of it (i.e., consume less). To achieve this, the world price of protected commodity drops in the constrained states relative to the unconstrained. Consistent with the small open economy results, the domestic price of the protected good p_Q rises above the world price in the constrained states, otherwise remains as in the unconstrained economy. However, because of the possibility of constrained states occurring in the future, the long-lived stock price of the quota-protected industry is always decreased in the unconstrained country N and increased in the quota-constrained country Q. This result obtains even if country Q is currently exporting and there are no concurrent effects on good prices. Finally, even though the aggregate wealth in the world economy is unchanged by the presence of quota restrictions, there is a wealth redistribution amongst the countries. The total wealth of the quota-constrained country increases while that of the unconstrained country decreases in all states. This result is in line with the standard optimal tariff argument made in the literature.

Remark 4 (Both Countries Import-Quota-Constrained). In the case where both countries are quota constrained, since clearing guarantees that when one country is importing the other must be exporting (and vice versa), equilibrium in the protected good market still falls into three regions. Either country 1 imports, there is no trade, or country 2 imports. Since there is an additional friction present, the no-trade region expands at the expense of the import region of the newly-constrained country. Both import regions are impacted by one additional cost term, the expected severity of the constraint of the appropriate importing country. Since both countries have an extended space of (ξ, ε) over which they are unwilling to trade, the state price over the no-trade region is indeterminate, but lies between two bounds.

4.2. Country-wide Effects of Quota Protection

A country's *price level* is quoted relative to a numeraire basket of commodities, representing the "typical" consumption goods in the economy, with appropriate weights.[12] In our analysis so far, such a basket consisted of one unit of the composite good. While this choice is appropriate for a country where the protected good constitutes an insignificant fraction of the economy, it is not the case where the good constitutes a considerable part. The latter economy would require the protected good to be included in the numeraire basket. To underscore the country-wide effects of quotas, we consider an extreme case of the basket consisting of the quota-protected good alone. (Alternatively, we could have adopted a more general and realistic choice of a basket consisting of α units of the quota-protected good and $1-\alpha$ units of the composite (e.g., Pavlova and Rigobon (2003)), which is straightforward to incorporate into our analysis below.)

The consumer's decisions are, of course, independent of the numeraire choice; only the price level gets affected. Now, the domestic state price density associated with the numeraire good, ξ_Q^*, depends on whether the country is importing, exporting or not trading, and is given by:

$$\xi_Q^*(t) = \begin{cases} \xi^*(t) + \frac{1}{y}E_t[\lambda(T)] & \text{if} \quad \xi(t) < \frac{1}{y}\left(u'(\varepsilon(t)) - E_t[\lambda(T)]\right), & \text{(import)} \\ \frac{1}{y}u'(\varepsilon(t)) & \text{if} \quad \frac{1}{y}\left(u'(\varepsilon(t)) - E_t[\lambda(T)]\right) \le \xi^*(t) < \frac{1}{y}\,u'(\varepsilon(t)), & \text{(no trade)} \\ \xi^*(t) & \text{if} \quad \frac{1}{y}\,u'(\varepsilon(t)) \le \xi^*(t), & \text{(export)}, \end{cases}$$

where $\xi^*(t)$ is the world state price density (quoted relative to the numeraire quota-protected good). Recall that the world state price density follows: $d\xi^*(t) = -r(t)\,\xi^*(t)\,dt - \theta(t)\,\xi^*(t)\,dw(t)$, where $r(t)$ is the interest rate and $\theta(t)$ the market price of risk. The dynamics of the domestic state price density in country Q, which can be represented as

$$d\xi_Q^*(t) = -r_Q(t)\,\xi_Q^*(t)\,dt - \theta_Q(t)\xi_Q^*(t)\,dw(t)\,,$$

where the interest rate $r_Q(t)$ and market price of risk $\theta_Q(t)$ in country Q, are now clearly different from those driving the world state price density. Indeed, the domestic interest rate and market price of risk in country Q can be shown to be

$$r_Q(t) = \begin{cases} \dfrac{1}{1 + E_t[\lambda(T)]/(y\xi^*(t))}\, r(t) & \text{if} \quad \text{import}\,, \\ A(\varepsilon(t))\varepsilon(t)\mu_\varepsilon(t) + \dfrac{1}{2}A(\varepsilon(t))C(\varepsilon(t))\varepsilon(t)^2||\sigma_\varepsilon(t)||^2 & \text{if} \quad \text{no trade}\,, \\ r(t) & \text{if} \quad \text{export}\,, \end{cases} \tag{27}$$

[12] As is well-recognized in the international economics literature, for example, Obstfeld and Rogoff (1996, p. 200), a country's price level index can depend on the choice of the numeraire basket. The importance of adequately describing the effects of instruments of international trade (e.g., quotas) on the price level calls for a careful selection of the appropriate basket. Obstfeld and Rogoff show how a weighting scheme determining the numeraire basket can be rigorously rationalized.

$$\theta_Q(t) \;=\; \begin{cases} \dfrac{1}{1+E_t[\lambda(T)]/(y\xi^*(t))}\,\theta(t) - \dfrac{\sigma_\lambda(t)}{y\xi^*(t)+E_t[\lambda(T)]} & \text{if} \quad \text{import}\,, \\ A(\varepsilon(t))\varepsilon(t)\sigma_\varepsilon(t) & \text{if} \quad \text{no trade}\,, \\ \theta(t) & \text{if} \quad \text{export}\,, \end{cases} \tag{28}$$

with $A(\varepsilon(t)) \equiv -u''(\varepsilon(t))/u'(\varepsilon(t))$, $C(\varepsilon(t)) \equiv -u'''(\varepsilon(t))/u''(\varepsilon(t))$, and $E_t[\lambda(T)]$ satisfies $dE_t[\lambda(T)] = \sigma_\lambda(t)dw(t)$.

As evident from (27)–(28), the domestic interest rate and market price of risk coincide with the world interest rate only if country Q is exporting. In the region over which the country is not trading in the quota-protected commodity, the prevailing interest rate and market price of risk are equal to their autarkic counterparts. Finally, in the import region, the interest rate in the quota constrained country is unambiguously lower than the world interest rate. The magnitude of the market price of risk in this region depends on the "quota constraint risk," σ_λ, arising due to additional uncertainty over time about the severity of the quota constraint. Whether this additional source of risk increases or decreases the market price of risk depends on whether $E_t[\lambda(T)]$ covaries positively or negatively with the endowment $\varepsilon(t)$.

