Adjustment Costs, Learning-by-Doing, and Technology Adoption under Uncertainty

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Abstract

We consider a variety of vintage capital models of a firm's choice of technology under uncertainty in the presence of adjustment costs and technology-specific learning. Similar models have been studied in a deterministic setting. Part of our objective is to examine the robustness of the implications of the certainty models to uncertainty. We find that the answer crucially depends on the specification of the costs of adoption of a new vintage of technology. In particular, if the cost comes only in terms of accumulated technology-specific expertise (cf. Parente (1994)), we demonstrate that the implications are robust for a variety of specifications of the firm's production function. However, once we develop a model in which each adoption requires a capital expenditure, predictions become increasingly different as uncertainty increases. The model implies that in booms, the firm accelerates adoptions of new technologies, delaying them in recessions. Adverse effects of a recession on the investment decisions are alleviated in part by the firm's expertise (or human capital). Compared to the deterministic benchmark, the firm increases the pace of adoptions, making a smaller technological advance each time it upgrades its technology. Overall, uncertainty negatively impacts growth and the firm value.

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1. Introduction

"Optimality of usually doing nothing" (as put by Bar-Ilan and Blinder (1992)) is rapidly becoming a conventional implication of economic models addressing optimal investment choice of a firm (or a plant). This result enjoys strong empirical support: studies by Doms and Dunne (1993) and Cooper, Haltiwanger and Power (1999) indicate that at the plant level, the investment rate displays a distinct spiked pattern, with periods of intensive investment being followed by long periods of inaction. Another important, though less emphasized, empirical observation indicates that most of the investment comes in the form of new capital goods (technologies) which *replace* the old capital rather than augment it (see e.g., Cooley, Greenwood and Yorukoglu (1997)). Finally, the plant-level data indicates that each burst of investment activity is typically followed by a drop in productivity. After the initial drop, productivity gradually rises, eventually exceeding the pre-investment level (see Klenow (1998) and references therein, Hugget and Ospina (2001)).

The reason for the spiked pattern is now well-understood: it is due to adjustment costs associated with investment activity. As demonstrated by e.g., Abel and Eberly (1994, 1998) and Dixit and Pindyck (1994), adjustment costs may deter a firm from amending its capital stock continuously. The second empirical observation suggests that capital goods acquired by a firm at different times (e.g, a computer bought five years ago and the one bought recently) may not be perfect substitutes in production − old capital goods eventually become obsolete. To capture this feature, one needs to deviate from the dominant assumption of standard neoclassical investment models (see Sargent (1987)) that capital is homogeneous, infinitely-lived and depreciates at a constant rate. In other words, one needs a model of a vintage capital variety, where adjustments to capital stock come in the form of scrapping old vintages of capital (or obsolete technologies) and replacing them by more advanced ones. The final stylized fact may be evidence of learning-by-doing. Learning-by-doing is technology-specific, which is why we observe an initial drop in productivity following a burst of investment activity, and, after that, a gradual increase in productivity as the firm learns to operate a new technology.

The three stylized facts have been motivating the rapidly growing vintage capital literature (see, e.g., Bahk and Gort (1993), Parente (1994), Yorukoglu (1998), Greenwood and Jovanovic (2001)). These authors give multiple reasons why traditional homogeneous-capital investment models cannot satisfactorily explain observed investment patterns, and argue in favor of vintage capital models as a more adequate description of reality. For tractability reasons, this strand of literature has been developed under the assumption of a fully deterministic economic environment. Due to this assumption, the models share a common shortcoming: predicted bursts of investment occur at regular intervals – a result which has very little empirical support. To this day, however, very few attempts have been made to extend the literature to include uncertainty and to see the extent to which implications of deterministic models are affected. This is part of the objective of this work.

To demonstrate that there is no clear-cut answer as to whether or not the implications of certainty models are unchanged in a stochastic setting, we consider two classes of models. The first class builds on the deterministic vintage capital model of Parente (1994). In his dynamic economy, there are two inputs in a firm's production function: technology (of a certain quality) and technology-specific expertise. While operating a technology, the firm accumulates expertise in it, a fraction of which is lost when the firm decides to make an investment – adopt a new technology. The more advanced the new technology is relative to the old one, the more expertise is lost in transition. There is no other cost associated with technology adoption but the *expertise-based cost* (EBC). Parente shows that the optimal policy for the firm is to continue operation of an already-installed technology until a certain amount of expertise is acquired; once this threshold level of expertise is reached, it is optimal to adopt a new technology and scrap the old one. Upon introducing uncertainty into the firm's problem via a multiplicative productivity shock, we arrive at a rather surprising conclusion: a realization of the shock does not affect the timing of technology adoptions at all. That is, it is inconsequential for the firm's decision making whether the economy is booming or is in a recession. It is solely the level of expertise that determines whether to adopt a new technology or not; since the learning curve is deterministic, adoptions are periodic and determined at the initial date. Hence, the main implications of Parente's deterministic model extend without much modification to the uncertainty scenario.[1] We examine a number of variations of the model – only to reinforce our conclusion of uncertainty not mattering.

In the remainder of the paper, we seek alternative ways of modeling a vintage capital economy in which uncertainty would play a role. The key feature that produces the desired effect is the *specification of adjustment costs*. In the EBC model, the only cost to adoption of a new technology is lost expertise; there is no capital expenditure associated with an adoption. Intro-

[1] The implications of a certainty model would be identical to those of an uncertainty one, once the growth rate of productivity is appropriately adjusted.

ducing a capital expenditure, or a *capital-based cost (CBC)* to adoption, we construct a model, in which the realization of uncertainty does enter decision rules. Our CBC model is very tractable, with analytical results extending well beyond specifying the Bellman equation. In contrast to its deterministic benchmark (nested within the model), technology adoptions occur at random intervals. In booms (high aggregate productivity shock), the firm accelerates adoptions of new technologies, delaying them in recessions. The firm's revenues increase in booms, which makes the cost of adoption smaller relative to the firm's revenue. Adverse effects of a recession on the investment decisions are alleviated in part by the firm's expertise (or human capital). The more expertise the firm has accumulated, the more will be carried over to a new technology it adopts. So, a productivity slowdown following an adoption is less detrimental for a firm with a high stock of human capital. Our model predicts that a firm has to have accumulated a high stock of human capital to be able to adopt a new technology in bad states of the world. Does uncertainty facilitate growth in the CBC model? Comparing the model implications to those of the deterministic benchmark, we find that the answer is negative. This result is in accord with empirical evidence (see Leahy and Whited (1996)). Our firm becomes more cautious when faced with increased uncertainty. Reluctant to take the risk of investing in expensive advanced technologies, it acquires cheaper and less productive ones. Somewhat surprisingly, the model predicts that the pace of adoptions accelerates with uncertainty. More frequent adoptions however do not compensate for smaller technological advances brought about by each adoption. Overall, the growth rate of the firm's output and value of the firm decrease. This result points out the direction of the bias arising in growth estimates made in the context of a deterministic model.

Our conclusions may shed some light on the controversy surrounding modeling of adjustment costs. Whereas different specifications of such costs (EBC and CBC) in a deterministic setting give rise to similar results, in an uncertain economy, their implications may be very different. We feel that this gives useful insight on the nature of adjustment costs, insight that cannot be captured in a deterministic model. The results of this paper suggest that one of the tests of, for example, Parente's specification would be to empirically examine the extent to which varying degrees of aggregate uncertainty affect investment at the plant level.

Besides Parente, another paper advocating a similar modeling framework is Greenwood and Jovanovic (2001). We deviate from their deterministic models by enriching the production process with a stochastic productivity shock, and focus mainly on the implications due to uncertainty. A related vintage capital model that does incorporate uncertainty is that of Klenow (1998).

Although in his model there is no transferability of expertise between technologies, and hence the firm never needs to decide on the size of a technology upgrade – it always adopts the most advanced technology – his results are quite similar to ours. Klenow solves his model using the value function iterations on the discretized state space for calibrated parameter values. The advantage of our analytical characterization is that we are able to derive some closed-form expressions for the quantities of interest and hence to disentangle the effects of our models' assumptions from the effects of specific parameterizations. Methodologically, our analysis adopts the approach of the real options literature (see Dixit and Pindyck (1994)) that employs stochastic optimal stopping and control theory. Dixit and Pindyck, along with Abel and Eberly (1994, 1995, 1998), also study adjustment costs, and show that for a specific class of such costs a firm optimally "does nothing" most of the time. However, these authors work with homogeneous capital, and look at issues different from ours. Alvarez and Stenbacka (2001) adopt the real options methodology to study the problem of optimal technology adoption under uncertainty.[2] Their model is quite different from ours. In the interest of tractability, they do not endogenize the investment volume, and do not model technology-specific learning. Finally, there is a related strand of literature on strategic technology adoption by rival firms (e.g., Fudenberg and Tirole (1985), Reinganum (1981)). These models focus on oligopolistic competition among large players whose profits depend on whether a rival firm has adopted a new technology or not. Our firm is a small competitive firm whose actions do not directly affect other agents in the economy.

The remainder of the paper is organized as follows. Section 2 examines various specifications of a vintage capital model along the lines of Parente and concludes that, under the proposed specification, realization of uncertainty does not enter the decision rules. Section 3 offers an alternative formulation of the cost structure which makes the dynamics of the economy uncertainty-dependent. In Section 4, we derive the growth implications of the economy introduced in Section 3 and provide comparative statics. Section 5 concludes, and the Appendix provides the proofs.

2. Expertise-Based Cost (EBC) Models

In this section we employ the real options methodology to extend a vintage capital model to uncertainty, and to see the extent to which the implications from a certainty model are affected. As a benchmark certainty model, we adopt that of Parente (1994), generalizing it later in the

[2]See also Wang (2001) for a related analysis within the Cox, Ingersoll and Ross (1982) production economy extended to include an option to expand the set of existing technologies.

section to reinforce our conclusions.

