On Trees and Logs*

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On Trees and Logs

Abstract

In this paper we contrast the main workhorse model in asset pricing theory, the Lucas (1978) tree model (LT-Model), to a benchmark model in financial equilibrium theory, the real assets model (RA-Model). It is commonly believed that the two models entail similar conclusions since the LT-Model is a special case of the RA-Model. But this is simply wrong: implications of these models can be strikingly at odds. Indeed, under the widely-used log-linear specification of households' preferences, we show that for a large set of initial endowments the LT-Model – even with potentially complete financial markets – admits only peculiar financial equilibria in which the stock market is completely degenerate, in that all stocks offer the same investment opportunity – and yet, allocation is Pareto optimal. We investigate why the LT-Model is so much at variance with the RA-Model, and uncover new results on uniqueness of financial equilibria and introduction of portfolio constraints obtaining in the LT-Model, but not in the RA-Model. *Journal of Economic Literature* Classification Numbers: D50, G00, G12.

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1. Introduction

One of the most commonly employed models in asset pricing theory is the Lucas [16] asset-market tree economy. Investment opportunities in this economy are represented by claims to exogenously specified stochastic dividend streams paid out by firms (Lucas trees) and long-lived real bonds. Households trade in goods, and shares of trees or, as we will call them, stocks and bonds so as to maximize their expected lifetime utility defined over intertemporal consumption. Initial endowments of the households are in terms of portfolios of shares of stocks and bonds. By imposing clearing in spot goods and asset markets, one obtains an environment for determining equilibrium asset prices.

In this paper we critically examine the Lucas tree model (LT-Model) extended to include heterogeneous agents and multiple goods.[1] Dividend streams of the trees are specified in terms of a particular good; different trees pay out in different goods. (In fact, our results are easily extended to cover trees which pay out in a variety of goods, as described at the end of Section 2 below.) We weigh equilibrium implications of the LT-Model against those of the benchmark real assets model (RA-Model) in financial equilibrium theory, in which (i) there is no production (and therefore there are no firms); (ii) households diversify risk by trading IOU's whose promised yields are specified in terms of commodity bundles (real assets); and (iii) initial endowments are also commodity bundles. At the outset, one would expect the two models to deliver similar implications since the LT-Model can be transformed into a special case of the RA-Model. Consequently, the wide array of equilibrium results developed in the context of the RA-Model should then readily apply to the LT-Model. It turns out that this is simply not correct: the LT-Model has certain embedded structure that makes it significantly different from the RA-Model, and part of our goal is to highlight this structure and the implications it may lead to.

In particular, specializing households' preferences to be additively separable (over time) as well as *log-linear*, we show that for a large set of initial endowments the LT-Model − even with potentially complete financial markets − admits a *peculiar financial equilibrium* (PFE) in which all stocks but one are redundant. Put differently, even though yields to the trees − one can think of these as the net output flows of firms involved in production of different commodities − are generally unrelated, goods prices always adjust to make the payoffs (yields in value terms) from traded claims to the trees perfectly correlated. This result is in sharp contrast to a fundamental implication of the RA-Model (see, in particular, Magill and Shafer [18] for the case of potentially

[1] Agents heterogeneity is by now a common feature in modern equilibrium models in asset pricing. The single-good assumption of Lucas is also becoming less prevalent, especially in international finance applications (e.g., Zapatero [23], Serrat [21]).

complete financial markets, and Duffie and Shafer [11], as refined by Bottazzi [4], for the case of intrinsically incomplete financial markets): under mild regularity conditions (satisfied in the LT-Model), the matrix of payoffs on the stocks has full column rank generically in initial endowments (i.e., except for a closed, measure-zero subset of endowments). Furthermore, while in the real asset economy households typically must trade in all assets to achieve equilibrium, in our Lucas tree economy trading in bonds only occurs at the initial date, and the desired objective from trading in stocks can always be achieved by means of a single, fixed portfolio of stocks (for example, consisting of just a single stock).

It then follows that since there are necessarily fewer non-redundant assets in equilibrium than there are states of the world, financial markets are always incomplete. In the RA-Model, when financial markets are incomplete, for given household preferences and asset yields, but for a generic subset of initial endowments, equilibrium allocations are never Pareto optimal (as can be argued, for example, along the lines of Geanakoplos, Magill, Quinzii and Drèze [12]). Strikingly, in the LT-Model, PFE allocations are always Pareto optimal. Also, for a large subset of initial endowments, this peculiar financial equilibrium in our model exists in general, while existence is only generic in the RA-Model (again see Magill-Shafer and Duffie-Shafer, as refined by Bottazzi).

The very peculiar characteristics of equilibria in our economy bring to the fore an important structural difference between the LT- and RA-Models. One of the key features driving our puzzling implications is the specification of endowments. While in the LT-Model, endowments are specified in terms of *shares* of stocks and bonds, in the RA-Model endowments are specified in terms of *commodities*. If in addition to portfolios of shares, households in our model were endowed with bundles of commodities, equilibrium would typically no longer be of the peculiar kind.

It is not unrealistic, however, to have endowments specified in terms of shares of assets. And, in fact, this specification may lead to a number of new results in equilibrium theory. In particular, equilibrium theorists have usually assumed that endowments are *nonnegative*. And while a non-negativity assumption is certainly very defensible in a model with commodity endowments, there is nothing contradictory in dropping this assumption in a model with share endowments, especially when there are no restrictions on asset trade. In our model we allow for short initial positions in some assets. Our log-linear utility specification best highlights one of the implications of this additional degree of freedom. It is a standard result in microeconomics that in a pure-distribution economy with (nonnegative) commodity endowments and log-linear utility, competitive equilibrium is always unique. In contrast, in our model we can find share endowments for which this is no longer true. In fact, we can show that there may even be a *continuum of equilibria*, all of the peculiar type, and supporting all of the Pareto set. The subset of initial endowments for which this can occur is of a smaller dimension than the space of all initial endowments, so getting a continuum of equilibria is atypical, but it is nonetheless a distinct theoretical possibility. We fully characterize the errant subset. The proposition about nonuniqueness does not require that there be multiple goods in the economy, it encompasses the one-good case as well.

We then explore the robustness of our results. In particular, we investigate whether the peculiar financial equilibria that we exhibit survive various types of restrictions on transactions in financial markets. We find that for a large class of portfolio constraints, our implications are robust. For example, if households are unconstrained in their bond trades and unconstrained in trading at least one stock (but face arbitrary portfolio constraints on the remaining stocks), the PFE still occur. This result is strikingly at odds with the implications of comparable single-good models with portfolio constraints (see Sundaresan [22] for a recent survey). This is because there are other markets which are open in addition to asset markets: spot goods markets. Spot goods markets offer an additional channel through which portfolio constraints can be alleviated. Our model is an example of the significance of this additional effect: it is possible to replicate the unconstrained equilibrium allocation by trading in one stock, bonds, and goods, thus fully circumventing portfolio constraints.

Finally, we investigate whether there are other (or ordinary) financial equilibria (OFE), apart from the peculiar ones, in our model. This exercise relies on exploiting the special structure of our model: via a simple transformation of units we demonstrate that a LT-economy with log-linear households' preferences (or, concisely, a trees and logs economy) is equivalent to a deterministic economy, in which, in the transformed units, the aggregate endowments of all commodities are state-independent. This observation establishes an unexpected parallel between trees and logs economies and sunspot economies of Cass and Shell [7]: both economies are deterministic, in the sense that preferences and aggregate endowments are state-independent, however a state-(sunspot)-dependent allocation may emerge because agents are allowed to trade in assets whose yields are state-(sunspot)-dependent. In the transformed trees and logs economy, allocation in a PFE is shown to be deterministic. Then, all PFE in the trees and logs economy can be identified with the (deterministic) nonsunspot equilibria of Cass-Shell, and the remaining equilibria − OFE − with their sunspot equilibria. Despite having fewer nonredundant assets than there are states of the world, somewhat surprisingly, our economy does not exhibit any OFE (sunspot equilibria). In other words, we show that all financial equilibria are PFE, or in the language of the sunspot literature, our transformed economy is immune to sunspots.[2] However, in the presence of portfolio constraints under which (some of) the PFE survive, there may also be OFE: this depends on the

[2]This result is related to findings of Mas-Colell [19] and Antinolfi and Keister [1] that the presence of redundant securities of a specific nature can make an economy immune to sunspots.

nature and scope of the constraints, as we illustrate by example.

Most of the above results have their analogues in continuous time. There, equilibria in the model are peculiar in the sense that, for arbitrary stochastic processes representing dividends paid by the trees, the volatility matrix of securities in the investment opportunity set of the agents is always degenerate. Continuous time offers additional tractability over the original two-period model: we are able to parameterize stochastic processes for the state prices and stochastic weighting for a representative agent in the economy. Stock prices, spot goods prices and equilibrium allocations are computed in closed-form, providing a structural model potentially useful for future empirical work and applications. We feel that the continuous-time extension may be particularly relevant for a further investigation of the effects of portfolio constraints on asset prices and goods allocations in our model. Of course, one has to exercise extreme caution in employing a trees and logs-type economy: the peculiar implications we highlight must not be a part of a viable equilibrium model. However, extended along pertinent dimensions so as to rule out peculiar equilibria, the model may find a variety of applications.

Closely related to our work is the analysis by Zapatero, who uncovers a financial equilibrium of the peculiar variety in the context of a two-country two-good model of asset prices and exchange rates. In fact, it was Zapatero's results which led us to thinking about our trees and logs model. In the same vein is the earlier work of Cole and Obstfeld [8], who also document occurrence of something like a PFE in an equilibrium international model. Also related is the strand of literature investigating the special structure of preferences belonging to the linear risk tolerance class (see Magill and Quinzii [17], Chapter 3, and the references contained therein). In the context of a one-good model, it has been shown that "effective" market completeness, and hence Pareto optimality obtains in an incomplete financial market when households' preferences display linear risk tolerance with the same coefficient of marginal risk tolerance.

The remainder of the paper is organized as follows. Section 2 describes the economy. Section 3 characterizes the set of equilibria and investigates its properties. Section 4 contains an extension in continuous time. Section 5 outlines avenues for future research, while two Appendices contain all proofs.

2. The Economic Environment

Most of our basic framework is very standard in the Finance literature. There are two periods, today and tomorrow, labeled (when useful) $t = 0, 1$ $(= T)$. Uncertainty tomorrow is represented by future states of the world, labeled $\omega = 1, 2, \ldots, \Omega < \infty$, so that it is also natural to represent today as the present state of the world, labeled $\omega = 0$. In our only major departure from the common convention in asset pricing theory (but the common convention in financial equilibrium theory), we assume here that there are many goods in each state, labeled $g = 1, 2, \ldots, G < \infty$.