4.3. Non-separable preferences

We now drop the assumption of the consumers' preferences being separable across the two goods, and consider instead time-additive, state-independent utility of the form $u(c,\, c^*)$ in $[0, T]$. Denote $J(\cdot,\, \cdot;\, q) \in \Re^2_+$ to be the inverse demand mapping; i.e., $(\cdot,\, \cdot;\, q)$ solves

$$u_1(J^1(x_1,\, x_2;\, q),\, J^2(x_1,\, x_2;\, q)) = x_1, \quad u_2(J^1(x_1,\, x_2;\, q),\, J^2(x_1,\, x_2;\, q)) = x_2\,,$$

where the superscripts ^{1} and ^{2} denote the first and the second elements of the mapping, respectively, and q is a vector of the parameters of the model. The optimal consumption policies of a small quota-constrained country can be shown to satisfy:

$$c_Q(t),\, c^*_Q(t) = \begin{cases} J^j(y\,\xi(t) + E_t[\lambda(T)],\, y\xi^*(t);\, y,\, E_t[\lambda(T)]) & \text{if } J^1(y\,\xi(t) + E_t[\lambda(T)],\, y\xi^*(t);\, y,\, E_t[\lambda(T)]) > \varepsilon(t), \\ \varepsilon(t),\; u_2^{-1}(\varepsilon(t),\, y\xi^*(t)) & \text{if } J^1(y\,\xi(t) + E_t[\lambda(T)],\, y\xi^*(t);\, y,\, E_t[\lambda(T)]) \le \varepsilon(t) \\ & \qquad \text{and } J^1(y\,\xi(t),\, y\,\xi^*(t);\, y) > \varepsilon(t), \\ J^j(y\,\xi(t), y\,\xi^*(t);\, y) & \text{if } J^1(y\,\xi(t),\, y\,\xi^*(t);\, y) \le \varepsilon(t), \end{cases}$$

where $j = \{1,\, 2\}$, $u_2^{-1}(\varepsilon(t),\, y\xi^*(t))$ solves $u_2(\varepsilon(t),\, x) = y\,\xi^*(t)$ and $y,\, E_t[\lambda(T)]$ solve (17).

Clearly, import restrictions affect not only consumption of the quota-protected good, but also that of the composite good. The additional cost to consumption, and the additional source of risk, captured by the expected severity of the quota constraint, $E_t[\lambda(T)]$, now enters the expression for consumption of the composite good. This will, of course, imply that in a world equilibrium, both the state price density associated with the quota-protected good ξ, and the state price density ξ^* are potentially affected by import quotas. That is, the effects of protection are not limited to a single sector: quotas affect the entire economy, including the stock markets.

5. Conclusion

We have examined dynamic consumption behavior, and equilibrium pricing under a dynamic quota constraint on imports. Such a constraint results in all quantities being driven by an additional term capturing the concurrent expectation of how severely the constraint will bind at the horizon. This term appears only when the country is importing the quota-protected commodity, as an additional cost to consumption, introducing path-dependency into the problem. We identify additional economic driving factors behind this cost: the utilization rate to date and time-remaining till the quota horizon. Only imports are subject to this additional cost implying a nonlinearity in the cost structure, hence generating an extended region over which no-trade occurs. In the no-trade and import regions, the domestic price of the quota-protected good exceeds the world price, regardless of whether the quota binds or not. Policy implications of our work would be obtained by quantifying the economic effects of quotas generated by our model. While we validate some of our main conclusions via a simple empirical analysis presented in the Appendix, we do not compute alternative estimates of quota protection. This exercise requires a more comprehensive empirical investigation, which is left for future research.

Appendix A: Proofs

Proof of Proposition 1. Substituting $\xi_P(t) = \xi^*(t)p_P(t)$, $\xi(t) = \xi^*(t)p(t)$, and $W(0) = S(0) + S^*(0) = E\left[\int_0^T \xi_P(t)\varepsilon(t)dt\right] + E\left[\int_0^T \xi^*(t)\varepsilon^*(t)dt\right]$ into (6), and simplifying yields the budget constraint $E\left[\int_0^T [\xi(t)\, c_P(t;\, y) + \xi^*(t)\, c_P^*(t;\, y)]\, dt\right] \leq E\left[\int_0^T [\xi(t)\, \varepsilon(t) + \xi^*(t)\, \varepsilon^*(t)\,]dt\right]$. Standard optimization with respect to c^* yields (11). To derive the expression for c_P, we adopt the common convex-duality approach (see, e.g., Karatzas and Shreve (1998)) to incorporate the quota constraint. The expression in Lemma 1 is the convex conjugate of u with an additional term capturing the quota constraint.

Lemma 1. *Expression (10) solves the following pointwise problem* $\forall t \in [0, T]$, $\forall \xi(t)$:

$$\max_c \{u(c) - y\, \xi(t)\, c - \eta(t)\, [(c - \varepsilon(t))^+ - k]\}$$

where $\eta(t) \equiv [u'(\varepsilon(t) + k) - y\, \xi(t)]\, 1_{\{c \geq \varepsilon(t)+k\}} \geq 0$.

Proof. The functional on which $\max\{\cdot\}$ operates could exhibit local maxima at $\hat{c} = I(y\,\xi(t))$ (only when $\hat{c} < \varepsilon(t) + k$) and/or $\hat{c} = \varepsilon(t) + k = I(y\,\xi(t) + \eta(t))$. When $\xi(t) < u'(\varepsilon(t) + k)/y$, then $I(y\,\xi(t)) > \varepsilon(t) + k$ so the former maximum is not allowed and the solution is $\hat{c} = \varepsilon(t) + k$. When $\xi(t) \geq u'(\varepsilon(t) + k)/y$, both maxima are allowed, but $u(I(y\,\xi(t))) - y\,\xi(t)\,I(y\,\xi(t)) > u(\varepsilon(t) + k) - y\,\xi(t)\,(\varepsilon(t) + k)$, so $\hat{c} = I(y\,\xi(t))$. *Q.E.D.*