We consider an infinite horizon, continuous-time economy with complete financial markets and one good serving as the numeraire. There is a single agent in the economy: a competitive firm.[3] The firm is represented by a production function $f(a, h)$ with the inputs $a \in \Re_+$ representing the *quality of technology* (or the vintage of capital) employed and $h \in [0, 1)$ representing the level of *expertise* accumulated in technology a. The firm can operate one technology at a time. Uncertainty enters into the economy via a multiplicative aggregate productivity shock z. To ensure non-negativity of z, we model it as a lognormally distributed random variable with parameters $(\mu - \sigma^2/2)t \in \Re$ and $\sigma^2 t \in \Re_+$:

$$z_t = z_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2} \right) t + \sigma w_t \right\} ,$$

where w_t, $t \in [0, \infty)$, is a one-dimensional Brownian motion driving the uncertainty. w_0 is assumed to be equal to zero. Throughout the paper, we will refer to μ as the "mean growth rate" and σ as the "volatility" of the shock process z_t. The time-t flow of output of the firm operating a technology of quality a with accumulated expertise h is then given by

$$y_t = z_t f(a_t, h_t) .$$

At any point in time, the firm may either continue operation of its present technology of quality a, or adopt a new one. Unless the firm chooses to adopt a new technology, a stays fixed. While operating a technology, the firm accumulates technology-specific expertise, or "learns-by-doing". There are diminishing returns to learning; the rate of accumulation of expertise in any technology a approaches zero as the firm continues its operation for a sufficiently long time:

$$\frac{dh_t}{dt} = \lambda - \lambda h_t , \quad \lambda > 0 , \tag{1}$$

where λ is the (exogenous) learning parameter, governing the speed of accumulation of expertise.

At an instant of an adoption, an old technology is scrapped (the scrap value is zero), and a new more advanced one is installed to replace it. Such adoption comes at a cost: some of the expertise associated with the old technology cannot be transferred to the new one. The cost consists of two components. First, a fixed percentage of accumulated expertise $\kappa \in (0, 1)$ is lost with each adoption. The second component is proportional to the factor by which the quality

[3]Unlike that of Parente, ours is a partial equilibrium model: we focus only on the firm's problem.

of current technology is increased: the more advanced the new technology is, the less expertise is carried over from the old technology. Accordingly, on adopting the more advanced technology at time τ, the quality increases as

$$a_{\tau+} = (1 + \nu_\tau) a_{\tau-} \, , \tag{2}$$

and the expertise is lost according to[4]

$$h_{\tau+} = h_{\tau-}(1 - \kappa) - \delta \nu_\tau, \quad \delta > 0. \tag{3}$$

The quality increment ν_τ, $\nu_\tau > 0$ is a choice variable of the firm: the trade-off is between a technology quality increase brought by an adoption and an amount of expertise lost therewith. Note that all adoption costs come solely in terms of lost expertise – this feature is crucial for the results that follow and is the reason why we call the models developed in this section "Expertise-Based Cost" models.

The market environment is described by the (stochastic) state-price density p_t, driven by the same Brownian motion w as the shock z. The Brownian motion generates the time-state space (analogous to the time-event tree in discrete-time stochastic dynamic models), on which by the complete markets assumption, there must be a unique price, corresponding to each time-state (t, ω). So, we define p_t to be the Arrow-Debreu state price per unit probability of one unit of the good in state ω at time t. p_t is assumed to be distributed lognormally and is given by

$$p_t = \exp \left\{ \left(-r - \frac{\theta^2}{2} \right) t - \theta w_t \right\} ,$$

where the interest rate $r \in \Re_+$ and market price of risk $\theta \in \Re_+$ are constants, and p_0 is normalized to 1. The benchmark certainty case results from setting the shock volatility σ and market price of risk θ equal to zero. Then, p_t becomes the familiar discount factor e^{-rt}.

Taking the state price process p as given, the firm chooses sequences $\{\tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots\}$ and $\{\nu_{\tau_1}^*, \nu_{\tau_2}^*, \ldots, \nu_{\tau_n}^*, \ldots\}$ – timing of adoptions and corresponding quality increments, respectively, to maximize the present value of its expected lifetime revenue given by

$$E \int_0^\infty p_t \, y_t \, dt \, . \tag{4}$$

[4]In the law of adjustment of expertise, h can in principle go negative. Greenwood and Jovanovic (2001) suggest a specification which does not have this shortcoming, but we chose not to depart from Parente's model, and at an optimum, verify that ν_τ is chosen so that h is always positive. In Parente's model, the rule of adjustment of expertise is $h_{\tau+} = h_{\tau-} - \kappa - \delta \nu_\tau$. For reasons which will become clear in the next section and without loss of substance, we model the fixed cost in percentage terms. The solution for Parente's formulation is very similar to ours.

2.1. Solution to Firm's Problem

In this subsection, we briefly review the methodology that is employed throughout the paper. A more elaborate discussion of the main steps can be found in Dixit and Pindyck (1994), or in Lund and Øksendal (1991) (with a more mathematically rigorous presentation).

The main difficulty in solving the model of Section 2 is posed by endogeneity of the time horizon (timing of adoptions) in the firm's problem, thus making it an optimal stopping and control problem. The value of the firm $I(a, p, z, h)$ satisfies the following Bellman equation:

$$I(a_t,\, p_t,\, z_t,\, h_t) = \max_{\tau,\nu_\tau} \frac{1}{p_t} E_t \left[\int_t^\tau p_s\, z_s f(a_s,\, h_s)\, ds + p_{\tau+} I((1+\nu_\tau)a_{\tau-},\, p_{\tau+},\, z_{\tau+},\, h_{\tau-}(1-\kappa) - \delta\nu_\tau) \right], \tag{5}$$

where the stopping time τ is the next time an adoption occurs, ν_τ is a corresponding quality increment and $E_t[\cdot]$ is expectation conditional on information available at time t. For notational convenience, throughout the paper we will work with the following transformation of the value function: $J(\cdot) = pI(\cdot)$ (the undiscounted value function, hereafter referred to as simply the "value function"). We note that if $J(a, p, z, h) > \max_{\nu_\tau,\, \nu_\tau > 0} J((1+\nu_\tau)a,\, p,\, z,\, h(1-\kappa) - \delta\nu_\tau)$, then it is optimal not to adopt a new technology immediately, but rather to continue operation of the existing one; the reverse is true if $J(a, p, z, h) \leq \max_{\nu_\tau,\, \nu_\tau > 0} J((1+\nu_\tau)a,\, p, z, h(1-\kappa) - \delta\nu_\tau)$. Strict inequality in the last expression is ruled out by continuity of the value function (which is required for optimality), so it is optimal to adopt as soon as the firm becomes indifferent between continuing with the old technology and adopting a new one. In the region of the state space where the existing technology is kept in operation, henceforth referred to as the "inaction region", the value function $J(a, p, z, h)$ satisfies the following partial differential equation:

$$0 = p\, z\, f(a, h) + J_z\, \mu z - J_p\, r\, p + J_h\, (\lambda - \lambda h) + \frac{1}{2}(J_{zz}\, \sigma^2\, z^2 + J_{pp}\, \theta^2\, p^2) - J_{zp}\, \sigma\, \theta\, z\, p. \tag{6}$$

Heuristically, the way to get this equation is to assume that in the inaction stage during a small enough time period Δt, there will not be any adoption of a new technology, and hence from the dynamic formulation of the problem (5) for time-t state vector $(a_t, p_t, z_t, h_t) = (a, p, z, h)$ we get

$$0 = E_t \left[\int_t^{t+\Delta t} p_s z_s f(a_s, h_s) ds + J(a,\, p + \Delta p,\, z + \Delta z,\, h + \Delta h) - J(a, p, z, h) \right].$$

Dividing through by Δt, taking a limit as $\Delta t \to 0$ and using Itô's Lemma to evaluate $(1/\Delta t)E_t[\Delta J(a, p, z, h)]$ we arrive at (6).

To complete the description of the solution, we need to specify what happens to the value function $J(\cdot)$ at the point of an exit from the inaction region (i.e., at an instant when it becomes optimal to adopt a new technology) − the boundary conditions. We already alluded to the first one: at an optimum, the values of the firm right before and immediately after an adoption are equal

$$J(a_{\mathcal{T}^*-}, p_{\mathcal{T}^*-}, z_{\mathcal{T}^*-}, h_{\mathcal{T}^*-}) = J((1+\nu_{\mathcal{T}}^*)a_{\mathcal{T}^*-},\, p_{\mathcal{T}^*+}, z_{\mathcal{T}^*+}\, h_{\mathcal{T}^*-}(1-\kappa) - \delta\nu_{\mathcal{T}}^*), \tag{7}$$

This condition is a value matching condition. Additional restrictions are needed to identify the boundary of the inaction region. These remaining conditions are referred to as smooth pasting conditions which assert that the value function must be continuously differentiable on the boundary of the inaction region. Instead of motivating smooth pasting conditions here, for the general case, we will do it later, for each specific case we consider. We will derive the conditions as the first order conditions for the firm's maximization.

To summarize, the methodology we use for solving firms' problems throughout the paper consists of specifying a partial differential equation for the value of the firm inside the inaction region (equation (6)) with appropriate value matching and smooth pasting conditions. Provided that certain regularity conditions (see Brekke and Øksendal (1994)) are satisfied, decision rules obtained this way are indeed optimal.

Unfortunately, an analytical solution to (6) subject to the appropriate boundary conditions for an arbitrary choice of f is not available, so we have to make some parametric assumptions about the production process. We now turn to examining few special cases.