Production is described by exogenous stochastic streams of output of each type of good, $\delta^g(\omega) > 0$, all g, all ω, what in Finance have traditionally been viewed as dividend streams from stocks, but more recently as real yields from Lucas trees. The main difference here is that our trees or – as we will usually refer to them – *stocks* correspond one-to-one with the goods, and are accordingly also labeled $g = 1, 2, \ldots, G$. Quantities of stocks are denoted s^{tg}, all g, all t, and are by definition each in initial positive net supply of one unit.

Stocks are the sole source of goods in the economy, as well as one type of investment opportunity. The only other type of investment opportunity is long-lived real *bonds*,[3] each of whose promised yields is also specified in terms of a single good, by definition one unit of that good in each state. The bonds are labeled $\tilde{g} = 1, 2, \ldots, \tilde{G} \leq G$ – where a yield from bond $\tilde{g}$ is specified in units of good $g = \tilde{g}$ – and are in zero net supply. Their quantities are denoted $b^{t\tilde{g}}$, all $\tilde{g}$, all t. Even though the yields from bonds are nonstochastic, later on it will be useful to denote them by the abstract notation $\delta^{\tilde{g}}(\omega) = 1$, all $\tilde{g}$, all ω.

Households are the consumer-investors in this economy, and are labeled $h = 1, 2, \ldots, H < \infty$. Each household is endowed with an initial portfolio of assets $(b^0_h, s^0_h) \in \mathbb{R}^{\tilde{G}} \times \mathbb{R}^G = \mathbb{R}^{\tilde{G}+G}$, and trades on a spot market for goods and assets at spot 0, and then again, after the future state of the world $\omega > 0$ has been realized, on a spot market for goods at spot ω. Purchase, and therefore also consumption of goods is denoted $c^g_h(\omega) \in \mathbb{R}^G_{++}$, all g, all ω, and the terminal portfolio $(b^1_h, s^1_h) \in \mathbb{R}^{\tilde{G}} \times \mathbb{R}^G = \mathbb{R}^{\tilde{G}+G}$, while spot goods, and bond, and stock prices are denoted $p^g(\omega) \in \mathbb{R}^G_{++}$, all g, all ω , and $(q_b,\ q_s) \in \mathbb{R}^{\tilde{G}} \times \mathbb{R}^G = \mathbb{R}^{\tilde{G}+G}$, respectively. Both borrowing and short sales of stocks are permitted (and have been permitted in the past), which explains why there are no sign restrictions on $(b^1_h,\ s^1_h)$ (or on $(b^0_h,\ s^0_h)$).

Each household evaluates its actions according to a von Neumann-Morgenstern utility function over present and prospective future consumption

$$u_h(c_h) = \sum_{\omega > 0} \pi(\omega) v_h(c_h(0), c_h(\omega)),$$

where $\pi(\omega) > 0$, $\omega > 0$, with $\sum_{\omega>0} \pi(\omega) = 1$, represent common prior probabilities, and $v_h : \mathbb{R}^{2G}_{++} \to \mathbb{R}$ represents the household's two-period certainty utility function. Expected utility is assumed to satisfy textbook regularity, monotonicity, and convexity assumptions, in particu-

[3]Our particular specification of the alternative available investments to stocks is chosen primarily for expositional convenience. In fact, our results generalize immediately to any real (Economics) or derivative (Finance) assets – as long as they are in zero net supply.

lar those (minimally) consistent with additively separable log-linear certainty utility: v_h is C^2, differentiably strictly increasing, and differentiably strictly concave, and satisfies the boundary condition, for every $(c_h^{0\prime}, c_h^{1\prime}) \gg 0$,

$$cl\{(c_h^0, c_h^1) \in \mathbb{R}_{++}^{2G} : v_h(c_h^0, c_h^1) \geq v_h(c_h^{0\prime}, c_h^{1\prime})\} \subset \mathbb{R}_{++}^{2G}.$$

Later on we will specialize to log-linear utility

$$v_h(c_h(0), c_h(\omega)) = \sum_g \alpha_h^{0g} \log c_h^g(0) + \beta_h \sum_g \alpha_h^{1g} \log c_h^g(\omega),$$

so that

$$u_h(c_h) = \sum_g \alpha_h^{0g} \log c_h^g(0) + \sum_{\omega>0} \pi(\omega)\beta_h \sum_g \alpha_h^{1g} \log c_h^g(\omega),$$

with $\alpha_h^{tg} > 0$, all g, and $\sum_g \alpha_h^{tg} = 1$, all t, and $\beta_h > 0$.

Since one of our primary concerns will be with the relationship between equilibrium allocation and Pareto optimality, for the most part we will concentrate on the case where $G+\tilde{G} = \Omega$ (so that $G < \Omega \leq 2G$), that is, where there are *potentially complete financial markets.* However, our main results do not depend on this assumption, and are equally true for the case where $G + \tilde{G} < \Omega$, so that there are *intrinsically incomplete financial markets* − as well as, obviously, the case where $G+\tilde{G} > \Omega$, that is, where there are *necessarily redundant assets.* Notice that when assets provide, effectively − as they do in this economy − both initial endowments (of goods) and investment opportunities, having necessarily redundant assets is not immaterial; such redundancy enlarges the set of possible initial endowments.

For certain purposes we will also concentrate on what we will refer to as *the leading example*, where $\Omega = 3$, $G = 2$, $\tilde{G} = 1$, and $H = 2$, the smallest dimensional case with more than one good in which a bond is required in order to provide potentially complete financial markets. This is purely for expositional purposes, where there is no especial insight gained by introducing more generality.

Incidentally, it is worth stressing that all but one of our main results are still valid when, instead of there being trees yielding distinct goods, described by $\delta^g(\omega) > 0$, all ω, there are actually firms (perhaps *hybrid trees*), labeled $f = 1, 2, \ldots, F < \infty$, paying off in terms of distinct bundles of goods, described, say, by

$$(\delta^g(\omega) y_f^g, \text{ all } g), \text{ all } \omega,$$

provided only that

$$\sum_f y_f^g = r^g > 0, \text{ all } g,$$

(so that, in particular, total resources are strictly positive). Under this interpretation, $Y_f = \{(y_f^g,$ all $g)\}$ can be viewed as a typical firm's (single point) production set,[4] and $\{\delta^g(\omega),$ all $g\}$, $\omega > 0$, as goods-specific multiplicative aggregate uncertainty − covering the specific, standard case where $\delta^g(\omega) = \delta(\omega)$, all g, $\omega > 0$, that is, where there is purely multiplicative aggregate uncertainty. Then the payoffs

$$\sum_g p^g(\omega)\delta^g(\omega)y_f^g, \text{ all } \omega$$

represent a typical firm's dividend stream (in terms, say, of some appropriate units of account).[5] Of course, under this generalization, a tree is simply a special kind of hybrid. Alternatively, a hybrid tree may be interpreted as a mutual fund. Households in the model would then be trading portfolios of mutual funds (some of which might be simply individual stocks). Replacing individual stocks in the investment opportunity sets of the households by mutual funds effectively imposes portfolio constraints: some stocks can be transacted only as a bundle. The nature of the (rather mild) restrictions on the mutual funds trade required for all of our results to go through will become apparent from the analysis in section 3.3.

We will appeal to this generalization during the course of the following analysis; in fact, the sequencing of this development and the choice of arguments used in our proofs are partly dictated by a desire to encompass this simple but nonetheless important, interesting extension.

Finally, we again emphasize that − except for the assumption of many goods − this model, including log-linear utility, is a standard workhorse in Finance, even more so when there are intrinsically incomplete financial markets, or institutionally imposed portfolio restrictions.

3. Characterization of Equilibrium

3.1. Preliminaries

3.1.1. Notation

We adopt the obvious convention for forming vectors (and, similarly, matrices) from indexed scalars or vectors: simply suppress the common index, and write the corresponding set of indexed

[4] Of course, when some good is an input for some firm, additional restrictions on initial portfolios may be required in order to guarantee, for instance, that households have positive initial endowments of at least some commodities.

[5] In fact, aside from issues concerning the assumptions required to guarantee existence of equilibrium, Y_f can be taken to be an arbitrary compact, convex set, in which case the typical firm should choose

$$y_f(\omega) = \arg\max_{y_f \in Y_f} \sum_g p^g(\omega)\delta^g(\omega)y_f^g, \quad \text{all } \omega.$$

A slight further possible extension is to assume that the typical firm's production set depends on the date (though not on the event).

scalars or vectors in their natural order. Thus, for instance,

$$\begin{aligned} &p(\omega) = (p^g(\omega),\ \text{all } g) \text{ and } p = (p(\omega),\ \text{all } \omega),\ \text{while} \\ &c_h(\omega) = (c_h^g(\omega),\ \text{all } g),\ c_h = (c_h(\omega),\ \text{all } \omega),\ \text{and } c = (c_h, \text{all } h). \end{aligned}$$

Also, modifying the standard convention in mathematics that $x \in \mathbb{R}^n$ is an n-dimensional column vector, we will treat price (e.g., $p(\omega)$) and price-like (e.g., α_h^t) vectors as rows rather than columns of their elements. The remaining vectors are understood as column vectors.