Now let $\{c, c^*\}$ be any candidate solution which satisfies the budget constraint (6) and each import constraint. We have

$$
\begin{aligned}
& E\left[\int_0^T [u(c_P(t)) + v(c_P^*(t))]dt\right] - E\left[\int_0^T [u(c(t)) + v(c^*(t))]dt\right] \\
= \; & E\left[\int_0^T [u(c_P(t)) + v(c_P^*(t))]dt\right] - E\left[\int_0^T [u(c(t)) + v(c^*(t))]dt\right] \\
& -y\,E\left[\int_0^T [\xi(t)\,\varepsilon(t) + \xi^*(t)\,\varepsilon(t)\,]\,dt\right] + y\,E\left[\int_0^T [\xi(t)\,\varepsilon(t) + \xi^*(t)\,\varepsilon(t)]\,dt\right] \\
\geq \; & E\left[\int_0^T [u(c_P(t)) + v(c_P^*(t))]dt\right] - E\left[\int_0^T [u(c(t)) + v(c^*(t))]dt\right] \\
& - E\left[\int_0^T [y\,\xi(t)\,c_P(t) + y\,\xi^*(t)\,c_P^*(t)]\,dt\right] + E\left[\int_0^T [y\,\xi(t)\,c(t) + y\,\xi^*(t)\,c^*(t)]\,dt\right] \\
& - E\left[\int_0^T \eta(t)\{(c_P(t) - \varepsilon(t))^+ - k\}\,dt\right] + E\left[\int_0^T \eta(t)\{c(t) - \varepsilon(t))^+ - k\}\,dt\right] \;\geq 0\,,
\end{aligned}
$$

where the former inequality follows from the budget constraint holding with equality, complementary slackness of the import constraint, and all constraints holding with inequality for c. The latter inequality follows from Lemma 1 and optimality of c_P^*. Hence c_P is optimal.

To prove $y \leq y_N$, observe that $c_P(t;\,y) < c_N(t;\,y)$, $c_P^*(t;\,y) = c_N^*(t;\,y)$, $t \in [0, T]$, and that both $c_P(t;\,y)$ and $c_P^*(t;\,y)$ are nonincreasing in y. Hence the result is deduced from (12). *Q.E.D.*

Proof of Proposition 2. For domestic good market equilibrium, equating the consumption allocations determined by the country $c_P(\xi(t);\,y)$ in equation (10), with that optimally determined by the representative consumer $c_P(\xi_P(t);\,y_P)$ in equation (8), we back out the state price density associated with the quota-protected good as perceived by the consumer, ξ_P (13). We finally note that $y_P = y$. The multiplier y_P is determined from equation (9), which upon substitution of the transfer δ_P, $W(0) = E\left[\int_0^T \xi_P(t)\varepsilon(t)dt\right] + E\left[\int_0^T \xi^*(t)\varepsilon^*(t)dt\right]$, $\xi_P(t) = \xi^*(t)p_P(t)$ and $\xi(t) = \xi^*(t)p(t)$, is equivalent to equation (12). Hence $y_P = y$. *Q.E.D.*

Proof of Proposition 3. Substituting $\xi_Q(t) = \xi^*(t)p_Q(t)$, $\xi(t) = \xi^*(t)p(t)$, and $W(0) = S(0) + S^*(0) = E\left[\int_0^T \xi_Q(t)\varepsilon(t)dt\right] + E\left[\int_0^T \xi^*(t)\varepsilon^*(t)dt\right]$ into (6), and simplifying yields the budget constraint $E\left[\int_0^T [\xi(t)\,c_Q(t;\,y) + \xi^*(t)\,c_Q^*(t;\,y)]\,dt\right] \leq E\left[\int_0^T [\xi(t)\,\varepsilon(t) + \xi^*(t)\,\varepsilon^*(t)]\,dt\right]$. Standard optimization with respect to c^* yields (16). To derive the expression for c_Q, we use the following.

Lemma 2. *Expression (15) solves the following pointwise problem* $\forall t \in [0,T]$, $\forall \{\xi(s),\, s \geq t\}$:

$$\max_c \{u(c) - y\,\xi(t)\,c - E_t[\lambda(T)]\,(c - \varepsilon(t))^+\}\ , \tag{A.1}$$

where $\lambda(T) \geq 0$ *solves (17).*

Proof. The functional on which $\max\{\cdot\}$ operates could exhibit local maxima at $\hat{c} = I(y\,\xi(t)) < \varepsilon(t)$ and/or $\hat{c} = \varepsilon(t)$ and/or $\hat{c} = I(y\,\xi(t) + E_t[\lambda(T)]) > \varepsilon(t)$. If $\xi(t) < \frac{1}{y}\,[u'(\varepsilon(t)) - E_t[\lambda(T)]]$, the first maximum cannot occur. Then, since $I(y\,\xi(t) + E_t[\lambda(T)])$ maximizes the functional $u(c) - [y\,\xi(t) + E_t[\lambda(T)]]c$, we have that the global maximum occurs at $I(y\,\xi(t) + E_t[\lambda(T)])$. If $\frac{1}{y}[u'(\varepsilon(t)) - E_t[\lambda(T)]] \leq \xi(t) < \frac{1}{y}\,u'(\varepsilon(t))$, the first and last maxima cannot occur, so the solution is $\varepsilon(t)$. If $\frac{1}{y}\,u'(\varepsilon(t)) \leq \xi(t)$, the first and second maxima could occur, but the solution is $I(y\,\xi(t))$ since this maximizes the functional $u(c) - y\xi(t)c$. *Q.E.D.*

Now let $\{c,\, c^*\}$ be any candidate solution which satisfies the budget constraint (6) and each import constraint. We have

$$\begin{aligned}
& E\left[\int_0^T [u(c_Q(t)) + v(c_Q^*(t))]dt\right] - E\left[\int_0^T [u(c(t)) + v(c^*(t))]dt\right] \\
= \;& E\left[\int_0^T [u(c_Q(t)) + v(c_Q^*(t))]dt\right] - E\left[\int_0^T [u(c(t)) + v(c^*(t))]dt\right] \\
& -y\,E\left[\int_0^T [\xi(t)\,\varepsilon(t) + \xi^*(t)\,\varepsilon(t)\,]\,dt\right] + y\,E\left[\int_0^T [\xi(t)\,\varepsilon(t) + \xi^*(t)\,\varepsilon(t)]\,dt\right] - E[\lambda(T)\,K] + E[\lambda(T)\,K] \\
\geq \;& E\left[\int_0^T [u(c_Q(t)) + v(c_Q^*(t))]dt\right] - E\left[\int_0^T [u(c(t)) + v(c^*(t))]dt\right] \\
& - E\left[\int_0^T [y\,\xi(t)\,c_Q(t) + y\,\xi^*(t)\,c_Q^*(t)]\,dt\right] + E\left[\int_0^T [y\,\xi(t)\,c(t) + y\,\xi^*(t)\,c^*(t)]\,dt\right] \\
& - E\left[\int_0^T E_t[\lambda(T)]\,(c_Q(t) - \varepsilon(t))^+\,dt\right] + E\left[\int_0^T E_t[\lambda(T)]\,(c(t) - \varepsilon(t))^+\,dt\right] \;\geq 0\,,
\end{aligned}$$

where the former inequality follows from the budget constraint holding with equality, complementary slackness of the quota constraint, and both constraints holding with inequality for c. The second inequality follows from Lemma 2 and optimality of c_Q^*. Hence c_Q is optimal.