2.2. Linear production function

In order to compare firm's behavior in our model to that of Parente, we first restrict our analysis to his specification of the production function:

$$f(a,h) = ah.$$

Assume that $E\int p_t y_t dt < \infty$. Equation (6) can be substantially simplified due to the following lemma:

Lemma 1. *For the linear production function $f(a,h) = ah$, $J(a,p,z,h)$ is homogeneous of degree 1 in $z\,a\,p$.*

We can now conjecture that the value function for this problem has the form

$$J(a,p,z,h) = z\,a\,p\,V(h),$$

where $V(\cdot)$ is a (continuous) function of expertise. Accordingly, equation (6) becomes

$$0 = h + \pi V(h) + V'(h)\,\lambda\,(1-h)\,, \tag{8}$$

where $\pi = \mu - r - \sigma\theta$. Note that h is the only relevant state variable in equation (6). From now on, where possible, we will drop the time subscript since the problem is time-homogeneous (in particular, time does not explicitly enter into (8)). Equation (8) is a simple linear ordinary differential equation whose solution is given by

$$V(h) = C\,(1-h)^{\frac{\pi}{\lambda}} + \frac{1-h}{\pi-\lambda} - \frac{1}{\pi}\,, \tag{9}$$

where C is a constant of integration, determined by the boundary conditions. Recall from Section 1.1 that the boundary conditions – value matching and smooth pasting – together with equation (8) fully characterize the value function as well as the boundary of the inaction region. Solving for the optimal $\{\tau_1^*,\, \tau_2^*, \ldots, \tau_n^*, \ldots\}$ is equivalent to identifying this boundary: adoption times τ_i^* correspond to the times (and states of the world) when the state vector $(a,\, p,\, z,\, h)$ takes a value on the boundary of the inaction region, thus making an adoption of a new technology optimal. Upon adoption, the vector $(a,\, p,\, z,\, h)$ is forced back inside the inaction region. The quality increment $\nu_{\tau_i}^*$ reflects the extent to which $(a,\, p,\, z,\, h)$ is altered by an adoption (in particular, how the technology quality a and expertise h are altered). The case of $\{\nu_{\tau_i}^* \equiv 0,\, \tau_i^* \in \{\tau_1^*,\, \tau_2^*, \ldots, \tau_n^*, \ldots\}\}$ corresponds to adopting an infinitesimal amount just enough to force the state vector back inside the inaction region. It is intuitive and can be verified formally, that this type of policy would have been optimal if the fixed cost κ associated with an adoption were zero. To determine $\{\nu_{\tau_i}^*\, (\equiv \nu^*$due to time homogeneity), $i = 1, 2, \ldots\}$ in our model, we first identify the inaction region.

Note that for the linear production function case, the value matching condition becomes

$$zapV(\overline{h}) = za(1+\nu^*)pV(\underline{h}), \tag{10}$$

where $\overline{h}$ and $\underline{h}$ denote levels of expertise right before and right after an adoption ($h_{\tau^*-} = \overline{h},\, h_{\tau^*+} = \underline{h},\, \forall \tau_i^*$), thereafter referred to as "trigger" and "target" values, respectively. Observing that z, a and p enter both sides of the equation (10) symmetrically, and do not enter equation (8), we conclude that it is the level of expertise h that triggers an adoption. The optimal quality increment ν^* is determined by the (yet unknown) trigger $\overline{h}$ and target $\underline{h}$:

$$\nu^* = \frac{\overline{h}(1-\kappa) - \underline{h}}{\delta}. \tag{11}$$

The remaining conditions, which are needed to identify $\overline{h}$ and $\underline{h}$, are smooth pasting conditions. In the context of the linear production function, these are just the conditions for maximization of $V(\cdot)$. Note that $J(a,\, p,\, z,\, h)$ is proportional to $V(h, C(\overline{h}, \underline{h}))$ and depends on the boundaries $\overline{h}$ and $\underline{h}$ only through $V(\cdot)$. So, optimally, $\overline{h}$ and $\underline{h}$ are chosen such that $V(h, C(\overline{h}, \underline{h}))$ is maximized (for all h) with respect to $\overline{h}$ and $\underline{h}$ subject to constraint (10).

The following lemma summarizes the discussion above.

Lemma 2. *Assume $f(a,h) = ah$. Then the firm's value function is given by*

$$J(a,p,z,h) = z\,a\,p\,V(h),$$

where

$$V(h) = C\,(1-h)^{\frac{\pi}{\lambda}} + \frac{1-h}{\pi-\lambda} - \frac{1}{\pi}\;,$$

with $\pi = \mu + \lambda - \sigma\theta$. The constant of integration C, as well as the optimal trigger and target $\overline{h}$ and $\underline{h}$ are determined by the following boundary conditions

Value matching:

$$V(\overline{h}) = \left(1 + \frac{\overline{h}(1-\kappa) - \underline{h}}{\delta}\right) V(\underline{h}), \tag{12}$$

Smooth pasting:

$$\left\{ \begin{array}{rcl} V'(\overline{h}) & = & \frac{1-\kappa}{\delta}\, V(\underline{h}), \\ V(\underline{h}) & = & \delta\left(1 + \frac{\overline{h}(1-\kappa)-\underline{h}}{\delta}\right) V'(\underline{h}). \end{array} \right. \tag{13}$$

Lemma 2 implies that the problem of finding a value function and decision rules is reduced to a much simpler task of completing the specification of $V(h)$.

Substituting the functional form for $V(\cdot)$ in (12)–(13), we get a system of three equations in three unknowns (C, $\overline{h}$, and $\underline{h}$):

$$\begin{array}{rcl} C(1-\overline{h})^{\frac{\pi}{\lambda}} + \dfrac{1-\overline{h}}{\pi-\lambda} - \dfrac{1}{\pi} & = & \left(1 + \dfrac{\overline{h}(1-\kappa) - \underline{h}}{\delta}\right) \Big\{ C\,(1-\underline{h})^{\frac{\pi}{\lambda}} + \dfrac{1-\underline{h}}{\pi-\lambda} - \dfrac{1}{\pi} \Big\}\;, \\ -C\,\dfrac{\pi}{\lambda}(1-\overline{h})^{\frac{\pi-\lambda}{\lambda}} - \dfrac{1}{\pi-\lambda} & = & \dfrac{1-\kappa}{\delta} \Big\{ C\,(1-\underline{h})^{\frac{\pi}{\lambda}} + \dfrac{1-\underline{h}}{\pi-\lambda} - \dfrac{1}{\pi} \Big\}\;, \\ C(1-\underline{h})^{\frac{\pi}{\lambda}} + \dfrac{1-\underline{h}}{\pi-\lambda} - \dfrac{1}{\pi} & = & \delta\left(1 + \dfrac{\overline{h}(1-\kappa) - \underline{h}}{\delta}\right) \Big\{ -C\dfrac{\pi}{\lambda}(1-\underline{h})^{\frac{\pi-\lambda}{\lambda}} - \dfrac{1}{\pi-\lambda} \Big\}. \end{array} \tag{14}$$

We are now ready to conclude our analysis with the main result of this section.

Proposition 1. *Assume $f(a,h) = ah$. Then the optimal policy is determined solely by the level of expertise:*

$$\begin{array}{ll} \textit{if } h_t < \overline{h} & \textit{then continue operation of current technology } a, \\ \textit{if } h_t = \overline{h} & \textit{then adopt a technology of quality } (1+\nu^*)a, \end{array}$$

where $\nu^* = \frac{\overline{h}(1-\kappa)-\underline{h}}{\delta}$. *The optimal timing of adoptions* $\{\tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots\}$ *is given by*

$$\begin{aligned} \tau_1^* &= -\frac{1}{\lambda} \ln\left(\frac{1-\overline{h}}{1-h_0}\right), \\ \tau_{i+1}^* &= \tau_i^* - \frac{1}{\lambda} \ln\left(\frac{1-\overline{h}}{1-\underline{h}}\right), \quad i = 1, 2, \ldots \end{aligned}$$

The trigger and target values of expertise $\overline{h}$ *and* $\underline{h}$ *are determined by (14).*[5]

According to Proposition 1, after the first adoption, adoptions of new technologies are periodic, and the quality increment ν^*, associated with each adoption, is constant over time. Somewhat strikingly, the realization of uncertainty does not affect the dynamics of the economy. Essentially, there is no genuine dynamic decision in this setup: the adoption plan is accepted at time 0, and the firm follows it from then on regardless of the state of the world. This result further reinforces Parente's (1994) conclusions.[6] One the other hand, one would be interested in exploring its sensitivity to the model's assumptions. In Section 3 we consider a variation on our model where in a stochastic environment realization of uncertainty plays a role: z_t enters the decision rules.

2.3. Extensions

It seems useful to show that it is not a specific (linear) form of the production function that drives the main results in Proposition 1. This subsection considers a more general specification of the production function (as studied by, for example, Cooley, Greenwood and Yorukoglu (1997)) and demonstrates that adoption policy is still independent of realization of uncertainty.

Consider a Cobb-Douglas production function of the form

$$f(a,h) = a^{\alpha} h^{1-\alpha}, \qquad \alpha \in (0,1).$$

The value function is no longer linear in zap, however one can show that

$$J(a,p,z,h) = za^{\alpha}\, p\, V(h)\,. \tag{15}$$

[5] Assuming that the system of equations (14) has a solution.

[6] The effects of uncertainty are limited to adjusting the trigger and target values $\overline{h}$ and $\underline{h}$, and hence the frequency of technology adoptions. Note that the volatilities σ and θ enter (14) only through $\pi = \mu - r - \sigma\theta$. So, the uncertainty model implications are the same as those of an otherwise identical deterministic model where the growth rate of productivity is equal to $\mu - \sigma\theta$.

The proof is very similar to that of Lemma 1. Upon substitution of $J(a, p, z, h)$ into equation (6), we get the following ordinary differential equation for the value function inside the inaction region:

$$0 = h^{1-\alpha} + \pi\, V(h) + V'(h)\, \lambda\, (1-h)\,. \tag{16}$$

The boundary conditions are now given by

Value matching:
$$V(\overline{h}) \;=\; \left(1 + \frac{\overline{h}(1-\kappa) - \underline{h}}{\delta}\right)^{\alpha} V(\underline{h}),$$

Smooth pasting:
$$\begin{cases} V'(\overline{h}) &=& \frac{\alpha(1-\kappa)}{\delta}\, V(\underline{h}), \\ \alpha V(\underline{h}) &=& \delta\left(1 + \frac{\overline{h}(1-\kappa)-\underline{h}}{\delta}\right) V'(\underline{h}). \end{cases}$$

Again, the state variables z, a and p enter neither the differential equation (16), nor the boundary conditions, making the trigger and target values $\overline{h}$ and $\underline{h}$, and hence $\{\tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots\}$ and $\{\nu_{\tau_1}^*, \nu_{\tau_2}^*, \ldots, \nu_{\tau_n}^*, \ldots\}$, independent of realization of uncertainty.

Additionally, we would like to remark that allowing for a more general specification of f is not going to affect the nature of the solution. Thus, if a production function has the form $f(a, h) = a^{\alpha}\, h^{\beta},\ \alpha,\, \beta \in \Re_+$, the only place where this problem would differ from the one above is equation (16). The first term in this equation would be h^{β} rather than $h^{1-\alpha}$.