3.1.2. Financial Equilibrium

From each household's viewpoint, the yield from an asset is simply a vector of goods – albeit a particular, possibly a very special vector of goods – and their initial portfolio (of assets) represents their initial endowment (of goods). For this reason it is useful to begin by formulating the concept of *financial equilibrium* (FE) in terms of the real asset equivalents of bonds and stocks, initial endowments, and net changes in portfolio holdings. Such a general formulation also highlights the differences between the LT-model and the RA-model, and facilitates comparing properties of their equilibria. Let

$$\Delta_b(\omega) = \overset{\widetilde{G}}{\begin{bmatrix} \begin{bmatrix} \ddots & & 0 \\ & \delta^{\tilde{g}}(\omega) & \\ 0 & & \ddots \end{bmatrix} \\ 0 \end{bmatrix}} \begin{matrix} \widetilde{G} \\ \\ G - \widetilde{G} \end{matrix} \qquad \left(= \begin{bmatrix} I \\ 0 \end{bmatrix} \right)$$

and

$$\Delta_s(\omega) = \overset{G}{\begin{bmatrix} \ddots & & 0 \\ & \delta^{g}(\omega) & \\ 0 & & \ddots \end{bmatrix}} \; G$$

be the $(G \times \widetilde{G})$- and $(G \times G)$-dimensional matrices representing the goods yields from bonds and stocks, respectively, so that

$$e_h(\omega) = [\Delta_b(\omega)\Delta_s(\omega)](b_h^0, s_h^0)$$

is the initial endowment of household h in state w. Also let

$$z_h = (z_{b,h},\ z_{s,h}) = ((b_h^1 - b_h^0),\ (s_h^1 - s_h^0))$$

be the net change in the portfolio holdings of household h. Then, (p, c, q, z) is a FE if

- households optimize, i.e., given (p, q) (and $\Delta = [\Delta(\omega), \text{ all } \omega] = [[\Delta_b(\omega)\Delta_s(\omega)], \text{ all } \omega]$, according with our convention), for every h, (c_h, z_h) is an optimal solution to the problem

$$\begin{array}{llll} \text{(H)} & \text{maximize}_{c_h, z_h} & u_h(c_h) & \text{with multipliers} \\ & \text{subject to} & p(0)(c_h(0) - e_h(0)) + qz_h = 0 & \lambda_h(0) \\ & \text{and} & p(\omega)(c_h(\omega) - e_h(\omega)) - p(\omega)\Delta(\omega)z_h = 0, \omega > 0, & \lambda_h(\omega) \end{array}$$

  and

- spot goods and asset markets clear, i.e.,

$$\text{(M)} \quad \begin{array}{l} \sum_h (c_h - e_h) = 0, \text{ and} \\ \sum_h z_h = 0. \end{array}$$

For the purpose of presenting and interpreting our main results concerning the structure of FE, it is necessary to introduce two auxiliary concepts: first, the concept of a *certainty equilibrium* (CE) − which is the Walrasian equilibrium in a related two-period, pure-distribution economy that we will refer to as the certainty economy (see Cass-Shell, pp. 207-8) − and second, the device for relating FE to CE, the concept of a puzzling or *peculiar financial equilibrium* (PFE).

3.1.3. Certainty Equilibrium

Consider the two-period, pure-distribution economy without uncertainty for which utility functions, initial endowments, and consumption for each household are v_h, $\bar{e}_h = (\bar{e}_h^0, \bar{e}_h^1)$, and $\bar{c}_h = (\bar{c}_h^0, \bar{c}_h^1)$, respectively, and goods prices (on overall goods markets in period 0) are $\bar{p} = (\bar{p}^0, \bar{p}^1)$. In such a certainty economy, $(\bar{p}, \bar{c})$ is a CE (otherwise known as a Walrasian, competitive, or general equilibrium) if

- households optimize, i.e., given $\bar{p}$, for every h, $\bar{c}_h$ is the optimal solution to the problem

$$\begin{array}{llll} (\overline{\text{H}}) & \text{maximize}_{\bar{c}_h} & v_h(\bar{c}_h) & \text{with multiplier} \\ & \text{subject to} & \bar{p}(\bar{c}_h - \bar{e}_h) = 0 & \bar{\lambda}_h \end{array}$$

  and

- overall goods markets clear, i.e.,

$$(\overline{\text{M}}) \quad \sum_h (\bar{c}_h - \bar{e}_h) = 0.$$

It will be convenient, when analyzing existence of FE, to have a means of referring to the set of certainty endowments for which CE exists. So, given total resources $\bar{r} = (\bar{r}^0, \bar{r}^1) = \mathbf{1}$, let

$$\overline{E} = \{\bar{e} \in (\mathbb{R}^{2G})^H : \sum_h \bar{e}_h = \bar{r} \text{ and there is a CE}\}.$$

Note that here there is a major departure from the mainstream Walrasian tradition: we consider all conceivable certainty endowments, and, specifically, do not require that they lie in each household's consumption set.

3.1.4. Peculiar Financial Equilibrium

Our first main result concerns the particular kind of FE we refer to as PFE in an economy in which (as in the original economy, the economy described in Section 2)

$$e_h(\omega) = [\Delta_b(\omega)\Delta_s(\omega)](b_h^0, s_h^0), \text{ all } \omega, \text{ all } h, \tag{3.1}$$

but (in sharp contrast to the original economy)

$$\delta^{\widetilde{g}}(\omega) > 0, \text{ all } \widetilde{g}, \text{ and } \delta^g(\omega) = 1, \text{ all } g, \text{ all } \omega, \tag{3.2}$$

that is, $\Delta_b(\omega)$ is essentially unrestricted while $\Delta_s(\omega) = I$. The crucial implication of the second assumption is that, in this economy, total resources, denoted r, are stationary across states

$$r = [r(\omega), \text{ all } \omega] = [\Delta_s(\omega)\mathbf{1}, \text{ all } \omega] = \mathbf{1}.$$

We are now ready to formalize the concept of a peculiar financial equilibrium.

Definition. When $\Delta_s(\omega) = I$, all ω, a FE is a PFE if

(i) *irrelevancy*: $z_{bh} = -b_h^0$, all h, i.e., households completely liquidate their initial portfolio of bonds;

(ii) *degeneracy*: rank $[p(\omega)\Delta_s(\omega), \ \omega > 0]$ = rank $[p(\omega), \omega > 0] = 1$, i.e., households are completely indifferent to which (equally valued) terminal portfolio of stocks they hold; and yet

(iii) *optimality*: rank $[\lambda_h, \text{ all } h] = 1$, i.e., the goods allocation is Pareto optimal.

It may be unclear at this point what the economy with constant total resources as well as the certainty economy have to do with the original economy of section 2. These auxiliary economies are quite important: in fact, they capture the essence of the special structure of our model. Thanks to a simple transformation of units, the original economy will soon be shown to map into a setting in which all stocks' payoffs are independent of the state of the world.

3.2. Existence

The key feature of a PFE which permits a simple characterization is that, effectively, the spot market budget constraints in a FE collapse to the Walrasian budget constraint in a CE with certainty endowments given by the formulas

$$\begin{aligned}\bar{e}_h &= (e_h(0), \sum_{\omega>0} \pi(\omega) e_h(\omega)) \\ &= ([\Delta_b(0)\Delta_s(0)](b_s^0, s_h^0), \sum_{\omega>0} \pi(\omega)[\Delta_b(\omega)\Delta_s(\omega)](b_h^0, s_h^0)), \text{ all } h.\end{aligned} \tag{3.3}$$

This will become obvious when we detail the proof of Proposition 1 in Appendix A. So now let

$$\bar{E}_\Delta = \{\bar{e} \in \bar{E} : \text{ for some } (b_h^0, s_h^0), \text{ all } h, \text{ such that } \sum_h (b_h^0, s_h^0) = (0, \mathbf{1}), \bar{e} \text{ satisfies (3.3)}\}.$$

Note that, generically in Δ, dim $\bar{E}_\Delta = (H-1)(\tilde{G}+G)$, which in the leading example equals 3.

For simplicity, normalize prices so that $p^1(\omega) = 1$, all ω, and $\bar{p}^{11} = 1$ (later on we will find that another price normalization is more useful when analyzing the nature of PFE).

Proposition 1 (Existence of PFE). *Consider an economy which satisfies (3.1) and (3.2), together with the related certainty economy which satisfies (3.3).*

(i) *If (p, c, q, z) is a PFE, then $\overline{e} \in \overline{E}_\Delta$ and there is a CE $(\overline{p}, \overline{c})$ such that*

$$\begin{aligned}\overline{p} &= \left(p(0), \ \frac{\lambda_1(\omega)}{\pi(\omega)\lambda_1(0)} p(\omega)\right) = \left(p(0), \ \frac{\lambda_1(1)}{\pi(1)\lambda_1(0)} p(1)\right) \\ &\textit{and} \\ \overline{c}_h &= (c_h(0), c_h(\omega)) = (c_h(0), c_h(1)), \quad \textit{all } \omega, \ \textit{all } h.\end{aligned} \tag{3.4}$$

(ii) *If $\overline{e} \in \overline{E}_\Delta$ and $(\overline{p}, \overline{c})$ is a CE, then there is a PFE (p, c, q, z) such that*

$$\begin{aligned}p(\omega) &= \begin{cases} \overline{p}^0, & \omega = 0 \\ \overline{p}^1/\overline{p}^{11}, & \omega > 0 \end{cases} \\ &\textit{and} \\ c_h(\omega) &= \begin{cases} \overline{c}_h^1, & \omega = 0 \\ \overline{c}_h^2, & \omega > 0, \quad \textit{all } h. \end{cases}\end{aligned} \tag{3.5}$$

Returning now to consideration of the original economy, we observe that if units of goods are converted into per-stock-yield units, that is, if, in each state ω, one unit of good g becomes $1/\delta^g(\omega)$ units of good g, then the yield matrix for stocks $\Delta_s(\omega)$ becomes simply the identity matrix. Furthermore, with log-linear utility functions, each household's utility in the old and the new units is identical up to an additive constant. This leads immediately to a characterization of FE in such an economy, which we can state succinctly in terms of the "trees and logs" of the paper's title.

Corollary to Proposition 1 (PFE with Trees and Logs). *The characterization of PFE in Proposition 1 applies to an economy with trees and logs after conversion to per-tree-yield units of goods.*

An economy in which stocks yield the same amount of goods in each state is itself not really very interesting. On the other hand, the *trees and logs model* (TL-Model) is intrinsically interesting and − as it turns out − much can be inferred about the finer structure of FE in this model. For this reason we now focus exclusively on the TL-Model, assuming conversion to per-tree-yield units (so that hereafter, $\delta^{\widetilde{g}}(\omega) > 0$, all $\widetilde{g}$, all ω, while $\Delta_s(\omega) = I$, all ω). At the same time we will also occasionally concentrate on the leading example. We must emphasize, however, that the Corollary to Proposition 1 is valid for arbitrary dimensionality − including the general case of intrinsically incomplete markets, as well as the special case commonly considered in the Finance literature, where there is a single good.

Before turning to questions of uniqueness and, say, *exclusivity* − that is, whether there are *other* (or "ordinary") *financial equilibria* (OFE) in the TL-Model[6] − it is quite instructive to highlight the peculiarity of the PFE. We accomplish this by contrasting the results reported in Proposition 1 with well-known properties of the RA-Model.