To show that (15) evaluated at $t = T$ implies (14) under additional regularity conditions as stated in footnote 5, make use of the complementary slackness in (17) to deduce that:

$$\begin{aligned}
\text{Either}\quad & \lambda(T) = 0,\; c_Q(T) = I(y\,\xi(T)) \leq \varepsilon(T) \quad \text{and} \quad \xi(T) \geq \tfrac{1}{y}\,u'(\varepsilon(T) + \dot{X}(T)), \\
& \lambda(T) = 0,\; c_Q(T) = I(y\,\xi(T)) > \varepsilon(T) \quad \text{and} \quad \xi(T) \geq \tfrac{1}{y}\,u'(\varepsilon(T) + \dot{X}(T)), \\
\text{or}\quad & \lambda(T) \geq 0,\; c_Q(T) = I(y\,\xi(T)) < \varepsilon(T) \quad \text{and} \quad K - \textstyle\int_0^T (c_Q(s) - \varepsilon(s))^+ ds = 0\,, \\
\text{or}\quad & \lambda(T) \geq 0,\; c_Q(T) = \varepsilon(T) > I(y\,\xi(T)) \quad \text{and} \quad K - \textstyle\int_0^T (c_Q(s) - \varepsilon(s))^+ ds = 0\,, \\
\text{or}\quad & \lambda(T) = u'(\varepsilon(T) + \dot{X}(T)) - y\,\xi(t),\; c_Q(T) = \varepsilon(T) + \dot{X}(T) \;\text{ and } \xi(T) < \tfrac{1}{y}\,u'(\varepsilon(T) + \dot{X}(T)).
\end{aligned}$$

The first three cases make up the latter region of (14) while the last two the former region.

We now show that the quota cannot be exhausted before time T. Suppose that the quota can be exhausted at time $t < T$ making $\lambda(T) < \infty$ $\mathcal{F}_t$-measurable. Consider a stopping time $\tau \in (t, T]$ such that $\xi(\tau) < \frac{1}{y}(u'(\varepsilon(\tau)) - \lambda(t))$. (This condition is satisfied with positive probability for some τ since both ε and ξ have support $(0, \infty)$.) Then, from (15), it must be the case that $c_Q(\tau) = I(y\xi(\tau)) + \lambda(t))$. On the other hand, since the quota is fully exhausted at τ, $c_Q(\tau) \leq \varepsilon(\tau)$. Therefore, $I(y\xi(\tau)) + \lambda(t)) \leq \varepsilon(\tau)$, contradicting the definition of τ.

To prove $y \leq y_N$, observe that $c_Q(t;\, y) < c_N(t;\, y)$, $c^*_Q(t;\, y) = c^*_N(t;\, y)$, $t \in [0, T]$, and that both $c_Q(t;\, y)$ and $c^*_Q(t;\, y)$ are nonincreasing in y. Hence the result is deduced from the budget constraint in (17). Property (iii) of the Proposition is then immediate. *Q.E.D.*

Proof of Proposition 4. For domestic good market equilibrium, equating the consumption allocations determined by the country $c_Q(\xi(t),\, y,\, E_t[\lambda(T)])$ in equation (15), with that optimally determined by the representative consumer $c_Q(\xi_Q(t);\, y_Q)$ in equation (8), we back out the state price density associated with the quota-protected good as perceived by the consumer ξ_Q (18). Finally note that $y_Q = y$. The multiplier y_Q is determined from equation (9), which upon substitution of the transfer δ_Q, $W(0) = E\left[\int_0^T \xi_Q(t)\varepsilon(t)dt\right] + E\left[\int_0^T \xi^*(t)\varepsilon^*(t)dt\right]$, $\xi_Q(t) = \xi^*(t)p_Q(t)$ and $\xi(t) = \xi^*(t)p(t)$, is equivalent to equation (17). Hence $y_Q = y$. *Q.E.D.*

Proof of Proposition 5. The good market clearing (19), together with the countries' optimal consumption, (15)–(17), as well as $c_N(t) = I_N(\xi(t))$ and $c^*_N(t) = I^*_N(\xi^*(t))$ imply (20), (21) and (25). We have used the definition of $G(\,\cdot\,;\, y,\, E_t[\lambda(T)])$ in the "Q imports" region, and the fact that the inverse of $U'(c;\, y)$ is $I(h;\, y) = I_N(h) + I_P(y\, h)$ in the "Q exports" region. (22) follows from (18) with equilibrium ξ from (20) substituted in. Substitution of (20) into the countries' optimal consumption (15)–(16), $(I_N(\xi(t)),\, I^*_N(\xi(t)))$ and (19) yields the equilibrium consumption allocations (23)–(24). Under further regularity conditions (footnote 5), either by evaluating the equations (20) and (23) at the horizon or by imposing clearing in the quota-constrained good market together with the optimal horizon consumption of the quota-constrained good (14) and $c_N(T) = I_N(\xi(T))$, we obtain, at the horizon,

$$\xi(T) = \begin{cases} u'_N\Big(\varepsilon_N(T) - \dot{X}_Q(T)\Big) & \text{if} \quad u'_N\Big(\varepsilon_N(T) - \dot{X}_Q(T)\Big) < \frac{1}{y}\, u'_Q\Big(\varepsilon_Q(T) + \dot{X}_Q(T)\Big) \quad (Q \text{ binding}) \\ U'(\varepsilon_N(T) + \varepsilon_Q(T);\, y) & \text{otherwise} \qquad\qquad (Q \text{ not binding}) \end{cases}$$

which upon substitution into optimal horizon consumptions leads to

$$c_N(T) = \begin{cases} \varepsilon_N(T) - \dot{X}(T)\,, \\ I_N\Big(U'(\varepsilon_N(T) + \varepsilon_Q(T);\, y)\Big)\,, \end{cases} \qquad c_Q(T) = \begin{cases} \varepsilon_Q(T) + \dot{X}(T) & \text{if} \quad Q \text{ binding}, \\ I_Q\Big(y\, U'(\varepsilon_N(T) + \varepsilon_Q(T);\, y)\Big) & \text{if} \quad Q \text{ not binding}. \end{cases}$$