Finally, an extension to the production function which incorporates the firm's choice for unskilled labor $-$ $y = a^{\alpha} h^{\beta_1} l^{\beta_2},\ \alpha,\, \beta_1,\, \beta_2 \in \Re$, where l denotes unskilled labor, optimization problem for which is static $-$ still delivers a similar result.

In the following remarks we embed uncertainty into the factors of production and comment on the robustness of the EBC Model's implications.

Remark 1 (Vintage-specific productivity shocks). In the EBC Model, we assume that the quality of technology a remains constant between adoptions. A natural question to ask is whether the presence of vintage-specific productivity shocks will make technology adoptions aperiodic. Vintage-specific shocks will effectively make the technology quality a stochastic. For example, a may have dynamics

$$da_t = \mu_a\, a_t\, dt + \sigma_a\, a_t\, dw_{a_t},$$

where the expected mean growth $\mu_a \in \Re$ and the volatility $\sigma_a \in \Re_+$ of the vintage-specific shock are constants and w_a is a (possibly correlated with w) Brownian motion. In this setup, our "uncertainty does not matter" result carries through: decision rules are still functions of expertise only, and are independent of realizations of both z_t and (now stochastic) a_t.

Remark 2 (Stochastic Learning). In the EBC Model, accumulation of expertise is deterministic. A stochastic component can be added in the learning curve in the following way:

$$dh_t = (\lambda - \lambda h_t)\, dt + \sigma_h(h_t)\, dw_{h_t},$$

where $\sigma_h(h_t)$ is volatility of learning and w_h is a Brownian motion. Decision rules are still functions of the level of expertise, but now learning is stochastic, and hence realization of w_h is going to matter for adoption decisions. In this case, the time interval between any two successive adoptions is stochastic.

To summarize, in this section we presented a number of special cases of the model, inspired by Parente's work. All the specifications, apart from stochastic learning, share a common implication: the decision rules are not affected by a business cycle (realization of uncertainty). The next section identifies the assumption that drives the result, and explores the consequences of relaxing it.

3. Capital-Based Cost (CBC) Model

In the EBC Models of Section 2 we assumed that the cost associated with an adoption comes only in terms of lost expertise. It turns out that, even under uncertainty, technology adoptions are periodic $-$ a result which has very little empirical support. It seems valuable to develop a tractable model with more realistic implications.

In an EBC-type Model, new technologies are free (apart from expertise-based costs), however it is reasonable to assume instead that the firm has to actually purchase a new technology it installs. We are going to show that embedding this additional cost in the model has significant consequences: unlike a cost in terms of expertise, a nontrivial purchase price of a new technology makes an adoption decision dependent not only the expertise h, but also on the realization of uncertainty z. It is important to emphasize that this result does not require the production structure to be parameterized in the way presented below. Our choice of the structure allows us to progress quite far analytically, and helps illustrate how the evolution of the firm under uncertainty is different from the benchmark deterministic case. The only modeling feature that is essential for making the firm's choice dependent on the realization of uncertainty is that the firm has to give up a fraction of its revenue in exchange for a new technology.

We consider the following modification to the model of Section 2. The production function belongs to the class considered in Section 2.3, it is Cobb-Douglas, with constant returns to scale. The flow of output is given by

$$y = z\, a^{\alpha}\, h^{1-\alpha}\,, \qquad \alpha \in (0,1).$$

Unlike that in the EBC Model, accumulation of technology-specific expertise is not subject to diminishing returns:[7]

$$\frac{dh_t}{dt} = \lambda\, h_t\,, \quad \lambda > 0. \tag{17}$$

The main difference from the EBC Models lies in the cost structure. A cost of adoption now consists of two components. First, some of the firm's expertise is lost:

$$h_{\tau+} = h_{\tau-}\,(1-\kappa)\,,$$

where, as before, $\kappa \in [0,1)$ is a fixed cost.[8] Second, a new technology is not free: the firm pays $qa_{\tau+}$ for it where q denotes a constant price per unit of quality of a new technology.[9]

The value of the firm is now

$$J(a_0, p_0, z_0, h_0) \;=\; \max_{\{\tau,\, \nu_\tau\}} E\left\{ \int_0^\infty p_t\, y_t dt - \sum_i p_{\tau_i}\, q\, a_{\tau_i +} \right\}. \tag{18}$$

The second term in the firm's problem is the present value of purchases of new technologies.

3.1. Solution to Firm's Problem

We start with the homogeneity property of the value function.

Lemma 3. *Under the CBC Model assumptions, $J(a,p,z,h) = a\, p\, V(\frac{\tilde{z}\, h}{a})$, where $\tilde{z}_t \equiv (z_t/q)^{1/(1-\alpha)}$ and $V(\cdot)$ is some (yet to be determined) function. $\tilde{z}_t$ is distributed lognormally with parameters*

[7] Assumption of no diminishing returns to learning was previously employed by Lucas (1988). We make it here for tractability reasons – the assumption is *not* crucial for the results that follow.

[8] It is more common in the literature to model the fixed cost in absolute rather than in percentage terms ($h_{\tau+} = h_{\tau-} - \kappa$). Our specification prevents the fixed cost from becoming negligible when h is high and very significant when h is low and we believe that it is an appropriate way to address costs of losing expertise. Since it is no longer required for finiteness of the solution to the firm's problem (and for tractability reasons), the proportional cost of expertise component is removed.

[9] Alternatively, we can model the price q as a geometric Brownian motion process to account, for example, for the secular decline in the price of new technologies. We consider this extension in Remark 3.

$(\mu_z - \frac{\sigma_z^2}{2})t$ and $\sigma_z^2 t$:

$$\tilde{z}_t = z_0^{\frac{1}{1-\alpha}} \exp\left\{ \left(\mu_z - \frac{\sigma_z^2}{2}\right) t + \sigma_z w_t \right\} ,$$

where $\mu_z = \mu/(1-\alpha) - \frac{1}{2}\frac{\alpha}{(1-\alpha)^2}\,\sigma^2$ and $\sigma_z = \sigma/(1-\alpha)$.

In the next lemma, we exploit the result of Lemma 3 which allows us to reduce the partial differential equation for the value function inside the inaction region (analogous to equation (6)),

$$0 = p\, a^{\alpha}\, (\tilde{z}\, h)^{1-\alpha} - J_p\, r\, p + J_z\, \mu_z \tilde{z} + J_h\, \lambda h + \frac{1}{2} J_{pp}\, \theta^2\, p^2 + \frac{1}{2} J_{zz}\, \sigma_z^2\, \tilde{z}^2 - J_{zp}\, \sigma_z\, \theta\, \tilde{z}\, p\,, \qquad (19)$$

to an ordinary differential equation. Denote $x \equiv \dfrac{\tilde{z}\, h}{a}$.

Lemma 4. *Assume that*

$$-r - \sigma_z \theta + \mu_z + \lambda < 0 \qquad (20)$$

and

$$\mu_z + \lambda - \frac{\sigma_z^2}{2} > 0. \qquad (21)$$

Then, in the inaction region, $V(x)$ satisfies

$$0 = x^{1-\alpha} - r\, V(x) + (\mu_z + \lambda - \sigma_z\, \theta)\, x\, V'(x) + \frac{1}{2}\, \sigma_z^2\, x^2 V''(x)\,. \qquad (22)$$

A solution to (22) is given by

$$V(x) = H\, x^{1-\alpha} + B\, x^{\gamma_p}\,, \qquad (23)$$

where γ_p is the positive root of the polynomial

$$\chi(\gamma) = -\frac{1}{2}\, \sigma_z^2\, \gamma^2 - (\lambda + \mu_z - \sigma_z\, \theta - \frac{\sigma_z^2}{2})\, \gamma + r\,,$$

$H = 1/\chi(1-\alpha)$, and B is a (yet to be determined) constant of integration.

The form of the value function given by Lemma 4, $J = paH\, x^{1-\alpha} + paB\, x^{\gamma_p}$, is very intuitive: the first term would be the (undiscounted) value of the firm if it were not allowed to switch technologies (since the second term would have to be equal to zero due to the transversality condition). The option to upgrade, captured by the second term, increases the value of the firm by some positive amount. Condition (20) is a sufficient condition for the integral in (18) to be well-defined; as a by-product, it ensures that the polynomial $\chi(\gamma)$ has two distinct roots, one of which is negative, and the other one, γ_p, is positive and greater than one. The role of assumption (21) will be clear from the analysis in Section 4.

To complete the description of the solution to the firm's problem, we need to specify the boundary conditions. Making use of Lemma 3, we can state the value matching condition as

$$a_{\tau^*-}\, p_{\tau^*}\, V(x_{\tau^*-}) = (1+\nu^*_\tau) a_{\tau^*-}\, p_{\tau^*-}\, V(x_{\tau^*+}) - a_{\tau^*-} p_{\tau^*}\,, \tag{24}$$

with the last term on the right hand side reflecting the purchase price of new technology paid at time τ^*. Note that the quality increment ν^*_τ has a convenient representation in terms of x_{τ^*-} and x_{τ^*+}:

$$\nu^*_\tau = \frac{x^*_{\tau-}}{x^*_{\tau+}}(1-\kappa) - 1\,.$$

Again, there is a single state variable entering equations (22) and (24): x. As in Section 2, we derive the optimal boundaries of the inaction region, the trigger $\overline{x}$ ($x_{\tau^*-} = \overline{x}$, $\forall \tau^* \in \{\tau^*_1, \tau^*_2, \ldots, \tau^*_n, \ldots\}$) and target $\underline{x}$ ($x_{\tau^*+} = \underline{x}$, $\forall \tau^* \in \{\tau^*_1, \tau^*_2, \ldots, \tau^*_n, \ldots\}$) from smooth pasting conditions, which are now the first order conditions for maximization of $V(x, B(\overline{x}, \underline{x}))$ with respect to $\overline{x}$ and $\underline{x}$ subject to (24). The boundary conditions are given by:

Value matching:
$$V(\overline{x}) = \frac{\overline{x}}{\underline{x}}(1-\kappa)\, V(\underline{x}) - \frac{\overline{x}}{\underline{x}}(1-\kappa), \tag{25}$$

Smooth pasting:
$$\begin{cases} \underline{x}\, V'(\overline{x}) &=& (1-\kappa)\,(V(\underline{x})-1), \\ \underline{x}\, V'(\underline{x}) &=& V(\underline{x})-1. \end{cases} \tag{26}$$

Substituting $V(x) = H\, x^{1-\alpha} + B\, x^{\gamma_p}$ in, we obtain the full characterization of the value function:

$$B = \frac{\underline{x}^{-\gamma_p}\,(H\,\alpha\,\underline{x}^{1-\alpha} - 1)}{\gamma_p - 1}, \tag{27}$$

$$\overline{x} = \underline{x}\left(1 - \frac{\underline{x}^{\alpha-1}}{H\,\alpha}\right)^{\left(-\frac{1}{-1+\alpha+\gamma_p}\right)}, \tag{28}$$

where $\underline{x}$ solves

$$F(\underline{x}) \equiv \frac{H}{\gamma_p - 1}\left\{\underline{x}^{1-\alpha}\left(1 - \frac{1}{\underline{x}^{1-\alpha}\,H\,\alpha}\right)^{-\frac{\alpha}{1-\alpha-\gamma_p}}(\alpha+\gamma_p-1) \;-\; (1-\kappa)\,\underline{x}^{1-\alpha}\,(\gamma_p+\alpha-1)\right\} + \frac{(1-\kappa)\,\gamma_p}{\gamma_p-1} = 0. \tag{29}$$

To show existence of a solution to our problem, we need to show that there is a finite $\underline{x} > 0$ satisfying equation (29) and a corresponding trigger $\overline{x}$, given by equation (28), with the property that $0 < \underline{x} < \overline{x} < \infty$.