3.2.1. The LT-Model v. the RA-Model

The "well-known" properties asserted here can be found − or easily inferred following the lead of related results in the RA-Model literature.[7] We contrast these to the results reported in Proposition 1 applied to the TL-Model. For this purpose, when presenting a result which is (within a well-specified conventional context) true without any qualification we will use the term "general" or "generally". Otherwise, when a result is true generically (on some open, full measure set of parameters: in the RA-Model, endowments of commodities, given yields), we will use the term "typical" or "typically" (in contrast to "exceptional" or "exceptionally"). Of course, a general result is also typical, but not vice versa. We refer to a typical result which is not also general as "only typical." We also use self-explanatory tables to describe the RA-Model literature. Bear in mind that, looking ahead to subsection 3.5 below (where we establish exclusivity, that OFE can never occur), it is accurate to simply identify PFE with FE in the TL-Model.

[6] We should mention explicitly, that for the economy of Proposition 1, it is fairly straightforward to establish exclusivity, by slightly modifying Mas-Colell's argument, which builds on Cass-Shell's argument that sunspots can't matter with complete markets. We are indebted to Paolo Siconolfi for bringing this to our attention. However, such an argument relies on a construction which cannot be applied in the case of hybrid trees unless $F = G$ and rank $[y_f^g,$ all $f,$ all $g] = F$, and also in the case of various portfolio constraints, one instance of which is explicitly analyzed later on. So we have chosen instead to provide an alternative proof which can be so applied without such strong restrictions (and which relies heavily on the specification of log-linear utility). Our particular argument can also be adapted for other purposes, such as applied work relying explicitly on the structure of the equations characterizing equilibria, though the full extent to which this yields interesting results remains to be seen.

[7] In fact, many of the counterexamples are so obvious, or so easily constructed based on other results in the financial equilibrium literature that they are hard to find explicit cites for. We will refer to such "well-known" results as "folklore".

1. **Existence**

Existence of FE

FM are / Existence is	typical	only typical
Potentially Complete	Magill-Shafer	Hart [13]
Intrinsically Incomplete	Duffie-Shafer	Cass [6]

In the TL-Model, on the other hand, the operative condition in Proposition 1 – $\overline{e} \in \bar{E}_\Delta$ – characterizes the very large set of initial portfolios for which there is generally a PFE (depending, of course, on the other parameters of the model, in particular, $\delta^{\widetilde{g}}(\omega)$, all $\widetilde{g}$, all ω). It is important, and we stress the point, that this condition clearly encompasses much more than just the initial portfolios for which $\overline{e} >> 0$ (see subsection 3.4 below).

2. **Optimality**

By virtue of Arrow's Equivalency Theorem [2], for the RA-Model, when financial markets are potentially complete, Pareto optimality is closely related to the rank of the matrix of asset yields in value terms, or *payoffs*.[8] So we tabulate both optimality and rank properties for this model.

Optimality of FE

FM are / Pareto Optimality is	typical	only typical	exceptional
Potentially Complete	Magill-Shafer	folklore	no
Intrinsically Incomplete	no	no	Geanakoplos et al

Matrix of Asset Payoffs

FM are / Full Rank is	typical	only typical
Potentially Complete	Magill-Shafer	folklore
Intrinsically Incomplete	Duffie-Shafer	folklore

The TL-Model obviously turns all this on its head: *It is a general result in the TL-Model that the matrix of asset payoffs never has full rank, and yet allocation is always Pareto optimal*! FE

[8] For example, in the TL-Model, this matrix is

$$[p(\omega)[\Delta_b(\omega)\Delta_s(\omega)], \omega > 0].$$

in this model are very, very peculiar, indeed.

3. **Trade in Assets**

Using the fact that, typically, in the RA-Model the matrix of asset payoffs has full rank, it is a routine application of the Transversality Theorem to show that, again typically, all assets must be traded by all households. In the TL-Model, contrarily, financial markets are quite inactive. In the first place, households transact on the bond market only to the extent that they completely liquidate their initial positions. In a model with many periods, that is, where $T > 1$, this means that, beyond today, bond markets are completely inactive.[9] In the second place − the point of Proposition 1 − only a single stock market need be active, though, obviously they all can be. So, in this respect as well, PFE are also very, very peculiar!

4. **The Explanation**

Why such striking disparity between the two models? The answer is both very simple and obvious. The TL-Model is an extraordinarily atypical specification of the RA-Model, for two basic reasons: *First, after conversion to per-tree-yield units, tree yields, and hence total resources are identically one in each state of the world. Second, more importantly, initial endowments must both (i) lie in the span of the matrix of asset yields, and (ii) add up to the tree yields in each state of the world.* While the first property is specific to a trees and logs economy, the second one applies more generally, to any LT-Model. Any model in which endowments are specified purely in terms of portfolio shares and are marketed (spanned by the assets), is very special. It may display properties which are true only on a measure zero set of commodity endowments, given asset yields, in the RA-Model, and hence produce implications which are not robust. In particular, when there are potentially complete financial markets, it must be the case that if households own (independent) initial endowments, in addition to initial portfolios, then all the anomalies revealed above (typically) simply disappear.[10]

We now turn to consideration of another very important implication of the fact that degeneracy and incompleteness of financial markets are part and parcel of the PFE.

[9] By the way (and this should really go without saying!) all the results concerning the discrete date-event version of our model are easily generalized to many periods − provided all assets can be retraded. "Many periods" and "asset retrade" (what is labelled "dynamically ..." in Finance) are of course inherent in the continuous date-event version of the model; see section 4 below.

[10] In the leading example, for instance, this will be the case if and only if, for some $\omega', \omega'' > 0$, $\delta^1(\omega') \neq \delta^1(\omega'')$; see Magill and Shafer, pp. 174-5.

3.3. Portfolio Constraints

Financial markets with portfolio constraints have recently become a major area of research in asset pricing theory (see Karatzas and Shreve [15], Sundaresan [22] and references contained therein). The main bulk of this analysis is undertaken in the context of a single-good economy. Rather surprisingly, however, very little is known about the robustness of various implications within a multiple-good setting.

Our objective here is to illustrate the interaction between the spot goods markets and portfolio constraints, and to see to what extent the possibility of trade in the real markets can alleviate frictions in the financial markets. Toward that end, we present a straightforward implication of the arguments in the proof of Proposition 1.

Proposition 2 (Portfolio Constraints). *Consider a class of portfolio constraints under which it is feasible for the households in the economy to liquidate their initial bond holdings in period 0 and invest the proceeds (net of $p(0)c_h(0)$) in some (fixed) portfolio of the stocks. Then in this constrained economy, as long as it is feasible for the households to jointly hold one share of each stock, the relevant PFE – in particular, including those for which initial portfolios also satisfy the constraints, and saving takes place in terms of the (fixed) portfolio of stocks – still obtain.*

In particular, Proposition 2 encompasses the case of restricted participation in the stock market.

Corollary to Proposition 2 (Restricted Participation). *Suppose that* $b_h^{t\widetilde{g}} \in \mathbb{R}$, all $\widetilde{g}$, and $s_h^{tg} \in \mathbb{R}$, *some* g, $t = 0$, 1, all h. *Then, for arbitrary constraints on the remaining stocks, as long as market clearing in those stocks is feasible, the relevant PFE still obtain.*

This result is in striking contrast to the implications of a single-good model with multiple stocks. Portfolio constraints in the TL-economy can be fully circumvented by households trading in the spot goods markets (nonexistent in a single-good model). In a single-good world, the only way of achieving a desired state-contingent allocation is through trade in financial assets. If such trades are constrained, a household would typically suffer from impaired risk-sharing opportunities. In a multi-good world, a household can make both financial market transactions and spot market transactions. If trades in a specific financial asset(s), say shares of the "wheat tree," are constrained, the household would still achieve its (unconstrained) optimal state-contingent wheat consumption by buying additional shares in the "oil tree," receiving dividends in terms of oil, and then selling oil and buying wheat on the spot exchange market. Spot markets thus offer an additional channel through which a desired state-contingent allocation can be attained.

3.4. Uniqueness

In the TL-Model the question of uniqueness of PFE for given initial portfolios is equivalent to the question of uniqueness of CE for the corresponding initial endowments (3.1). This question has a very straightforward answer.

It is a routine exercise given in the graduate microeconomic theory sequence to show the following: in the standard 2×2 model of pure distribution with log-linear utility, Walrasian equilibrium is unique. This property stems from the fact that, in this example, the prices which support allocations in the Pareto set define lines which are either parallel − in the borderline case of identical log-linear utility − or intersect outside the Edgeworth-Bowley box. In other words, the only initial endowments for which there are multiple equilibria must lie outside the households' consumption sets − and this violates the spirit of the model.

In the certainty model equivalent of the TL-Model, however, there is absolutely no reason, given the opportunities of both borrowing and short-selling, that initial endowments must lie in the households' consumption sets. This yields an interesting result for the leading example.

Proposition 3 (Uniqueness of PFE). *For the leading example, the CE, and hence the PFE is unique*

- *in the borderline case where $\alpha_1^t = \alpha_2^t$, $t = 0, 1$ and $\beta_1 = \beta_2$, for all initial endowments $\bar{e} \in \bar{E}_\Delta$, but otherwise*

- *in the general case, for all initial endowments except possibly those which lie on a line segment, say, $\bar{e} \in \bar{L}_\Delta \subset \bar{E}_\Delta$.*

Also, for $\bar{e} \in \bar{L}_\Delta$, every Pareto optimal allocation is supported as a PFE.

In other words, either the PFE is unique, or there are PFE corresponding to each allocation in the Pareto set (on a relatively small subset of possible initial portfolios, to be sure!).

The intuition behind this result is presented in Figure 1 for the case in which $G = \Omega = 1, \widetilde{G} = 1$, and $H = 2$; the redundant bond is required so that $\bar{e}_1'$ is consistent with portfolio choice (otherwise, for an initial portfolio consisting of just one stock, it must be the case that $\bar{e}_1^0 = \bar{e}_1^1 = s_1^0 > 0$, and the PFE is unique). Note also that in this example, since $G = 1$, $\alpha_h = 1$, $h = 1, 2$.

3.5. Exclusivity

When one first encounters the pervasiveness of PFE − mainly because these financial equilibria are so strange − an immediate, natural reaction is to ask "Just how important is this peculiar phenomenon, anyway?", or more objectively, "Are there other FE which have significant presence as well?" In this subsection we establish that the answer is a blunt and clear "No!"

Proposition 4 (Exclusivity of PFE). *The only FE are PFE.*

Our particular method of proof for this result (which fully exploits the trees and logs structure) admits an immediate corollary concerning restricted participation.[11]

Corollary to Proposition 4 (Restricted Participation I: Some households are barred from transacting in some stocks). *Suppose that, for* $h < H$, *there is* $G_h \subset \{2, 3, \ldots, G\}$, *such that, for* $g \in G_h$, *household* h *faces the constraint* $s_h^{tg} = 0,\ t = 0, 1$, *while household* H *is unconstrained. Then the only FE are PFE.*

However, matters may become quite complicated with more general portfolio constraints, as the following result indicates.