Finally, we derive the conditions for the regions in the equilibrium consumption of the quota-constrained good. For the "Q imports" region, from clearing, N exports the quota-constrained good implying $c_N(t) = I_N(\xi(t)) < \varepsilon_N(t)$. This, together with the condition for Q to import (15) imply $\exists\, \xi(t)$ such that $\xi(t) < \frac{1}{y}\Big(u'_Q(\varepsilon_Q(t)) - E_t[\lambda(T)]\Big)$ and $\xi(t) > u'_N(\varepsilon_N(t))$, or $\frac{1}{y}\Big(u'_Q(\varepsilon_Q(t)) -$

$E_t[\lambda(T)]\Big) > u_N'(\varepsilon_N(t))$. For the "$Q$ exports" region, from clearing, N imports the quota-constrained good implying $c_N(t) = I_N(\xi(t)) \geq \varepsilon_N(t)$. This, together with the condition for Q to export (15) imply $\exists\, \xi(t)$ such that $\xi(t) < \frac{1}{y}\, u_Q'(\varepsilon_Q(t))$ and $\xi(t) > u_N'(\varepsilon_N)$, or $u_N'(\varepsilon_N(t)) \geq \frac{1}{y}\, u_Q'(\varepsilon_Q(t))$. The condition for the "no-trade" region follows by directly substituting $\xi(t)$ of (20) into the condition for Q to not trade in the quota-constrained good (15). The proof of the last statement is a variation on Karatzas, Lehoczky and Shreve (1990) to include nonredundant positive net supply securities (e.g., see Basak (1995)) and the transfer within the quota-constrained country. *Q.E.D.*

Proof of Proposition 6. We prove the first statement by contradiction. Suppose $1/y < 1/y^u$. In the unconstrained economy the world state price density associated with the quota-constrained good is given by $\xi^u(t) = U'(\varepsilon(t);\, y^u)$. From Proposition 5, ξ can be expressed as

$$\xi(t) = U'(\varepsilon(t);\, y) + M(t)\,,$$

where $M(t) \equiv \min\Big\{0,\, \max\{u_N'(\varepsilon_N(t)) - U'(\varepsilon(t);\, y),\, G(\varepsilon(t); y, E_t[\lambda(T)]) - U'(\varepsilon(t); y)\}\Big\} \leq 0$. To show that $\xi(t) < \xi^u(t)$, it then suffices to demonstrate that $U'(\varepsilon(t);\, y)$ is decreasing in y. The inverse of $U'(c;\, y)$ is $I(h;\, y) = I_N(h) + I_Q(y\, h)$, i.e., $I(U'(c;\, y); y) = c$. Differentiating the latter expression with respect to y yields $I'(U'(c;\, y);\, y)\, U_y'(c;\, y) + I_y(U'(c;\, y);\, y) = 0$ implying

$$U_y'(c;\, y) = -U''(c;\, y)\, I_y(U'(c;\, y); y) < 0$$

where the equality uses the fact that $I' = 1/U''$ and the inequality that $I_y(h;\, y) = h/u_Q''$. It can be shown analogously that $V'(\varepsilon^*(t);\, y)$ is decreasing in y. Hence, since $U'(\varepsilon(t);\, y) < U'(\varepsilon(t);\, y^u)$, $M(t) \leq 0$ and $V'(\varepsilon^*(t);\, y^u)$, we have

$$\xi(t) < \xi^u(t) \quad \text{and} \quad \xi^*(t) < \xi^{*u}(t)\,. \tag{A.2}$$

Consequently, $c_N(t) = I_N(\xi(t)) > I_N(\xi^u(t)) = c_N^u(t)$, and $c_N^*(t) = I_N^*(\xi^*(t)) > I_N^*(\xi^{*u}(t)) = c_N^{*u}(t)$. It then follows from goods market clearing (19) that

$$c_Q(t) < c_Q^u(t) \quad \text{and} \quad c_Q^*(t) < c_Q^{*u}(t)\,. \tag{A.3}$$

(A.2), (A.3) and Q's budget constraint in the unconstrained economy imply $E[\int_0^T[\xi(t)(c_Q(t) - \varepsilon_Q(t)) + \xi^*(t)(c_Q^*(t) - \varepsilon_Q^*(t))]dt] < E[\int_0^T[\xi^u(t)(c_Q^u(t) - \varepsilon_Q(t)) + \xi^{*u}(t)(c_Q^{*u}(t) - \varepsilon_Q^*(t))]dt] = 0$. So, the budget constraint of country Q cannot hold with equality in the quota-constrained economy.

The above argument can be easily adapted to rule out the case of $1/y = 1/y^u$. It suffices to demonstrate that it is not possible to have $M(t) \leq 0,\ \forall t$. Indeed, this would correspond to the quota-constrained country exporting at all times and states, which is in contradiction to its budget constraint. Then, $\xi(t) < \xi^u(t)$, while $\xi^*(t) = \xi^{u*}(t)$. Similar steps to above lead to this contradicting Q's budget constraint holding with equality in the quota-constrained economy.

We now specialize preferences to be logarithmic and prove (i)-(iv).
(i) In the unconstrained economy $\xi^u(t) = \frac{1+1/y^u}{\varepsilon(t)}$ and $\xi^{*u}(t) = \frac{1+1/y^u}{\varepsilon^*(t)}$. In the quota-constrained economy, $\xi(t) = \frac{1+1/y}{\varepsilon(t)} + M(t)$ and $\xi^*(t) = \frac{1+1/y}{\varepsilon^*(t)}$ with $M(t) \leq 0$. Hence

$$p(t) = \frac{\xi(t)}{\xi^*(t)} = \frac{(1+1/y)/\varepsilon(t) + M(t)}{(1+1/y)/\varepsilon^*(t)} = \frac{1/\varepsilon(t)}{1/\varepsilon^*(t)} + \frac{M(t)}{1/\varepsilon^*(t)(1+1/y)} \leq \frac{1/\varepsilon(t)}{1/\varepsilon^*(t)} = p^u(t)\,.$$