Lemma 5. *Solution to equation (29) exists and is unique.*

We conclude our analysis and present the complete characterization of the firm's optimal behavior in Proposition 2.

Proposition 2 (CBC Model). *Under the CBC Model assumptions, the optimal policy is determined solely by the state variable x, $x \equiv \frac{\tilde{z}h}{a}$, and is given by:*

$$\begin{array}{ll} \textit{if } x_t < \overline{x} & \textit{then continue operation of current technology } a, \\ \textit{if } x_t = \overline{x} & \textit{then adopt a technology of quality } (1+\nu^*)a, \end{array}$$

where $\nu^* = \frac{\overline{x}}{\underline{x}}(1-\kappa) - 1$. The trigger and target values $\overline{x}$ and $\underline{x}$ are determined by equations (28) and (29), respectively.

According to Proposition 2, the decision on whether to adopt a new technology is dependent not only on expertise h, but also on the quality of the technology in place, a, as well as realization of uncertainty, z, thereby making the optimal policy dependent on the business cycle. In booms (high aggregate productivity shock), the firm accelerates adoptions of new technologies, delaying them in recessions. Adverse effects of a recession on the firm's investment decision are alleviated in part by the firm's expertise (or human capital) $-$ another variable that positively influences a decision to adopt.

Remark 3 (Declining Price of New Technologies). It has been documented that the US has witnessed the secular decline in the price of equipment relative to output in the post-war period. In particular, Greenwood and Jovanovic (2001) argue that the price has declined steadily, at the mean rate of 4% per year. This feature can be incorporated in the CBC Model by allowing q to decline with time as

$$dq_t = -\mu_q \, q_t \, dt \, ,$$

where μ_q represents the mean rate of decline. Alternatively, one can account for stochastic deviations of q_t from its exponential trend by introducing a Brownian motion component w_q with the corresponding volatility σ_q:

$$dq_t = -\mu_q \, q_t \, dt + \sigma_q \, q_t \, dw_{qt} \, .$$

Both specifications preserve the model's tractability. Under the former, all our expressions remain valid up to a replacement of μ_z by $\mu_z + \mu_q$ and r by $r + \mu_q$. Under the latter specification, we need to account for an additional state variable q, which introduces another source of (aggregate) risk in the firm's problem. For brevity, we omit the analysis here (details are available upon request).

To summarize, investment in the new technologies in the CBC Model is not only lumpy (similarly to the EBC Models), but also positively correlated with the uncertainty in the economy (a feature, absent in the EBC and deterministic models). In Section 4, we derive the moments of the distribution of a time interval between two successive adoptions.

3.2. Benchmark Deterministic Model

Since the main body of the technology adoption literature considers deterministic environments, we find it useful to solve a non-stochastic version of our model, which we will refer to as the benchmark.

Consider an economy with deterministic productivity growth $\mu \in \Re$, so that

$$z_t = z_0 e^{\mu t}.$$

Future prices are known with certainty

$$p_t = e^{-rt}.$$

In other words, σ and θ of the baseline model are set equal to zero.

In the Appendix, we solve the benchmark model using traditional techniques and report the results in Proposition 3. Of course, the same results would be obtained by simply setting $\sigma = \sigma_z = \theta = 0$ in the CBC Model.

Proposition 3 (Benchmark Deterministic Model). *Assume $0 < \mu_z + \lambda < r$. Consider the deterministic firm's problem with $\sigma = \theta = 0$:*

$$J(a_0, z_0, h_0, t) \quad = \quad \max_{\{\tau, \nu_\tau\}} \left\{ \int_0^\infty e^{-rt}\, z_t\, a_t^\alpha\, h_t^{1-\alpha}\, dt - \sum_i e^{-r\tau_i}\, q\, (1 + \nu_{\tau_i})\, a_{\tau_i -} \right\}. \tag{30}$$

The value of the firm in the unique steady state is given by

$$J(a_t, z_t, h_t, t) = a_t\, e^{-rt}\, V(x_t),$$

where $\tilde{z} = (z/q)^{1/(1-\alpha)}$, $x \equiv \frac{\tilde{z}h}{a}$, $x \in [\underline{x}, \overline{x}]$, and

$$V(x) = \hat{H}\, x^{1-\alpha} + B\, x^{\hat{\gamma}}\,, \tag{31}$$

with $\hat{\gamma} = \frac{r}{\mu_z + \lambda}$, $\hat{H} = \frac{1}{r - (\mu_z - \lambda)(1-\alpha)}$. The optimal policy is determined by x and is given by:

$$\begin{array}{ll} \textit{if } x_t < \overline{x} & \textit{then continue operation of current technology } a, \\ \textit{if } x_t = \overline{x} & \textit{then adopt a technology of quality } (1+\nu^*)a, \end{array}$$

where $\nu^* = \frac{\overline{x}}{\underline{x}}(1-\kappa) - 1$. *The optimal timing of adoptions* $\{\tau_1^*, \tau_2^*, \ldots, \tau_n^*, \ldots\}$ *is:*

$$\begin{aligned} \tau_1^* &= \frac{1}{\mu_z + \lambda} \ln\left(\frac{\overline{x}}{\underline{x}}\right), \\ \tau_{i+1}^* &= \tau_i^* + \frac{1}{\mu_z + \lambda} \ln\left(\frac{\overline{x}}{\underline{x}}\right), \quad i = 1, 2, \ldots \end{aligned}$$

The coefficient B, *trigger point* $\overline{x}$ *and target* $\underline{x}$ *are given by (27)–(29) upon substitution of* $\hat{\gamma}$ *and* $\hat{H}$ *for* γ *and* H, *respectively.*

Indeed, the deterministic model is nested in the stochastic one.

Of course, in the benchmark case we again have periodicity of technology adoptions. However, in contrast to an EBC Model, this implication does not extend to uncertainty. In other words, two seemingly similar deterministic models (EBC and CBC) give rise to very different conclusions under uncertainty.

4. CBC Model Implications

In this section, we derive additional analytical implications of the CBC Model, which allow us to compare the long-run mean growth rates of output in the two economies: the benchmark and the stochastic CBC-economy. The discussion is followed by numerical analysis, where we present comparative statics and discuss the implications due to uncertainty.

4.1. Long-run Growth in the CBC Economy

We define the long-run mean growth rate g_j of a variable j to be

$$g_j = \lim_{t \to \infty} \frac{E \ln\left(\frac{j_t}{j_0}\right)}{t}.$$

Recall that the steady-state flow of output y is given by

$$y = a\,(\tilde{z}h/a)^{1-\alpha} = a\,x^{1-\alpha},$$

and hence

$$g_y = g_a + (1-\alpha) g_x.$$

The state variable x is bounded, so $g_y = g_a$. Recall that with each technology adoption, the machine quality a gets augmented by the factor $1+\nu^* = \frac{\overline{x}}{\underline{x}}(1-\kappa)$. Let $T_i \equiv \tau_i^* - \tau_{i-1}^*$, $i = 1, 2, \ldots$, and denote $\#T_i(t)$ to be the maximal i such that $\sum_i T_i \le t$ (the number of adoptions prior to and including time t). Then

$$a_t = a_0 \left(\frac{\overline{x}}{\underline{x}}(1-\kappa)\right)^{\#T_i(t)}. \tag{32}$$

It can be derived from (32) that the long-run mean growth of output in the benchmark economy is equal to:

$$g_y = (\mu_z + \lambda)\left[1 + \frac{\ln(1-\kappa)}{\ln(\frac{\overline{x}}{\underline{x}})}\right]. \tag{33}$$

In the uncertainty case, the length of a period of inaction, T_i, is a random variable. The following lemma provides useful characterization of T_i, which we will use for evaluating the growth rate of output.

Lemma 6. *If $\mu_z + \lambda - \frac{\sigma_z^2}{2} > 0$ then the distribution of the time between two successive adoptions T_i has moments summarized by the moment generating functional*

$$u(\beta) = -\left(\frac{\overline{x}}{\underline{x}}\right)^{\eta_p},$$

where η_p is the positive root of $\frac{\sigma_z^2}{2}\eta^2 + (\mu_z + \lambda - \frac{\sigma_z^2}{2})\eta - \beta$. In particular, the first moment is given by

$$E[T_i] = \frac{1}{\mu_z + \lambda - \frac{\sigma_z^2}{2}} \ln\left(\frac{\overline{x}}{\underline{x}}\right), \qquad i = 1, 2, \ldots \tag{34}$$

and the variance of T_i is

$$Var[T_i] = \frac{\sigma_z^2}{\left(\mu_z + \lambda - \frac{\sigma_z^2}{2}\right)^2} E[T_i], \qquad i = 1, 2, \ldots.$$

If $\mu_z + \lambda - \frac{\sigma_z^2}{2} \le 0$ then

$$E[T_i] = \infty, \qquad i = 1, 2, \ldots.$$

Restating (32) in logs and taking expectations, we have

$$\begin{aligned} g_y &= \lim_{t\to\infty} \frac{1}{t} E\#T_i(t) \ln\left(\frac{\overline{x}}{\underline{x}}(1-\kappa)\right) \\ &= \frac{1}{E[T_i]} \ln\left(\frac{\overline{x}}{\underline{x}}(1-\kappa)\right) \\ &= (\mu_z + \lambda - \frac{\sigma_z^2}{2})\left[1 + \frac{\ln(1-\kappa)}{\ln(\frac{\overline{x}}{\underline{x}})}\right]. \end{aligned} \tag{35}$$

The second equality follows from the fact that $T_i,\ i = 1, 2, \ldots,$ are i.i.d. random variables.