Cautionary to Proposition 4 (Restricted Participation II: Some households face general constraints on transacting in some stocks). *For the leading example, consider the possibility that Ms. 1 faces a constraint of the form* $\phi(s_1^{12}) \geq 0$, *where* $\phi : \mathbb{R} \to \mathbb{R}$ *is differentiable and strictly quasi-concave. Then*

- *if* $\phi(s_1^{12}) = 0 \Rightarrow s_1^{12} \in [0,\, 1]$, *the only FE are PFE, otherwise*
- *for some economy (specified, in part, by* ϕ*) there are some OFE as well as PFE.*

3.6. The TL-Model vis-à-vis the sunspot model

For one familiar with the literature on the sunspot model (as one of us, anyway, surely is!), the parallel between PFE and nonsunspot equilibria is inescapable. Both types of equilibrium exhibit stationarity in the precise sense that they are equivalent to CE. Moreover, both are, in their respective economic environments, the only equilibria for which goods allocations are Pareto optimal. This suggests another possible interesting parallel, that between OFE and sunspot equilibria. In the original sunspot model, there is typically a distinct sunspot equilibrium in the leading example with an incomplete financial market (this can be inferred from the analysis in Cass [5] together with Balasko and Cass [3], pp. 145–9; when assets are specified in terms of payoffs there is typically even a continuum of distinct sunspot equilibria). While markets are incomplete in the TL-Model, however, as evidenced by our exclusivity results (Proposition 4 and its corollary)

[11] By generalizing to hybrid trees we also get an immediate corollary concerning the situation where households can freely transact in bonds, but can only trade stocks in terms of a fixed portfolio with equal shares.

an OFE does not emerge just in the model with unrestricted participation, but also in the model with restricted participation. This is related to results of Mas-Colell, and Antinolfi-Keister who show that in the RA-Model (as opposed to the nominal assets model of Cass) the presence of a sufficient number of particular redundant assets, essentially futures or options contracts, rules out sunspot equilibria. Stocks in the TL-Model play a similar role: in the transformed (per-tree-yield units) economy, stocks are redundant securities paying one unit of a corresponding commodity, but equal payoffs, and there are as many stocks as there are commodities.

At the outset, one would not expect (we certainly did not!) there to be a parallel between trees and logs and sunspot models. But once this parallel has been established, it seems that there may be some synergies between the sunspot literature and pertinent models in Finance. By the latter we mean models in which our simple transformation of units goes through, and hence can be mapped into the sunspot model. The insight from the sunspot literature for Finance is identifying economic environments in which OFE's are likely to emerge (incomplete markets, constraints, differences of opinion). Conversely, some findings in Finance may be useful for identifying financial market structures that guarantee sunspot immunity.

4. Extension in Continuous Time

We now consider a continuous-time variation on our leading example. Results presented this section are parallel to those of the discrete date-event version. For that reason this section is going to be intendedly terse. The economy has a finite-horizon, $[0, T]$. Uncertainty is represented by a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathcal{P})$, on which is defined a two-dimensional Brownian motion $w(t) = (w_1(t),\, w_2(t))$, $t \in [0, T]$. For simplicity, the Brownian motions w_1 and w_2 are taken to be independent of each other. All stochastic processes are assumed adapted to $\{\mathcal{F}_t; t \in [0, T]\}$, the augmented filtration generated by w. All stated equalities involving random variables hold $\mathcal{P}$-almost surely. Note that this continuous-time specification of the state space is in parallel to that of the discrete-time leading example: there, a random variable was represented by three possible future realizations corresponding to the three branches of the date-event tree; in the continuous-time version, each process would be parameterized by a triple $(\mu,\, \sigma_1,\, \sigma_2)$ $-$ the drift and two diffusion processes (one for each Brownian motion).

The risky stocks pay out the strictly positive dividend stream at rate δ^g, in good g, following an Itô process

$$d\delta^g(t) = \delta^g(t)[\mu^g_\delta(t)\, dt + \sigma^g_\delta(t)\, dw(t)], \qquad g = 1,\, 2\,,$$

where μ^g_δ and $\sigma^g_\delta \equiv (\sigma_{\delta 1},\, \sigma_{\delta 2})^\top$ are arbitrary stochastic processes. The relative price of good 2 (in terms of good 1, the numeraire), p, will be shown in equilibrium to have dynamics

$$dp(t) = p(t)[\mu_p(t)dt + \sigma_p(t)dw(t)],$$

where μ_p and $\sigma_p \equiv (\sigma_{p1}, \sigma_{p2})^\top$ are (endogenous) drift and volatility processes.

Analogously to those in the discrete-time leading example, investment opportunities are represented by three securities: an instantaneously riskless bond, $q_b^{\widetilde{g}}$, $\widetilde{g} = 1$, in zero net supply, and two risky stocks, q_s^g, $g = 1, 2$, in unit supply. The bond price and stock prices are posited to follow

$$\begin{aligned} dq_b^1(t) &= q_b^1(t) r^1(t) dt \,, \qquad q_b^1(0) = 1 \,, \\ dq_s^1(t) + \delta^1(t)dt &= q_s^1(t)[\mu_s^1(t)dt + \sigma_s^1(t)dw(t)] \,, \\ dq_s^2(t) + p(t)\,\delta^2(t)dt &= q_s^2(t)[\mu_s^2(t)dt + \sigma_s^2(t)dw(t)] \,, \end{aligned}$$

where the interest rate r^1, the drift (μ_s^1, μ_s^2), and the diffusion (σ_s^1, σ_s^2) processes are to be determined in equilibrium. Under this specification of the investment opportunities, financial markets are potentially dynamically complete.

The two households maximize their expected lifetime log-linear utility

$$u_h(c_h) = E\left[\int_0^T e^{-\rho_h t} v_h(c_h(t))dt\right] \qquad h = 1, 2, \tag{4.6}$$

where $v_h(c_h(t)) = \alpha_h^1 \log c_h^1(t) + \alpha_h^2 \log c_h^2(t)$ and $\rho_h > 0$, subject to the dynamic budget constraint

$$\begin{aligned} dW_h(t) =& W_h(t)r^1(t)dt - (c_h^1(t) + p(t)c_h^2(t))dt + s_h(t)^\top I_s(t) \begin{pmatrix} \mu_s^1(t) - r^1(t) \\ \mu_s^2(t) - r^1(t) \end{pmatrix} dt \\ &+ s_h(t)^\top I_s(t) \begin{pmatrix} \sigma_s^1(t) \\ \sigma_s^2(t) \end{pmatrix} dw(t), \qquad W_h(0) = b_h(0) + s_h(0)^\top q_s(0), \end{aligned} \tag{4.7}$$

where $W_h(t) \equiv b_h(t)q_b^1(t) + s_h(t)^\top q_s(t)$ is the wealth process of household h and I_s is a 2×2 diagonal matrix with diagonal elements q_s^1 and q_s^2.[12] The volatility matrix in the representation of the investment opportunity set,

$$\Sigma(t) \equiv \begin{pmatrix} \sigma_s^1(t) \\ \sigma_s^2(t) \end{pmatrix},$$

is not necessarily invertible. If it is, then the two risky stocks are linearly independent and markets are complete; otherwise the two stocks represent the same investment opportunity and markets are intrinsically incomplete. Appealing to the martingale methodology, standard in asset pricing,

[12]For simplicity of exposition, we assume that α_h^1 and α_h^2 are constants. It is straightforward to extend the our analysis to incorporate time-dependent (deterministic) coefficients.

we deflate household h's wealth by the state-price deflator ξ_h in order to convert its dynamic budget constraint into a static Arrow-Debreu-like budget constraint of the form

$$E\left[\int_0^T \xi_h(t)[c_h^1(t)+p(t)c_h^2(t)]dt\right] = E\left[\int_0^T \xi_h(t)[e_h^1(t)+p(t)e_h^2(t)]dt\right]. \tag{4.8}$$

The process $\xi_h(t,\omega)$, $t \in [0,T]$, $\omega \in \Omega$, a household's state-price deflator, is the continuous-time analog of the quantity $\lambda_h(\omega)/\pi(\omega)$, $t \in \{0,1\}$, the household's Lagrange multiplier divided by probability, that we employed in our earlier analysis. The state-price deflator ξ_h is formally defined as a strictly positive process such that (under the standard regularity conditions) deflated gains processes, $\xi_h q_b^1$ and $\xi_h q_s^g + \int \xi_h \delta^g dt$, $g = 1,\, 2$, are martingales. The ratio $\xi_h(t,\, \omega)/\xi_h(0)$ then can be interpreted as the Arrow-Debreu-like state price of a unit of the numeraire good in state ω at time t, per unit probability $\mathcal{P}$, faced by household h. Analogously to the discrete date-event version, Pareto optimality of an allocation obtains if $\xi_h(t,\, \omega)/\xi_h(0)$ are household-independent. In this case, we use notation $\xi(t) \equiv \xi_1(t)$, and hence $\xi_2(t) = \xi_2(0)/\xi_1(0)\, \xi(t)$. The endowments e_h^g, $g = 1,\, 2$, are, as before, the dividend streams from the initial shareholdings.

A *financial equilibrium* is defined as a collection of prices $(\xi_h,\, p,\, q,\, h = 1,\, 2)$ and associated optimal policies $(c_h,\, b_h,\, s_h,\, h = 1,\, 2)$ solving the optimization problem in (4.6)–(4.7) such that the goods, bond and stock markets clear, i.e., $\forall\, t \in [0,T]$, for $g = 1,\, 2$:

$$\begin{aligned} \sum_h c_h^g(t) &= \delta^g(t), \\ \sum_h b_h^g(t) &= 0, \\ \sum_h s_h^g(t) &= 1. \end{aligned} \tag{4.9}$$

For facilitate comparison with our discrete-time characterization of equilibria (in appendix A), we introduce a representative agent with utility

$$U(c;\eta) = E\left[\int_0^T v(c(t),\eta)dt\right],$$

where

$$v(c;\, \eta) = \max_{c_1+c_2=c} \eta_1 e^{-\rho_1 t} v_1(c_1) + \eta_2 e^{-\rho_2 t} v_2(c_2),$$

and $\eta_h > 0,\ h = 1,2$, may be stochastic. If in an equilibrium, η_1 and η_2 are constants, then the allocation is Pareto optimal, otherwise it is not. Since the weights for the representative agent are unique up to a multiplicative constant, we adopt the normalization $\eta_1 = \eta$, $\eta_2 = 1-\eta$, $\eta \in (0,1)$.