The inequality is due to $M(t) \leq 0$. In the "Q exports" region, $M(t) = 0$, resulting in $p(t) = p^u(t)$; in the remaining regions, $M(t) < 0$, and hence $p(t) < p^u(t)$.
(ii) It follows from Proposition 5 that $\xi_Q(t)$ can be expressed as

$$\xi_Q(t) = U'(\varepsilon(t);\, y) + R(t)\,,$$

where $R(t) \equiv \max\left\{0,\, \min\{u'_N(\varepsilon_N(t)) - U'(\varepsilon(t);\, y),\, G(\varepsilon(t); y, E_t[\lambda(T)]) - U'(\varepsilon(t); y)\}\right\} \geq 0$ and for logarithmic utilities $U'(\varepsilon(t); y) = (1 + 1/y)/\varepsilon(t)$. Hence,

$$p_Q(t) = \frac{\xi_Q(t)}{\xi^*(t)} = \frac{(1+1/y)/\varepsilon(t) + R(t)}{(1+1/y)/\varepsilon^*(t)} = \frac{1/\varepsilon(t)}{1/\varepsilon^*(t)} + \frac{R(t)}{1/\varepsilon^*(t)(1+1/y)} \geq \frac{1/\varepsilon(t)}{1/\varepsilon^*(t)} = p^u(t)\,,$$

where the inequality follows from $R(t) \geq 0$. In the "Q exports" region, $R(t) = 0$, resulting in $p_Q(t) = p^u(t)$; in the remaining regions, $R(t) > 0$, and hence $p_Q(t) > p^u(t)$.

The prove the stock price results for $S_N(t)$ and $S_Q(t)$, note that in the unconstrained economy

$$S_i^u(t) = \frac{1}{\xi^{*u}(t)} E_t\left[\int_t^T \xi^u(s)\varepsilon_i(s)ds\right] = \varepsilon^*(t) E_t\left[\int_t^T \frac{\varepsilon_i(s)}{\varepsilon(s)}ds\right]\,, \quad i \in \{N,\, Q\}\,.$$

We can then use the definition of the state price density associated with the protected good and the expressions for $\xi(t)$ and $\xi_Q(t)$ to obtain

$$S_N(t) = \varepsilon^*(t)\, E_t\left[\int_t^T \frac{\varepsilon_N(s)}{\varepsilon(s)}\, ds\right] + \frac{\varepsilon^*(t)}{1+1/y}\, E_t\left[\int_t^T M(s)\varepsilon_N(s)ds\right]\,,$$

$$S_Q(t) = \varepsilon^*(t)\, E_t\left[\int_t^T \frac{\varepsilon_Q(s)}{\varepsilon(s)}\, ds\right] + \frac{\varepsilon^*(t)}{1+1/y}\, E_t\left[\int_t^T R(s)\varepsilon_Q(s)ds\right]\,,$$

where $M(s) < 0$ and $R(s) > 0$ over a region where Q is not exporting. Since this region occurs with a strictly positive probability $\forall s$, we have the stated result.
(iii) Stock prices $S_N^*(t)$ and $S_Q^*(t)$ are unchanged in the quota-constrained economy since

$$S_i^*(t) = \frac{1}{\xi^*(t)} E_t\left[\int_t^T \xi^*(s)\varepsilon_i(s)ds\right] = \varepsilon^*(t) E_t\left[\int_t^T \frac{\varepsilon_i^*(s)}{\varepsilon^*(s)}ds\right] = S_i^{*u}(t)\,, \quad i \in \{N,\, Q\}\,.$$

(iv) Finally, we prove the wealth result. In both economies, the total wealth of country N is equal to $\hat{W}_N(t) = E_t\left[\int_t^T [\xi(s)\, c_N(s) + \xi^*(s)\, c_N^*(s)]\, ds\right]/\xi^*(t) = 2(T-t)/\xi^*(t)$, while the total wealth of the quota-constrained country is $\hat{W}_Q(t) = E_t\left[\int_t^T [\xi_Q(s)\, c_Q(s) + \xi^*(s)\, c_Q^*(s)]\, ds\right]/\xi^*(t) = 2(T-t)/(y\xi^*(t))$. Substituting $\xi^*(t)$, we deduce that $\hat{W}_N(t) + \hat{W}_Q(t) = 2(T-t)\varepsilon^*(t)$. Analogous substitution of $\xi^{*u}(t)$ in the unconstrained world's wealth leads to an identical expression. The inequalities $\hat{W}_N(t) < \hat{W}_N^u(t)$ and $\hat{W}_Q(t) > \hat{W}_Q^u(t)$ follow from $1/y > 1/y^u$. *Q.E.D.*

Proof of Corollary 1 (Appendix C). Using the definition of a license and the no-arbitrage valuation principle, we have $\ell(t) = \frac{1}{\xi^*(t)} \sup_{\tau\in[t,T]} E_t[\xi^*(\tau)\max(p_Q(\tau) - p(\tau), 0)]$. Since in the domestic equilibrium $p_Q(\tau) - p(\tau) \geq 0$, $\xi(t) = \xi^*(t)p(t)$ and $\xi_Q(t) = \xi^*(t)p_Q(t)$, this is equivalent to $\ell(t) = \frac{1}{\xi^*(t)} \sup_{\tau\in[t,T]} E_t[\xi_Q(\tau) - \xi(\tau)]$. A quota license is exercised only in the states when the country is importing. Substituting ξ_Q from (18) in the import region and noting that $E_\tau[\lambda(T)]$ is a martingale leads to the required result. *Q.E.D.*

Appendix B: Numerical Analysis

We adopt the following parametrization of the consumers' preferences and the underlying investment opportunity set. Consumer preferences are represented by $u(c) = \log(c)$ and $v(c^*) = \log(c^*)$, and the investment opportunities by a bond B and a stock S following

$$dB(t) = r\,B(t)\,dt\,, \quad dS(t) = S(t)[\mu_S\,dt + \bar{\sigma}_S\,dw(t)]\,,$$

where r, μ_S and $\bar{\sigma}_S = (\bar{\sigma}_{S1},\ \bar{\sigma}_{S2})$ are constant, and $w(t)$ is two-dimensional. We have two dimensions of uncertainty to allow for less than perfect correlation between pertinent processes. The endowment process follows a geometric Brownian motion process