Comparing the mean growth rate of the uncertainty economy to that of a benchmark one, we first note that uncertainty does affect growth. In a number of recent papers, vintage capital models under certainty have been used as measurement tools for growth accounting in the US (e.g., Greenwood and Jovanovic (2001), Gort, Greenwood and Rupert (1998)). In particular, it has been argued that vintage capital models provide a much more accurate estimate for the rate of technological change (g_a in our setup) compared to the conventional homogeneous capital models. As we can see from our analysis, volatility σ enters explicitly into equation (35), so the estimates made in the context of the benchmark model are biased in the presence of uncertainty. We show below that the direction of the effect of uncertainty can be identified: *ceteris paribus* uncertainty has a *negative* effect on the long-run mean growth of output. The extent to which the growth rate is affected is explored in Section 4.2.

4.2. Comparative statics

We have been unable to sign pertinent derivatives over the whole parameter space analytically, however numerical analysis provides some unambiguous conclusions.

4.2.1. Parameter Space

We consider a seven-dimensional set of parameters centered around the calibrated values reported in Table 1. The range of this set is $+/-$ 20 percent of a calibrated parameter value for each of the seven parameters. Calibration of all parameters but the shock mean growth rate μ and volatility σ is more or less standard.[10] From the results of Section 4.1, we can see that calibrating the productivity shock is not a matter of simply extracting the growth rate and volatility of output y from the U.S. data, and then matching the computed values to those of our artificial economy. Retrieving μ and σ from the output process parameters is difficult since the mean growth rate and volatility of output in our model are complicated functions involving, along with μ and σ, the trigger and target points $\overline{x}$ and $\underline{x}$ (which are themselves functions of μ and σ). However, since the emphasis of this work is on the qualitative issues rather than measurement, we just assign some values which we find reasonable and consider large (up to $+/-$ 75 percent of a parameter

[10] α is set such that the share of human capital (expertise) in production is 0.7, κ is chosen somewhat arbitrarily (the results do not show much sensitivity to κ), $\lambda = 0.002$ is set to be in line with the learning curve in Lucas (1988), for the market parameters, see, e.g., Obstfeld (1994).

value) admissible ranges for both of them.

Parameters			Value	Range (+/−)
Production	Technology share	α	0.3	20 percent
Expertise	Fixed Cost	κ	0.1	20 percent
	Learning parameter	λ	0.002	20 percent
Market	Interest rate	r	0.01	20 percent
	Market price of risk	θ	0.37	20 percent
Shock	Mean growth rate	μ	0.012	75 percent
	Volatility	σ	0.1	75 percent

Table 1. Parameter values.

4.2.2. Expected Inaction Time and Size of Upgrade

We summarize our first set of comparative statics results in the following proposition.

Proposition 4. *In each point of the seven-dimensional set of parameter values described in Table 1, we have*

Parameters			*Effects on Expected Inaction Time,* $E[T_i]$*, and Size of Upgrade,* ν^*
Expertise	*Fixed Cost*	κ	$+$
	Learning parameter	λ	$+$
Market	*Interest rate*	r	$-$
	Market price of risk	θ	$-$
Shock	*Mean growth rate*	μ	$+$
	Volatility	σ	$-$

The benchmark case yields the same results, except for the sensitivities to σ *and* θ *which are not defined in the deterministic economy.*

The direction in which the expected inaction time and the size of upgrade are affected by the parameters of the model is the same. It can be easily seen from the expression for $E[T_i]$ (equation (34)) why this is the case.

It is intuitive that a smaller fixed cost of adoption κ increases the frequency of technology upgrades. Each upgrade however will be smaller in size. The size cannot become infinitesimal, however, even if $\kappa = 0$. This is due to the the purchase price of new technologies $q > 0$, which acts as an additional cost to adoption, and hence creates lumpiness in the firm's investment behavior. The inaction range shrinks as the interest rate increases. This means that under high interest rates, the firm optimally chooses frequent but small upgrades of the technology it operates. Contemplating an adoption of a new technology, the firm is trading off a capital expenditure for an increase in revenues over a period of time in the future. The lower the interest rate, the higher the present value of the future revenues, and hence the more likely the firm is to incur a larger cost of adoption.

Growth rates μ and λ affect the economy in an similar way. As the rate of growth of the (external to the firm) aggregate productivity shock and the rate of accumulation of expertise go up, the firm's adoption policy becomes in a sense more aggressive: instead of making cautious small upgrades of its technology fairly frequently, it can now afford a longer wait followed by a larger advance. As we can see from our analytical expression for the mean growth rate g_y of the firm's output (35), larger technological leaps do compensate for longer waits. Indeed, the growth rate increases in both μ and λ. This observation gives an interesting insight into evaluating the growth potential of a firm: a prolonged operation of an old technology should be interpreted as "calm before the storm". It is not a firm that frequently takes advantage of small growth opportunities that grows faster, but it is a firm that waits longer, accumulating the potential for a significant technological advance. Our numerical results indicate that the value of the latter firm is higher.

How does uncertainty affect growth? As aggregate uncertainty in the economy (σ) increases, *ceteris paribus*, the size of the upgrade ν^*, associated with each adoption, as well as the long-run mean growth rate of output, decrease. This result advocates for a more cautious behavior of firms under uncertainty, which is to be expected. To provide some illustration, we simulate our economy for different values of σ and present the results in Figure 1. The figure plots (log of) technology quality a versus time. As one can see from the figure, the pace of technology adoptions accelerates with uncertainty, but due to the smaller upgrade sizes the firm ends up growing slower than that in the corresponding deterministic world. The value of the firm is also adversely affected by uncertainty.

5. Conclusion

In this paper we consider a variety of vintage capital models of a firm's choice under uncertainty in the presence adjustment costs and technology-specific learning. Part of our objective is to demonstrate how the main results of deterministic models change in an uncertain economic environment. We find that the answer crucially depends on the way costs of adjustment are modeled: if an adjustment cost comes only in terms of vintage-specific expertise (cf. Parente (1994)), we prove that the results are robust for a variety of specifications of the production process. However, once a capital expenditure associated with an adoption (capital-based cost) is introduced, predictions of an uncertainty model depart from its deterministic benchmark. We develop and solve a tractable model with capital-based costs, derive a formula for growth accounting and discuss the effects of the model's parameters on the pace of technology adoptions and sizes of technology upgrades. In booms, our firm accelerates adoptions of new technologies, delaying them in recessions. This finding may provide a foundation for explaining why aggregate investment is procyclical. The more expertise (the larger stock of human capital) the firm has, the more likely it is to adopt a new technology. In our model, uncertainty adversely affects growth and firm value. Quantitative statements would require a more careful calibration. We leave this as an exercise for the future.

Appendix

Proof of Lemma 1. Exploiting the structure of the problem, we reduce the dimension of the state space by introducing a new state variable $v = z\,a\,p$; now, $J(a,p,z,h) = J(v,h)$. The representation of the budget set in terms of v is as follows.
Between adoptions, v has dynamics

$$dv_t = (-r + \mu - \theta\sigma)\,v_t dt + (\sigma - \theta)\,v_t dw_t.$$

At a point of a switch,

$$v_{\tau+} = (1 + \nu_\tau)v_{\tau-}.$$

Clearly, for any trajectory $(v,\,h)$ which satisfies the constraints of the problem, $(\psi\,v,\,h)$, $\psi > 0$ would also satisfy these constraints.

Suppose now that (v_t^*, h_t^*) is an optimal policy for the original problem. Optimality of $(\psi v_t^*, h_t^*)$ follows from the observation that $E\,[\int_0^\infty \psi p_s\, z_s\, a_s\, h_s\, ds] = \psi E\,[\int_0^\infty p_s\, z_s\, a_s\, h_s\, ds]$. *Q.E.D.*

Proof of Lemma 3. *Linearity in p.* Since $p_t = p_0 \exp(-(r + \theta^2/2)t - \theta\, w_t)$,

$$\begin{aligned} J(a_0, p_0, z_0, h_0) \quad = \quad & \max_{\{\tau,\,\nu_\tau\}} q\, E_0 \Big\{ p_0 \int_0^\infty \exp(-(r + \theta^2/2)t - \theta\, w_t)\, \frac{1}{q} z_t\, a_t^\alpha\, h_t^{1-\alpha}\, dt \\ & -p_0 \sum_i \exp(-(r + \theta^2/2)\tau_i - \theta\, w_{\tau_i})\,(1 + \nu_{\tau_i})\, a_{\tau_i -} \Big\}. \end{aligned} \qquad (36)$$

Note that p_t, $\forall t$, does not enter the evolution equations of a_t, z_t and h_t, therefore any trajectory $(\{a\}, \{z/q\}, \{h\})$ feasible for problem (18) with initial conditions $(a_0, p_0, z_0/q, h_0)$ is also feasible if initial conditions were $(a_0, \rho p_0, z_0/q, h_0)$, $\forall \rho > 0$, and vice versa. It follows then from (36) that $J(a_0, \rho p_0, z_0/q, h_0) = \rho J(a_0, p_0, z_0/q, h_0)$, $\forall \rho > 0$.

It is verified in a similar fashion that the value of the firm is homogeneous of degree one in $\tilde{z}\,h$ (or $(z_t/q)^{1/(1-\alpha)}\,h$) and a, hence he have $J(a,p,z/q,h) = p\,a\,V(\frac{\tilde{z}\,h}{a})$. *Q.E.D.*

Proof of Lemma 4. First observe that equation (22) is obtained from (19) via a change of variable: $x \equiv \frac{\tilde{z}h}{a}$.