We are now ready to characterize equilibria in the economy. In the interest of space, we do not provide conditions for existence; they can be obtained analogously to those in the discrete date-event version.

Proposition 5 (Characterization of PFE). *If a financial equilibrium exists in the leading example, it is a PFE. Equilibrium state-price deflators are the same up to a multiplicative constant ($\xi_1(t) = \xi(t)$, $\xi_2(t) = \xi_2(0)/\xi_1(0)\,\xi(t)$), with*

$$\xi(t) = \frac{\alpha_1^1 \eta e^{-\rho_1 t} + \alpha_2^1 (1-\eta) e^{-\rho_2 t}}{\delta^1(t)} \,. \tag{4.10}$$

The price of good 2 and the equilibrium consumption allocations are

$$p(t) = \frac{\alpha_1^2 \eta e^{-\rho_1 t} + \alpha_2^2 (1-\eta) e^{-\rho_2 t}}{\alpha_1^1 \eta e^{-\rho_1 t} + \alpha_2^1 (1-\eta) e^{-\rho_2 t}} \, \frac{\delta^1(t)}{\delta^2(t)} \,, \tag{4.11}$$

$$c_1^g(t) = \frac{\alpha_1^g \eta e^{-\rho_1 t} \delta^g(t)}{\alpha_1^g \eta e^{-\rho_1 t} + \alpha_2^g (1-\eta) e^{-\rho_2 t}} \,, \qquad g = 1, 2, \tag{4.12}$$

$$c_2^g(t) = \frac{\alpha_2^g (1-\eta) e^{-\rho_2 t} \delta^g(t)}{\alpha_1^g \eta e^{-\rho_1 t} + \alpha_2^g (1-\eta) e^{-\rho_2 t}} \,, \qquad g = 1, 2. \tag{4.13}$$

The constant weight η is determined from either household's static budget constraint with the optimal consumption allocations (4.12)–(4.13) substituted in, i.e.,

$$E\left[\int_0^T \xi(t)[c_1^1(t) + p(t)c_1^2(t)]dt\right] = E\left[\int_0^T \xi(t)[e_1^1(t) + p(t)e_1^2(t)]dt\right] \,. \tag{4.14}$$

Furthermore, the prices of stocks expressed in terms of goods they are claims to, and the interest rate are given by

$$q_s^1(t) = \frac{\rho_1 \alpha_1^1 \eta (e^{-\rho_1 t} - e^{-\rho_1 T}) + \rho_2 \alpha_2^1 (1-\eta)(e^{-\rho_2 t} - e^{-\rho_2 T})}{\rho_1 \rho_2 (\alpha_1^1 \eta e^{-\rho_1 t} + \alpha_2^1 (1-\eta) e^{-\rho_2 t})} \, \delta^1(t), \tag{4.15}$$

$$q_s^2(t)/p(t) = \frac{\rho_1 \alpha_1^2 \eta (e^{-\rho_1 t} - e^{-\rho_1 T}) + \rho_2 \alpha_2^2 (1-\eta)(e^{-\rho_2 t} - e^{-\rho_2 T})}{\rho_1 \rho_2 (\alpha_1^2 \eta e^{-\rho_1 t} + \alpha_2^2 (1-\eta) e^{-\rho_2 t})} \, \delta^2(t), \tag{4.16}$$

$$r^1(t) = \mu_\delta^1(t) - \frac{\alpha_1^1 \, \eta \, \rho_1 e^{-\rho_1 t} + \alpha_2^1 (1-\eta) \rho_2 e^{-\rho_2 t}}{\alpha_1^1 \, \eta e^{-\rho_1 t} + \alpha_2^1 (1-\eta) e^{-\rho_2 t}} - |\sigma_\delta^1(t)|^2$$

Conversely, if there exist ξ, p and η satisfying (4.10)–(4.11) and (4.14), then the associated optimal policies clear all markets.

It is easy to see that the equilibrium is a PFE. Analogously to the discrete date-event version, the relative price of the two goods is proportional to the ratio of the dividends. It is easy to see then that in equilibrium the volatility matrix in the representation of the investment opportunity set, $\Sigma(t)$, is degenerate, or, equivalently, the two stocks represent the same investment opportunity. The mapping into the certainty model is also apparent from the characterization in Proposition 5: in per-tree-yield units, optimal consumption and prices are deterministic functions of time. Furthermore, since the weight η is constant in equilibrium, the allocation is Pareto optimal.

We now turn to the nonuniqueness of peculiar equilibria.

Proposition 6 (Nonuniqueness). *Consider the set of initial endowments of household 1, e_1, satisfying:*

$$E\left[\int_0^T \left(\alpha_1^1 e^{-\rho_1 t}\,\frac{e_1^1(t)}{\delta^1(t)} + \alpha_1^2 e^{-\rho_1 t}\,\frac{e_1^2(t)}{\delta^2(t)}\right) dt\right] = \frac{1-e^{-\rho_1 T}}{\rho_1}, \tag{4.17}$$

$$E\left[\int_0^T \left(\alpha_1^1 e^{-\rho_2 t}\,\frac{e_1^1(t)}{\delta^1(t)} + \alpha_1^2 e^{-\rho_2 t}\,\frac{e_1^2(t)}{\delta^2(t)}\right) dt\right] = 0. \tag{4.18}$$

On this set of endowments, there is a continuum of PFE with the characterization given by (4.10)–(4.11) and (4.12)–(4.13) for all $\eta \in (0,1)$.

Proposition 6 is an exact analogue of Proposition 3 in the discrete date-event version. Note that for condition (4.18) to be satisfied it is necessary that household 1 be endowed with a short position in one of the securities.

The continuous-time formulation offers additional tractability over the discrete-time version in that one can parameterize the processes for state prices and stochastic weighting in the economy, which proves to be very useful for getting explicit formulas in economies with frictions. Comprehensive investigation of the effects of portfolio constraints in the TL-economy is beyond the scope of this paper. Here, we just concentrate on a specific constraint: restricted participation in one of the risky securities.

Proposition 7 (Restricted Participation). *Consider the economy where household 1 is restricted from investing in one of the risky stocks, e.g., stock 1, but can take an unrestricted position in the bond and stock 2. Household 2 is unconstrained. Equilibrium in this constrained economy coincides with that of the unconstrained, with the characterization given in Proposition 5.*

5. Final Remarks

Our thorough examination of the Lucas tree model when extended to include multiple goods uncovers a variety of puzzling characteristics. In particular, we show that under the maintained assumption of log-linear utility, the only equilibria in the model are peculiar financial equilibria, in which all the stocks represent the same investment opportunity − and yet, nonetheless, allocation is Pareto optimal. This result is at odds with well-established principles of financial equilibrium theory in Economics. The reason for this striking disparity is that endowments in the Lucas tree model are specified in share portfolios, rather than commodity bundles, and hence comprise a measure-zero subset of commodity endowments, given asset yields. However, the fact that the specification of the Lucas tree model is non-generic does not render it uninteresting; all we are saying is that it is clearly different from the setting used in Economics. It is a routine practice to send first-year Finance Ph.D. students to learn fundamentals of (financial) equilibrium theory from the Economics department. We thus feel that is important to point out that one simply cannot cite results from Economics in Finance applications. Our paper only establishes why workhorse models in Economics and Finance may be incompatible, but it does not provide a comprehensive theory of existence, uniqueness, financial market structure, and Pareto optimality in the Lucas tree setting. We are aware of other instances where having share endowments leads to puzzling (from an equilibrium theorist's viewpoint) implications: for example, for a class of preferences, markets are known to be "effectively complete" in the Lucas tree model – Pareto optimality obtains generically in incomplete markets. We feel that more work is needed to generalize our trees and logs and other examples to develop a unified foundational theory of financial equilibria in Finance, parallel to that existing in Economics.

Fairly complete analysis of the effects of portfolio constraints in the general trees and logs economy is a separate issue. In this paper, we merely demonstrate that for a certain large class of portfolio constraints – in contrast to a single-good model – their impact on the economy can be fully alleviated by the possibility of trade in the spot goods markets. This result however must be heavily qualified: even within this class of portfolio constraints, there may be additional financial equilibria in which allocation is not Pareto optimal. Another important class of constraints to consider is the one which leads to allocation which is not Pareto optimal (and therefore financial equilibria which are not peculiar). In this situation constraints on transactions could only, at best, be partially alleviated by trading in the spot goods markets, and it would be of interest to quantify the extent to which trade in goods can circumvent restrictions on trade in assets. Conversely, we should be able to use our framework to investigate the interaction between restrictions on transactions on goods markets (e.g., one cannot transact an unlimited quantity of a particular good, or certain goods have to be purchased concurrently) and transactions on asset markets.

In models of asset pricing, there is always a trade-off between generality and tractability. To keep our model tractable, we have maintained the assumption that all stocks pay out in different, stock-specific goods, as opposed to paying out in (potentially overlapping) bundles of commodities, making the model closer to a real assets model. Our specification generalizes the standard single-good Finance model to multiple goods and hence multiple risky stocks, and allows one to fully characterize equilibrium stock and goods prices and allocations in closed-form. We feel that such tractability might make the model useful for applications and empirical work, but only after the model has been modified so as to rule out peculiar equilibria, and the ensuing perfect correlation of stock prices. Extending the investment opportunity set to include stocks paying out in arbitrary overlapping bundles of commodities – the special case of which are our hybrid trees – would be one way to achieve this, but at the expense of being able to explicitly characterize all equilibrium quantities. Recently Pavlova and Rigobon [20] proposed an alternative dimension along which our trees and logs economy can be generalized, but its tractability still maintained: they allow for state-dependent utility weights on different goods, interpreted as “demand shocks.” This, and other potential generalizations may prove fruitful in constructing simple models suitable for analyzing the dynamics of financial and real markets and their interactions.

Appendix A: Proofs of the Main Results

We begin by writing down the extended system of equations which provides the whole basis for our formal analysis. This consists of the Lagrange conditions characterizing an optimal solution to each household's optimization problem (H) together with the spot goods and asset market clearing conditions (M). For the time being we will continue to assume that spot goods prices are normalized at each spot in terms of good 1 as the numeraire $p^1(\omega) = 1$, all ω, and all pertinent quantities are expressed in per tree-yield units. Also, to avoid unnecessary clutter, where it is obvious from context, "all ω," "$\omega > 0$," or "all h" are understood as given.