$$d\varepsilon(t) = \varepsilon(t)[\mu_e\,dt + \bar{\sigma}_e\,dw(t)]\,,$$

with parameters μ_e and $\bar{\sigma}_e = (\bar{\sigma}_{e1},\ \bar{\sigma}_{e2})$. Since the processes $\bar{w}_1(t) = (\bar{\sigma}_{S1}w_1(t)+\bar{\sigma}_{S2}w_2(t))/\sqrt{\bar{\sigma}_{S1}^2 + \bar{\sigma}_{S2}^2}$ and $\bar{w}_2(t) = (\bar{\sigma}_{e1}w_1(t) + \bar{\sigma}_{e2}w_2(t))/\sqrt{\bar{\sigma}_{e1}^2 + \bar{\sigma}_{e2}^2}$ are both (one-dimensional) standard Brownian motions, we can represent the dynamics of S and ε as: $dS(t) = S(t)[\mu_S\,dt + \sigma_S\,d\bar{w}_1(t)]$, with $\sigma_S = \sqrt{\bar{\sigma}_{S1}^2 + \bar{\sigma}_{S2}^2}$ and ε as $d\varepsilon(t) = \varepsilon(t)[\mu_e\,dt + \sigma_e\,d\bar{w}_2(t)]$, with $\sigma_e = \sqrt{\bar{\sigma}_{e1}^2 + \bar{\sigma}_{e2}^2}$. Denote the quadratic covariation of $\bar{w}_1(t)$ and $\bar{w}_2(t)$ to be ρt. Further assume $p(t) = p$ (constant) for all t.

The value function J is defined as the maximum of the expected utility of the country over the set of admissible controls c and π:

$$\begin{aligned}
J(W(0),\ X(0),\ \varepsilon(0),\ t=0) &= \max_{\{c,\,\pi\}} E\left[\int_0^T [\log(c(t)) + \log(c^*(t))]\,dt\right] \\
\text{subject to: } dW(t) &= r\,W(t)\,dt - (pc(t) + c^*(t))dt + \pi(t)(\mu_S - r)dt + \pi(t)\,\sigma_S\,d\bar{w}_1(t)\,, \\
d\varepsilon(t) &= \varepsilon(t)[\mu_e\,dt + \sigma_e\,d\bar{w}_2(t)]\,, \quad dX(t) = (c(t) - \varepsilon(t))^+dt\,.
\end{aligned}$$

Note the presence of an additional state variable, $X(t) = \int_0^t (c(s) - \varepsilon(s))^+ds$. Letting $\phi(t) = \pi(t)/W(t)$ be the proportion of wealth invested in the risky stock, we can cast the Hamilton-Jacobi-Bellman equation for the problem as

$$\begin{aligned}
0 \;=\; & J_t(t) + \max_{\{c,\,\pi\}} \Big\{ \log(c(t)) + \log(c^*(t)) + J_X(t)(c(t) - \varepsilon(t))^+ \\
& + J_W(t)(W(t)r - (pc(t) + c^*(t)) + W(t)\phi(t)^2(\mu_S - r)) + J_\varepsilon(t)\mu_e\varepsilon(t) \\
& + \frac{1}{2}J_{WW}(t)W(t)^2\phi(t)^2\sigma_S^2 + \frac{1}{2}J_{\varepsilon\varepsilon}(t)\varepsilon(t)^2\sigma_e^2 + J_{\varepsilon W}(t)W(t)\varepsilon(t)\phi(t)\sigma_S\sigma_e\rho \Big\}\,, \qquad \text{(A.4)}
\end{aligned}$$

where $J(t)$ and its derivatives are shorthand for $J(W(t),\ X(t),\ \varepsilon(t),\ t)$ and its derivatives.

The boundary (terminal) condition at time T is standard: $J(T) = 0$. The boundary for the cumulative imports $X(t)$ is given by K. If $X(t)$ equals to K, then we require that $c(t) \le \varepsilon(t)$. The parameters are chosen as follows: $K = 1$, $\varepsilon = 0.2$, $p = 1.5$, $r = 0.05$, $\mu_S = 0.1$, $\mu_e = 0.1$, $\sigma_S = 0.2$, $\sigma_e = 0.04$ and $\rho = 0.5$.

We solve (A.4) on the $W \times X \times \varepsilon \times t$ grid using an explicit finite differences scheme: a predictor-corrector method with artificial viscosity designed to aid problems arising due to non-differentiability of optimal consumption. The graphs in the body of the paper, for consumption drawn as a function of ξ, are produced by identifying $J_W(t)$ with $y\xi(t)$. This equivalence follows from matching the solution produced with the dynamic programming method with that obtained using the martingale method, presented in the body of the paper.

Appendix C: Simple Empirical Analysis

Our objective in this Appendix is not to undertake a comprehensive empirical analysis, which is beyond the scope of this paper, but to provide simple empirical support for some of the main theoretical implications of our model. Toward that end, we collected publicly available data on the prices of quota licenses traded on secondary markets. A license price is a good proxy for a price distortion due to a quota since it captures the exact amount a holder is willing to pay to take advantage of the protection-induced discrepancy between domestic and world prices. ITC (2002) also argues that a license price serves as the best proxy for the price impact of a quota. Formally, a quota license gives it holder the right to import a unit of the protected commodity at any time until the quota horizon. Within our model, Corollary 1 identifies a mapping between a quota license price and the additional cost to consumption due to the quota constraint, $E_t[\lambda(T)]$.

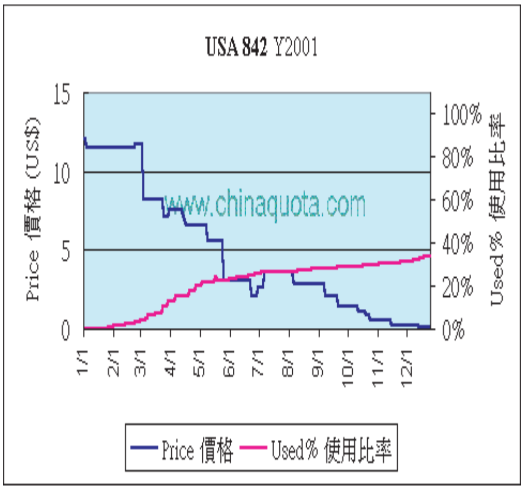
Corollary 1. *The price of a quota license that gives its holder the right to import into the quota-constrained country a unit of the protected commodity over* $[t, T]$*, is given by*