Solving the ordinary differential equation (22), we get the following expression for $V(x)$:

$$V(x) = H\,x^{1-\alpha} + B\,x^{\gamma_p} + D\,x^{\gamma_n}, \qquad (37)$$

where γ_p and γ_n are the two roots of the polynomial

$$\chi(\gamma) = -\frac{1}{2}\sigma_z^2\gamma^2 - (\lambda + \mu_z - \sigma_z\theta - \frac{\sigma_z^2}{2})\gamma + r\,,$$

$H = 1/\chi(1-\alpha)$, and B and D are constants of integration.

It follows from nonnegativity assumption on the interest rate and condition (20) that the two real roots of the quadratic $\chi(\gamma)$ γ_n and γ_p are distinct and $\gamma_n < 0$ and $\gamma_p > 1$ (observe that $\chi(0) = r > 0$ and $\chi(1) = r + \sigma_z\theta - \mu_z - \lambda > 0$). Since $\gamma_n < 0$, the constant of integration D has to be set equal to 0 to guarantee finiteness of the value function for small values of x (we will show later that x is bounded from above but can be infinitely small).

Imposing restriction (20) is equivalent to requiring the value of the firm to be finite provided that the optimal adoption policy of the firm is of trigger-target-type, i.e., the firm adopts a new vintage of technology as soon as x_t reaches its trigger value $x_{\tau^*-} = \overline{x}$; an instant after an adoption, x_t drops to $x_{\tau^*+} = \underline{x}$. To see why this is true, observe that

$$J(a_0, p_0, \tilde{z}_0, h_0) \leq E\int_0^\infty p_t\,a_t\,(\frac{\tilde{z}_t\,h_t}{a_t})^{1-\alpha}\,dt \leq E\,\overline{x}^{1-\alpha}\int_0^\infty p_t\,a_t\,dt\,. \tag{38}$$

Consistent with a trigger-target policy, a_t is a (non-decreasing) step function, each step corresponding to an adoption. At no circumstance, a can be bigger than $\frac{\tilde{z}\,h}{\underline{x}}$, where $\underline{x} > 0$.[11] At no time a non-negative policy process $\tilde{z}_t h_t$ in the presence of adjustment cost of expertise can be higher than $\tilde{z}_t h_t^* = \max_{\kappa \geq 0}\tilde{z}_t h_t$; the maximum is achieved for $\kappa = 0$. In that case, $\tilde{z}_t h_t^*$ is continuous and

$$p_t\tilde{z}_t h_t^* = p_0\tilde{z}_0 h_0^* \exp\left\{(-r - \theta\sigma_z + \mu_z + \lambda - \frac{(\sigma_z - \theta)^2}{2})t + (\sigma_z - \theta)w_t\right\}.$$

Consequently, a sufficient condition that we require for convergence of the integral in (38) is (20). *Q.E.D.*

Proof of Lemma 5. Denote the solution to $\left(1 - \frac{1}{\underline{x}^{1-\alpha}H\,\alpha}\right) = 0$ to be $\hat{\underline{x}}$;

$$\hat{\underline{x}} = \left(\frac{1}{H\,\alpha}\right)^{\frac{1}{1-\alpha}} > 0.$$

[11]The case where $\underline{x} = 0$ corresponds to the "stick to the same technology forever" plan. The value of the firm after an adoption becomes zero when $\underline{x} = 0$, so this outcome will be avoided at all cost. This would require having the trigger $\overline{x} \to \infty$. Clearly, for some set of parameters this policy would indeed be optimal, but this is not what we are looking for, so we put a condition ensuring that this is never the case.

From equation (29) we can see that the function $F(\cdot)$ is defined for $\underline{x} \in (\hat{\underline{x}}, \infty)$. Evaluating $F(\underline{x})$ at $\hat{\underline{x}}$, we find that $F(\hat{\underline{x}}) = -(1-\kappa)(1-\alpha)/\alpha < 0$. On the other hand, $F\left(\frac{\gamma}{H\,\alpha\,\kappa}^{\frac{1}{1-\alpha}}\right) > 0$, $\left(\frac{1}{H\,\alpha}\right)^{\frac{1}{1-\alpha}} < \left(\frac{\gamma}{H\,\alpha\,\kappa}\right)^{\frac{1}{1-\alpha}}$ which ensures that there is at least one (finite) solution to (29).

Uniqueness. It can be verified that $F'(\cdot) > 0$, therefore the solution to $F(\underline{x}) = 0$ is unique. *Q.E.D.*

Proof of Proposition 3. The (undiscounted) value of the firm at time t is given by

$$\begin{aligned} J(a_t, z_t, h_t, t) &= \max_{\tau,\,\nu_\tau} \Bigg\{ \int_t^\tau e^{-rs} a_s^\alpha (z_s\, h_s)^{1-\alpha}\, ds \\ &\quad + e^{-r\tau}\, J(a_t(1+\nu_\tau), z_\tau, h_{\tau-}(1-\kappa)) - e^{-r\tau} a_t(1+\nu_\tau) \Bigg\} \\ &= a_t \left(\frac{z_t\, h_t}{a_t}\right)^{1-\alpha} \max_{\tau,\nu_\tau} \Bigg\{ \int_t^\tau e^{-rs} e^{(\mu_z+\lambda)(1-\alpha)s}\, ds \\ &\quad + e^{-r\tau} J(a_t(1+\nu_\tau), z_\tau, h_{\tau-}(1-\kappa)) - e^{-r\tau} a_t(1+\nu_\tau) \Bigg\}, \end{aligned} \tag{39}$$

where $\tilde{z}_0$ is normalized so that prices of new machines are equal to unity.

Similarly to the uncertainty case, we take advantage of useful homogeneity of the value function which takes the form: $J(a_t, z_t, h_t, t) = e^{-rt}\, a_t\, V(x_t)$, where $x \equiv \frac{\tilde{z}h}{a}$, and $V(\cdot)$ does not explicitly depend on t. Manipulating (39), we get

$$V(x) = \max_{\tau,\,\nu_\tau} x^{1-\alpha} \frac{1 - e^{(-r+(\mu_z+\lambda)(1-\alpha))(\tau-t)}}{r - (\mu_z+\lambda)(1-\alpha)} + e^{-r(\tau-t)}\,(1+\nu_\tau)\,[V\left(\frac{x e^{(\mu_z+\lambda)(\tau-t)}\,(1-\kappa)}{1+\nu_\tau}\right) - 1]. \tag{40}$$

To be consistent with our main results, we rewrite the control problem in terms of $\overline{x}$ and $\underline{x}$ which replace the length of a period of inaction $T \equiv \tau_{i+1} - \tau_i$ and the quality increment $\nu_\tau = \nu$. The change of variable is as follows:

$$1 + \nu = \frac{\overline{x}}{\underline{x}}(1-\kappa) \qquad \text{and} \qquad \underline{x}\, e^{(\mu_z+\lambda)T} = \overline{x}.$$

This gives

$$V(x) = \max_{\overline{x},\,\underline{x}} \left\{ \frac{x^{1-\alpha}}{r - (\mu_z-\lambda)(1-\alpha)} + x^{\frac{r}{\mu_z+\lambda}} \left(-\frac{\overline{x}^{-\frac{r}{\mu_z+\lambda}+1-\alpha}}{r - (\mu_z-\lambda)(1-\alpha)} + \overline{x}^{-\frac{r}{\mu_z+\lambda}}\, \frac{\overline{x}}{\underline{x}}(1-\kappa)[V(\underline{x}) - 1] \right) \right\}. \tag{41}$$

Evaluating $V(x)$ (equation (41)) at $x = \overline{x}$, we get

$$V(\overline{x}) = \frac{\overline{x}}{\underline{x}}(1-\kappa)[V(\underline{x}) - 1], \tag{42}$$

which is just a restatement of the value matching condition (25).

First order conditions with respect to $\underline{x}$ and $\overline{x}$ are given by:

$$\begin{aligned}\underline{x}: \quad 0 &= \frac{\overline{x}}{\underline{x}}(1-\kappa)\,V'(\underline{x}) - \frac{\overline{x}}{\underline{x}^2}(1-\kappa)(V(\underline{x})-1), &(43)\\ \overline{x}: \quad 0 &= -\hat{H}(1-\alpha-\hat{\gamma})\,\overline{x}^{-\alpha-\hat{\gamma}} - \hat{\gamma}\overline{x}^{-\hat{\gamma}-1}\frac{\overline{x}(1-\kappa)}{\underline{x}}[V(\underline{x})-1] + \overline{x}^{-\hat{\gamma}}\frac{1-\kappa}{\underline{x}}[V(\underline{x})-1]. &(44)\end{aligned}$$

Finally, the envelope condition gives

$$V'(x) = \hat{H}(1-\alpha)x^{-\alpha} + \hat{\gamma}\,x^{\hat{\gamma}-1}\left(-H\,\overline{x}^{-\hat{\gamma}+1-\alpha} + \overline{x}^{-\hat{\gamma}}\frac{\overline{x}}{\underline{x}}(1-\kappa)[V(\underline{x})-1]\right). \tag{45}$$

Equation (43) is identical to (26), and substitution of $V'(x)$, evaluated at $\overline{x}$, from (45) into (44) yields (26). Hence, we have been able to reduce the problem to the special case[12] of the system of three equations in three unknowns we derived in the uncertainty model. *Q.E.D.*

Proof of Lemma 6. Notice that the domain of $x(t)$ is $(0, \overline{x}]$. Our strategy is to restate the problem in terms of $\ln x$ taking values on $(-\infty, \ln\overline{x})$, then consider an auxiliary problem where $\ln x$ is bounded from below by some constant $-K$, thus converting the problem to an analog of the well-known exercise of finding the expected time a Brownian motion with drift hitting either an upper or a lower boundary (either $-K$ or $\ln\overline{x}$ in our case). The final step is to let $K \to \infty$.