A.1 The Extended System of Equations

First-order conditions (FOC's)

$$\sum_{\omega>0} \pi(\omega) D_{c_h(0)} v_h(c_h(0), c_h(\omega)) - \lambda_h(0)p(0) = 0 \tag{A.1}$$

and

$$\pi(\omega) D_{c_h(\omega)} v_h(c_h(0), c_h(\omega)) - \lambda_h(\omega)p(\omega) = 0; \tag{A.2}$$

No-arbitrage conditions (NAC's)

$$\lambda_h(0)q_b - \sum_{\omega>0} \lambda_h(\omega)p(\omega)\Delta_b(\omega) = 0 \tag{A.3}$$

and

$$\lambda_h(0)q_s - \sum_{\omega>0} \lambda_h(\omega)p(\omega) = 0; \tag{A.4}$$

Budget constraints (BC's)

$$p(0)(c_h(0) - e_h(0)) + qz_h = 0 \tag{A.5}$$

and

$$p(\omega)(c_h(\omega) - e_h(\omega)) - p(\omega)\Delta(\omega)z_h = 0; \tag{A.6}$$

Market clearing conditions (MCC's)

$$\sum_h c_h(\omega) - \mathbf{1} = 0 \tag{A.7}$$

and

$$\sum_h z_h = 0. \tag{A.8}$$

Also bear in mind the definition of initial endowments,

$$e_h(\omega) = [\Delta_b(\omega) I](b_h^0, s_h^0). \tag{A.9}$$

Remarks:

1. By virtue of the NAC's (A.3)-(A.4), we can replace the first BC (A.5) by a personalized Walrasian-like BC
$$\sum_{\omega} \lambda_h(\omega)p(\omega)(c_h(\omega) - e_h(\omega)) = 0. \tag{A.5}$$
This fact will prove to be very useful in the course of most of our argument.

2. By virtue of the BC's (A.5)-(A.6) and the MCC's (A.7)-(A.8), $\Omega + 1$ of these equations are redundant (the analogue of Walras' law), for example, Mr. H's BC's. We will explicitly drop these particular redundant equations later on.

3. Taking account of the preceding remark together with the spot goods price normalizations, it follows that there are (at most)
$$\begin{aligned} J &= HG(\Omega+1) + H(\widetilde{G}+G) + (H-1)(\Omega+1) + G(\Omega+1) + (\widetilde{G}+G) \\ &= HG(\Omega+1) + H(\widetilde{G}+G) + H(\Omega+1) + (G-1)(\Omega+1) + (\widetilde{G}+G) \end{aligned}$$
independent equations in the J independent variables
$$c_h, z_h, \lambda_h, (p^g(\omega), g > 1, \text{ all } \omega), \text{ and } q.$$
Of course, at a solution corresponding to a PFE, and therefore a Pareto optimal allocation, the NAC's (A.3)-(A.4) are not independent. This means that, with potentially complete financial markets, all of the equations (A.1)-(A.8) can never be independent (since otherwise one would get an immediate contradiction based on Arrow's Equivalency Theorem), and this tends to complicate their analysis.

A.2 Proof of Proposition 1

(i) Suppose that (p, c, q, z) is a PFE. Then, by degeneracy, $p(\omega) = p(1)$, and by irrelevancy,
$$p(\omega)e_h(\omega) + p(\omega)\Delta(\omega)z_h = p(\omega)s_h^1,$$
so that (A.2) and (A.6) become simply
$$\pi(\omega)D_{c_h(\omega)}v_h(c_h(0), c_h(\omega)) - \lambda_h(\omega)p(1) = 0 \tag{A.10}$$
and
$$p(1)(c_h(\omega) - s_h^1) = 0. \tag{A.11}$$

From our textbook assumptions about v_h, it follows that (A.10), (A.11), and (A.7) describe an identical Walrasian equilibrium at each spot $\omega > 0$. Thus, from (A.10) it also follows that
$$c_h(\omega) = c_h(1) \text{ and } \lambda_h(\omega)/\pi(\omega) = \lambda_h(1)/\pi(1), \omega > 0,$$
and from optimality (or, equally well, the NAC (A.4)) that
$$\frac{\lambda_h(1)}{\pi(1)\lambda_h(0)} = \frac{\lambda_1(1)}{\pi(1)\lambda_1(0)}.$$

Hence, (A.5) becomes simply

$$p(0)(c_h(0) - [\Delta_b(0)I](b_h^0, s_h^0)) +$$

$$\frac{\lambda_1(1)}{\pi(1)\lambda_1(0)} p(1)(c_h(1) - \sum_{\omega>0} \pi(\omega)[\Delta_b(\omega)I](b_h^0, s_h^0)) = 0, \tag{A.12}$$

while (A1) and (A10) become simply

$$D_{c_h(0)} v_h(c_h(0), c_h(1)) - \lambda_h(0)p(0) = 0 \tag{A.13}$$

and

$$D_{c_h(1)} v_h(c_h(0), c_h(1)) - \lambda_h(0)\frac{\lambda_1(1)}{\pi(1)\lambda_1(0)} p(1) = 0. \tag{A.14}$$

Finally, making the identifications (3.3)-(3.4) together with $\overline{\lambda}_h = \lambda_h(0)$, we see that (A.12)-(A.14) characterize the optimal solution to $(\overline{H})$, and that these necessarily satisfy $(\overline{M})$, so that this half of the proof is complete.

(ii) Suppose that $\bar{e} \in \bar{E}_\Delta$, and that $(\bar{p}, \bar{c})$ is a CE. Then, given $\bar{p}$, $(\bar{c}_h, \bar{\lambda}_h)$ solves the analogues of the Lagrange conditions (A.12)-(A.14),

$$\bar{p}(\bar{c}_h - \bar{e}_h) = 0, \tag{A.12$'$}$$

$$D_{\bar{c}_h^0} v_h(\bar{c}_h^0, \bar{c}_h^1) - \bar{\lambda}_h \bar{p}^0 = 0, \tag{A.13$'$}$$

and

$$D_{\bar{c}_h^1} v_h(\bar{c}_h^0, \bar{c}_h^1) - \bar{\lambda}_h \bar{p}^1 = 0, \tag{A.14$'$}$$

with $\bar{e}_h$ satisfying (3.3) for some (b_h^0, s_h^0). Making the identifications (3.5) together with $\lambda_h(0) = \bar{\lambda}_h$,

$$\frac{\lambda_1(\omega)}{\pi(\omega)\lambda_1(0)} = \bar{p}^{11},$$

and, say,

$$s_h^1 = s_h^0 + (\Delta s_h^{01}, 0, \ldots, 0) \text{ such that } \bar{p}^1(c_h^1 - s_h^1) = 0,$$

one can then simply reverse the steps of the preceding argument. Since this procedure is obvious, we omit its details. ■

A.3 Reduction to The True Equations

From here on we will maintain the assumption of log-linear utility. This permits substantial simplification of the extended system of equations (A.1)-(A.8). We also drop Mr. H's BC's as being redundant.

With log-linearity, the FOC's (A.1)-(A.2) become, for all g,

$$\alpha_h^{0g}/c_h^g(0) - \lambda_h(0)p^g(0) = 0 \tag{A.15}$$

and

$$\pi(\omega)\beta_h\alpha_h^{1g}/c_h^g(\omega) - \lambda_h(\omega)p^g(\omega) = 0. \tag{A.16}$$

From (A.15) it follows that

$$\lambda_h(0)p(0)c_h(0) = 1 \tag{A.17}$$

and, together with (A.7) for $\omega = 0$, that

$$p(0) = \sum_h (1/\lambda_h(0))\alpha_h^0. \tag{A.18}$$

Similarly, from (A.16) it follows that, for $\omega > 0$,

$$\lambda_h(\omega)p(\omega)c_h(\omega) = \pi(\omega)\beta_h \tag{A.19}$$

and, together with (A.7), that

$$p(\omega) = \pi(\omega)\sum_h (\beta_h/\lambda_h(\omega))\alpha_h^1. \tag{A.20}$$

What this means − and this is the main advantage of assuming log-linear utility − is that, for all practical purposes, we can ignore the FOC's (A.15)-(A.16) as well as the MCC (A.7): the information these equations contain concerning the household's goods consumption can easily be recovered from the system of equations consisting of (A.3)-(A.6), (A.8), and the spot goods price equations (A.18) and (A.20) (SGP's).

It will be very convenient to record this fact formally, but only after first introducing two additional modifications, (i) substituting, in the appropriate places, for the Lagrange multipliers $\lambda_h(\omega)$ the so-called *stochastic weights*

$$\eta_h(\omega) = \beta_h/\lambda_h(\omega),$$

and (ii) substituting, in the the NAC's (A.3)-(A.4) for $h < H$, for the asset prices q defined by the NAC's (A.3)-(A.4) for $h = H$.

All this manipulation and consequent simplification then leaves us with what we only half-jokingly refer to as *The True Equations* (TTE).

Spot goods prices

$$p(0) = \sum_h \eta_h(0)(\alpha_h^0/\beta_h). \tag{A.21}$$

and

$$p(\omega) = \pi(\omega)\sum_h \eta_h(\omega)\alpha_h^1; \tag{A.22}$$

No arbitrage conditions (for $h < H$)

$$\sum_{\omega>0}(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega))p(\omega)\Delta_b(\omega) = 0 \tag{A.23}$$

and

$$\sum_{\omega>0}(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega))p(\omega) = 0; \tag{A.24}$$

Budget constraints (for $h < H$)

$$(1 + 1/\beta_h) - \sum_{\omega}(1/\eta_h(\omega))p(\omega)[\Delta_b(\omega)I](b_h^0, s_h^0) = 0 \tag{A.25}$$

and

$$\pi(\omega)\eta_h(\omega) - p(\omega)[\Delta_b(\omega)I](b_h^1, s_h^1) = 0; \tag{A.26}$$

Asset market clearing conditions

$$\sum_h (b_h^1, s_h^1) - (0, \mathbf{1}) = 0. \tag{A.27}$$

Finally, we will now find it much more useful to normalize prices according to the formulas

$$\sum_h \eta_h(\omega) = 1. \tag{A.28}$$

Remarks:

1. In deriving (A.25)-(A.26) we also used (A.9), (A.17), and (A.19).

2. The stochastic weights $\eta_h(\omega)$ owe their name to the fact that the FOC's (A.15)-(A.16) can be derived from the social welfare/social planner's problem of maximizing a fictitious representative agent's utility function of the form

$$\sum_h [\eta_h(0) \sum_g (\alpha_h^{0g}/\beta_h) \log c_h^g(0) + \sum_{\omega>0} \pi(\omega)\eta_h(\omega) \sum_g \alpha_h^{1g} \log c_h^g(\omega)]$$

   subject to feasibility of goods allocation (with associated multipliers p). Note that this fact implies that, for goods allocation to be Pareto optimal, it must be the case that

$$\eta_h(\omega) = \eta_h,$$

   which in turn implies that it must be the case that a FE is a PFE.