$$\ell(t) = \frac{1}{y\xi^*(t)} E_t[\lambda(T)]. \tag{A.5}$$

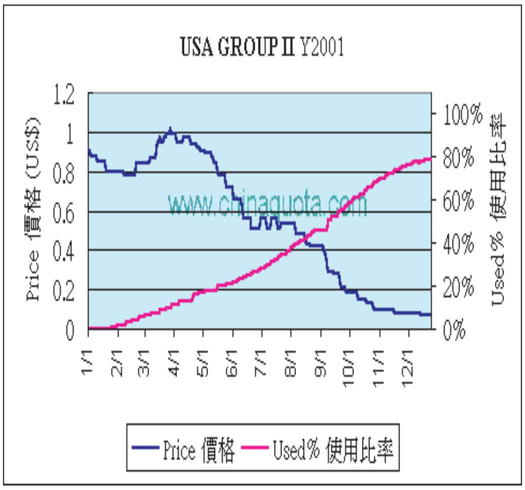
The derivation of this result, presented in Appendix A, bears similarity to the standard valuation of an American option with a payoff given by the difference between the domestic and world prices. This similarity is recognized in the license pricing literature.[13] In our model the stochastic evolution of this license payoff is endogenous, affected not only by (exogenous) shocks to the state price densities and endowment, but also by the optimal decision of the country to import, i.e., raise its cumulative imports to date. It can be inferred from Proposition 4 that the price difference, $p_Q(t) - p(t)$, is confined to stay within a band $[0, \ell(t)]$. The interior of the band corresponds to the no-trade region, where the price difference process is driven purely by exogenous quantities. The moment the upper boundary is reached, however, the process is no longer exogenous: imports are initiated, raising the country's cumulative imports to date and hence $E_t[\lambda(T)]$, leading the price difference to be reflected back inside the band (see the condition for the import region in equation (18)). Similarly, when the lower boundary is reached, exports are initiated. However, there is no additional cost to exporting, and hence no mechanism to force the process back into the no-trade region. The behavior described above, especially at the upper boundary, draws an analogy to the exchange rate target zones (Krugman (1991), Cadenillas and Zapatero (1999)) or shipping costs (Dumas (1992)) literatures, where pertinent processes are also confined to stay inside a band, and are reflected back each time a boundary of the band is reached. This analogy has a potentially important implication for the license pricing literature, where the specification of a license payoff is crucial. One may consider modeling this payoff as a regulated diffusion process, with a control applied each time the country imports thereby reducing the number of licenses outstanding.

[13] See Anderson (1987), Eldor and Marcus (1988), and Krishna and Tan (1996). Many modeling assumptions in these papers are inconsistent with our approach. The two most notable are that (i) licenses are replenished as soon as they are exercised and (ii) agents are risk-neutral. The first assumption removes the technically-challenging path-dependency, but also results in the current price of imports being independent of the cumulative imports to date. The second one leads to the conclusion that the expected return on a license equals to the interest rate, unless the quota binds before the horizon (which is counterfactual). None of these implications obtain in our model or in our data.

We first present some evidence in favor of our implication that non-binding quotas have economic effects and are not harmless. We collected monthly data on the prices of textile and apparel quota licenses and the corresponding utilization rates in China for the time period 1999–2002.[14]



(a) Category US 842



(b) US Group II

Figure 6: Chinese quota license prices for exporting to the US and the corresponding quota utilization rates during 2001: an illustration. The darker plot is the license price of US category 842 in panel (a), and the normalized average license price across all categories for US Group II in panel (b). The lighter plot is the utilization rate to date of US category 842 in panel (a), and the utilization rate averaged across all categories of US Group II in panel (b).

The secondary market, in which quota licenses are traded, exists only in few countries that have quota agreements with the US, restricting the scope of our investigation. However, China is by far the biggest trading partner with which the US has textiles and apparel quota agreement: for example, as reported by ITC (2002), in 1999, 9.6% of US imports of textiles and apparel came from China; Hong Kong was second with a 7.0% share. Figure 6 presents two graphs for typical trajectories of quota license prices and utilization rates. Figure 6a is for a specific quota-protected category (skirts, silk and vegetable blends), and Figure 6b is for the average of a typical quota-protected group (Group II, one of the four groups) during the year 2001. Evident from the plots is that neither the individual nor the average quota utilization rates at the year-end exceed 80%, and yet, in both graphs license prices are positive. This supports our conclusion that non-binding quotas have economic effects, and hence omitting product categories for which quotas are not binding from an analysis may significantly underestimate the economic impact of quotas. ITC (2002) argues that a theoretical utilization of 100% may be difficult to achieve in practice, and perhaps 85% utilized quotas should be considered binding. Linkins and Arce (2002) employ an 80% and a 90% threshold. In our data set, out of the total 362 observations across all listed categories over four years, at the year-end only 4 (1.38%) of the quotas were fully filled. Furthermore, a relatively small fraction of the quotas were "almost binding": only 91 (25.14%) were over 90% and 141 (38.84%) over 85% utilized. These figures are significantly lower if one

[14]These data are publicly available from http://www.chinaquota.com.

includes observations across all countries having a quota agreement with the US, as reported on the US customs webpage, http://www.customs.ustreas.gov. As evident from Figure 6a, however, even an under 40% utilized quota can have a positive price for most of the year − in full agreement with our observation that utilization should not be treated as a dichotomous variable, but rather, as a continuous measure.

The apparent high variability of license prices in the plots is also consistent with our license price equation (A.5), where one of the driving economic factors is the state of the economy. It is also suggestive that the price of a license decreases with the time-remaining till the quota horizon − this is in accord with our prediction of Section 3.4 that $E_t[\lambda(T)]$, and hence a quota license price, is increasing in the time remaining. Another testable implication of our model is that $E_t[\lambda(T)]$ increases with the utilization rate to date. To formally analyze this, we regress quota license prices on the utilization rate and time remaining, as well as protected-category and year dummies (accounting for fixed effects).[15]

	Dependent Variable: Quota License Prices, US dollars		
	Coefficient	Standard Error	t-Statistic
Utilization, X/K	19.13	1.79	10.68
Time remaining, $T-t$	16.98	1.63	10.42
Category fixed effects	Yes		
Year fixed effects	Yes		
Adjusted R^2	0.83		

Table 1: Regression analysis of Chinese quota license prices for exporting to the US. The sample consist of 4118 observations of reported quota license prices and utilization rates to date from 1999 to 2002. The data are monthly for each textiles and apparel category subject to a US import quota.

As predicted by our model, all else equal, higher utilization rates to date translate into higher license prices. This effect is both statistically and economically significant, supporting our conclusion that cumulative imports to date, or equivalently a quota utilization rate, is a driving state variable in the model. The other driving economic factor, the time remaining, is also highly significant: quota license prices are indeed for longer time remaining till the quota horizon.

[15] We pool observations across all reported US quota categories and four years of data. All standard errors are corrected for heteroskedasticity. Krishna and Tan (1996) insightfully conjecture an analogous empirical model in their study of Hong Kong quota license prices, however, their specification is not grounded in their theory.

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