It follows from Itô's lemma that

$$d\ln x_t = (\mu_z + \lambda - \frac{\sigma_z^2}{2})\,dt + \sigma_z d_t. \tag{46}$$

Consider an interval $[-K, \ln\overline{x}]$ and define $X_t = \ln x_t$. The moment generating functional $U(X) = E[e^{-\beta T^x}|X_0 = X],\ -\infty < X < \ln\overline{x}$, with T^x denoting time to reach the boundary $\ln\overline{x}$ when the process X is started at $X_0 = X$, satisfies the following ordinary differential equation (Karlin and Taylor (1981), p. 243):

$$\beta U(X) = (\mu_z + \lambda - \frac{\sigma_z^2}{2})\frac{dU(X)}{dX} + \frac{1}{2}\sigma_z^2\frac{d^2U(X)}{dX^2}, \qquad -K < X < \ln\overline{x} \tag{47}$$

with the boundary conditions $U(-K) = U(\ln\overline{x}) = 1$. The general solution to (47) is given by

$$U(X) = C_1 e^{\eta_p X} + C_2 e^{\eta_n X},$$

where η_p and η_n are the positive and the negative roots of the characteristic polynomial

$$\frac{\sigma_z^2}{2}\eta^2 + (\mu_z + \lambda - \frac{\sigma_z^2}{2})\eta - \beta.$$

[12]In the benchmark case system of equations, $\hat{\gamma}$ replaces γ_p and $\hat{H}$ replaces H; the rest is identical.

Substituting in the boundary conditions and passing the limit $K \to \infty$, we find that

$$U(X) = e^{\eta_p (X - \ln \overline{x})},$$

and hence, evaluated at $\ln \underline{x}$,

$$U(\ln \underline{x}) = -\left(\frac{\overline{x}}{\underline{x}}\right)^{\eta_p} \equiv u(\beta | \underline{x}, \overline{x}).$$

The first moment of T_i is found by differentiating the negative of $u(\beta)$ and evaluating the derivative at $\beta = 0$:

$$E[T_i] = -\frac{1}{\mu_z + \lambda - \frac{\sigma_z^2}{2}} \ln\left(\frac{\overline{x}}{\underline{x}}\right).$$

The second moment is given by $d^2 u(\beta)/d\beta^2$ evaluated at $\beta = 0$, which gives the following for the variance

$$\mathrm{Var}[T_i] = \frac{\sigma_z^2}{\left(\mu_z + \lambda - \frac{\sigma_z^2}{2}\right)^2} E[T_i].$$

Finally, we refer to the well-known result (see, for example, Heath, Orey, Pestien and Sudderth (1987)) that a Brownian motion with negative drift(equation (46)) a.s. does not reach a level $\overline{X}$ in finite time if started at some point $X_0 < \overline{X}$, which justifies the last assertion of the lemma. *Q.E.D.*

References

Abel, A. B., 1983, “Optimal investment under uncertainty,” American Economic Review 73, 228–233.

Abel, A. B. and Eberly, J. C., 1994, “A unified model of investment under uncertainty,” American Economic Review 84, 1369–1384.

Abel, A. B. and Eberly, J. C., 1995, “The effects of irreversibility and uncertainty on capital accumulation,” NBER Working paper #5363.

Abel, A. B. and Eberly, J. C., 1998, “The mix and scale of factors with irreversibility and fixed costs of investment,” Carnegie-Rochester Conference Series on Public Policy 48, 101–135.

Alvarez, L. and Stenbacka, R., 2001, “Adoption of uncertain multi-stage technology projects: a real options approach,” Journal of Mathematical Economics 35, 71–97.

Bahk, B.-H., and Gort, M., “Decomposing learning by doing in new plants,” Jornal of Political Economy 101, 561–583.

Bar-Ilan, A. and Blinder, A., S., 1992, “Consumer durables: Evidence on the optimality of usually doing nothing,” Journal of Money, Credit and Banking 24, 258–272.

Brekke, K., A. and Øksendal, B., 1994, “Optimal switching in an economic activity under uncertainty,” SIAM Journal of Control and Optimization 32, 1021–1036.

Cooley, T. F., Greenwood, J. and Yorukoglu, M., 1997, “The replacement problem,” Journal of Monetary Economics 40, 457–499.

Cooper, R. W., Haltiwanger, J. C. and Power, L., 1999, “Machine replacement and the business cycle: Lumps and bumps,” American Economic Review, 89, 921–946.

Cox, J., Ingersoll, J. and Ross, S., 1985, “An Intertemporal General Equilibrium Model of Asset Prices,” Econometrica, 53, 363–384.

Dixit, A. and Pyndick, R., 1994, “Investment under uncertainty,” Princeton University Press, Princeton, NJ.

Doms, M. and Dunne, T., 1993, “Capital adjustment patterns in manufacturing plants,” Center for Economic Studies, U.S. Bureau of the Census.

Fudenberg, D. and Tirole, J., 1985, “Preemption and rent equalization in the adoption of new technology,” Review of Economic Studies 52, 383–401.

Gort, M., Greenwood, J. and Rupert, P., 1999, “Measuring the rate of technological progress in structures,” Review of Economic Dynamics 2, 207-230.

Greenwood, J. and Jovanovic, B., 2001, “Accounting for growth, New Directions in Productivity Analysis,” edited by C. Hulten, Dean E., and Harper, M., The University of Chicago Press, IL.

Heath, D., Orey, S., Pestien, V. and Sudderth, W., 1987, “Minimizing or maximizing the expected time to reach zero,” SIAM Journal of Control and Optimization 25, 195–205.

Hugget, M. and Ospina, S., 1998, “Does productivity fall after the adoption of new technology?” forthcoming in Journal of Monetary Economics.

Karlin, S. and Taylor, S. M., 1981, “A second course in stochastic processes,” Academic Press, CA.

Klenow, P. J., 1998, “Learning curves and the cyclical behavior of manufacturing industries,” Review of Economic Dynamics 1, 532–550.

Leahy, J. V. and Whited, T. M., 1996, "The effect of uncertainty on investment," Journal of Money, Credit and Banking 28, 64–83.

Lucas, R. E., 1988, "On the mechanics of economic development," Journal of Monetary Economics 22, 3–42.

Lund, D. and Øksendal, B., eds, 1991, "Stochastic models and option values. Applications to resources, environment and investment problems," Elsevier Science Publishers, Netherlands.

Obstfeld, M., 1994, "Risk-taking, global diversification and growth," American Economic Review 84, 1310–1329.

Parente, S., 1994, "Technology adoption, learning-by-doing, and economic growth," Journal of Economic Theory 63, 346–369.

Reinganum, J., 1981, "On the diffusion of new technology: a game theoretic approach," Review of Economic Studies 153, 395–406.

Sargent, T. J., 1987, "Dynamic macroeconomic theory," Harvard University Press, Cambridge, MS and London, England.

Wang, T., 2001, "Equilibrium with new investment opportunities," Journal of Economic Dynamics and Control 25, 1751–1773.

Yorukoglu, M., 1998, "The information technology productivity paradox," Review of Economic Dynamics 1, 551–592.

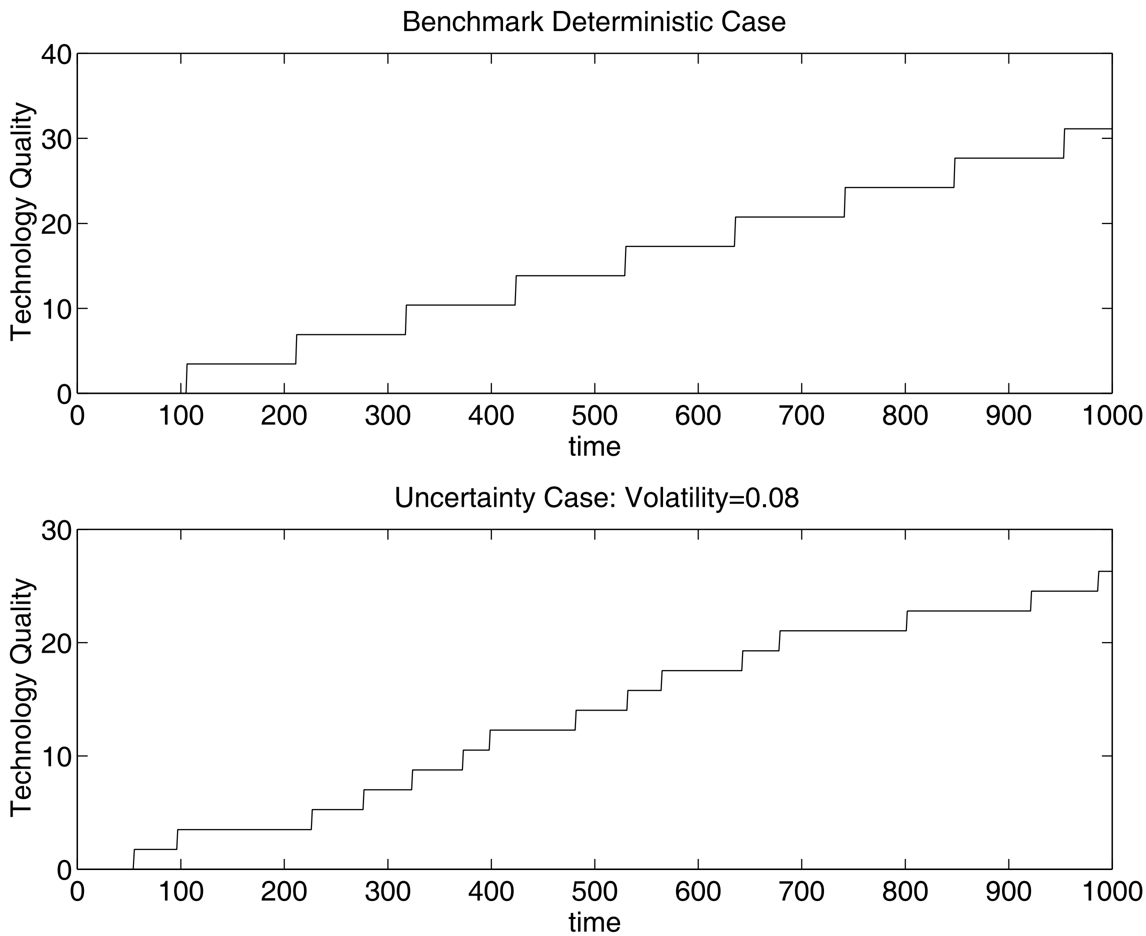

Figure 1. Simulated (log of) technology quality for the benchmark and uncertainty cases.