3. TTE preserve consistency of equations and variables. For this system of equations there are (at most)

$$\begin{aligned} K &= G(\Omega+1) + (H-1)(\tilde{G}+G) + (H-1)(\Omega+1) + (\tilde{G}+G) + (\Omega+1) \\ &= H(\tilde{G}+G) + H(\Omega+1) + G(\Omega+1) \end{aligned}$$

   independent equations in the K independent variables

$$(b_h^1, s_h^1), \eta_h, \text{ and } p.$$

A.4 Proof of Proposition 2 (and Corollary)

Obvious. ■

A.5 Proof of Proposition 3

For the leading example, in terms of just $\bar{e}_1$ (so that, by definition, $\bar{e}_2 = \mathbf{1} - \bar{e}_1$),

$$\{\bar{e}_1 \in \mathbb{R}^4 : 0 \ll \bar{e}_1 \ll \mathbf{1}\} \subset \bar{E} \subset \mathbb{R}^4,$$

that is, $\bar{E}$ is a full-dimensional subset of $\mathbb{R}^4$. On the other hand,

$$\begin{aligned}\bar{E}_\Delta \quad \subset \quad & \{\bar{e}_1 \in \bar{E} : \text{for some } (b_1^0, s_1^0), \bar{e}_1 = \\ & (\delta^1(0)b_1^0 + s_1^{01}, s_1^{02}, \sum_{\omega>0} \pi(\omega)\delta^1(\omega)b_1^0 + s_1^{01}, s_1^{02})\},\end{aligned}$$

that is, (given $\pi(\omega)$) generically in $\delta^1(\omega)$, all ω, $\bar{E}_\Delta$ is a full-dimensional subset of a 3-dimensional linear subspace in $\mathbb{R}^4$ (noting that, necessarily, $\bar{e}_1^{12} = \bar{e}_1^{02}$).

This said, in order to check for uniqueness of CE in terms of $\bar{e}_1$, we only need to consider solutions to Ms. 1's certainty BC (in terms of *just* her constant stochastic weight $0 < \eta_1 < 1$; see below), but given certainty endowments in the lower-dimensional subset $\bar{E}_\Delta$.

To formalize this problem, we begin by observing that the analogues of TTE in the certainty economy are identical to (A.21)-(A.28) when $\Omega = 1, G = H = 2$, and $\tilde{G} = 0$ (setting, say, $s_1^{12} = s_1^{02}$) after making appropriate changes in notation (replacing $p(0)$ with $\bar{p}^1, s_1^0$ with $\bar{e}_1^0$, and so on). Hence, after substituting from the SGP equations (A.21)-(A.22) into the BC (A.25) for $h = 1$, and also setting, for convenience, $0 < \eta_1^t = \eta < 1$ and $\eta_2^t = 1 - \eta_1^t = 1 - \eta$, $t = 0, 1$, finally we find that the question of *non*uniqueness of CE, and a fortiori, PPE boils down to this: when does the linear equation, for $\bar{e}_1 \in \bar{E}_\Delta$,

$$\eta(1 + 1/\beta_1) - [\eta(\alpha_1^0/\beta_1) + (1-\eta)(\alpha_2^0/\beta_2)]\bar{e}_1^0 - [\eta\alpha_1^1 + (1-\eta)\alpha_2^1]\bar{e}_1^1 = 0 \qquad \text{(A.29)}$$

admit every $0 < \eta < 1$ as a solution? But this will be the case if and only if the pair of equations

$$(\alpha_1^0/\beta_1 - \alpha_2^0/\beta_2)(\bar{e}_1^{01}, \bar{e}_1^{02}) + (\alpha_1^1 - \alpha_2^1)(\bar{e}_1^{11}, \bar{e}_1^{02}) - (1 + 1/\beta_1) = 0 \qquad \text{(A.30)}$$

and

$$(\alpha_2^0/\beta_2)(\bar{e}_1^{01}, \bar{e}_1^{02}) + \alpha_2^1(\bar{e}_1^{11}, \bar{e}_1^{02}) = 0 \qquad \text{(A.31)}$$

(together with the identity $\bar{e}_1^{12} = \bar{e}_1^{02}$) has a solution in $\bar{E}_\Delta$. Since (A.31) but not (A.30) is a homogeneous equation, this is possible only if

$$\text{rank}\begin{bmatrix} \alpha_1^{01}/\beta_1 - \alpha_2^{01}/\beta_2 & \alpha_1^{11} - \alpha_2^{11} & (\alpha_1^{02}/\beta_1 + \alpha_1^{12}) - (\alpha_2^{02}/\beta_2 + \alpha_2^{12}) \\ \alpha_2^{01}/\beta_2 & \alpha_2^{11} & \alpha_2^{02}/\beta_2 + \alpha_2^{12} \end{bmatrix} = 2,$$

that is, only if $\alpha_2^t \neq \alpha_1^t$, some t, or $\beta_2 \neq \beta_1$. The set of such solutions then defines the line segment $\bar{L}_\Delta \subset \bar{E}_\Delta$.[13] Note that, because the coefficients in (A.31) are all positive, any solution must have both positive and negative elements. ■

[13] Of course, there may be no solutions to (A.30)-(A.31) in $\bar{E}_\Delta$, as in the example depicted in subsection 3.4 when there is no redundant bond. However, for the leading example, it is easy to find parameter values for which there are solutions (using the analogues of TTE and the degrees of freedom afforded in choosing $\delta^1(\omega)$, all ω). It is also worth pointing out that (A.29) can also be exploited to give a precise description of $\bar{E}_\Delta$.

A.6 Exclusivity of PFE

A.6.1 Proof of Proposition 4

We want to show that the only solutions to TTE satisfy

$$\eta_h(\omega) = \eta_h(0), \quad \omega > 0, \text{ all } h. \tag{A.32}$$

Consider just the NAC's (A.23)-(A.24) together with the second period BC's (A.26) and the price normalization (A.28). Multiply (A.26) (after replacing h by h') by

$$(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega)),$$

sum over $\omega > 0$, and use (A.23)-(A.24) to simplify, which yields the equations

$$\sum_{\omega>0}(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega))\eta_{h'}(\omega)\pi(\omega) = 0, \quad h < H, \ h' < H. \tag{A.33}$$

But from the NAC (A.24) for $g = 1$ together with the SGP equation (A.22) for $g = 1$ (again after replacing h with h') and (A.33) it also follows that

$$\sum_{\omega>0}(\eta_h(0)/\eta_h(\omega) - \eta_H(0)/\eta_H(\omega))\eta_H(\omega)\pi(\omega) = 0, \quad h < H. \tag{A.34}$$

Focusing on (A.33) for just $h' = h$ (fixed) and (A.34) then yields the equations

$$\sum_{\omega>0}(\eta_h(\omega)/\eta_H(\omega))\pi(\omega) = \eta_h(0)/\eta_H(0) \tag{A.35}$$

and

$$\sum_{\omega>0}(\eta_H(\omega)/\eta_h(\omega))\pi(\omega) = \eta_H(0)/\eta_h(0). \tag{A.36}$$

So now letting $t(\omega) = \eta_h(\omega)/\eta_H(\omega)$, and defining $f(x) = 1/x$, for $x > 0$, a strictly convex function, (A.35)-(A.36) can be rewritten

$$f[\sum_{\omega>0} t(\omega)\pi(\omega)] = f[t(0)]$$

and

$$\sum_{\omega>0} f[t(\omega)]\pi(\omega) = f[t(0];$$

because of Jensen's inequality, this can only be true if

$$t(\omega) = t(0), \omega > 0,$$

or (because h is arbitrary)

$$\eta_h(\omega)/\eta_h(0) = \eta_H(\omega)/\eta_H(0), \quad \omega > 0, \ h < H. \tag{A.37}$$

Finally, to see that (A.37) implies (A.32), suppose that

$$\eta_h(\omega) = \theta(\omega)\eta_h(0) \text{ with } \theta(\omega) > 0, \quad \omega > 0, \text{ all } h.$$

Then (A.28) implies that

$$1 = \sum_h \eta_h(\omega) = \theta(\omega) \sum_h \eta_h(0) = \theta(\omega), \quad \omega > 0,$$

and the proof is complete. ■

A.6.1 Proof of the Corollary to Proposition 4

Since household H can transact freely in every stock, the relevant NAC's imply that (A.33) also obtains in this model. The rest of the argument is then identical to the foregoing. ■

A.6.2 Proof of the Cautionary to Proposition 4

This is detailed separately in Appendix B. ■

A.7 Continuous Time

A.7.1 Proof of Proposition 5

Step 1. We first show that if (ξ_h, p, q, c_h, b_h, s_h, $h = 1, 2$) is an equilibrium in the model, then the stocks represent the same investment opportunity.

Suppose there exists an equilibrium where none of the risky stocks is redundant. That is, each agent faces an investment opportunity set represented by (4.7) such that the volatility matrix $\Sigma(t) \equiv \begin{pmatrix} \sigma_s^1(t) \\ \sigma_s^2(t) + \sigma_p(t) \end{pmatrix}$ is invertible. Then then households face complete markets, and hence the martingale representation approach of Cox and Huang [9] and Karatzas, Lehoczky and Shreve [14] is applicable, and, in particular, $\xi_h(t)/\xi_h(0)$ are identical across households.

Household h maximizes (4.6) subject to (4.8). Restating (4.8) in term of ξ, we derive the following first-order conditions:

$$\alpha_h^1 e^{-\rho_h t}/c_h^1(t) = y_h \xi(t), \tag{A.38}$$

$$\alpha_h^2 e^{-\rho_h t}/c_h^2(t) = y_h \xi(t) p(t), \tag{A.39}$$

where y_h is the multiplier associated with (4.8). This together with goods market clearing (4.9) implies

$$\xi(t) = \frac{\alpha_1^1 e^{-\rho_1 t}/y_1 + \alpha_2^1 e^{-\rho_2 t}/y_2}{\delta^1(t)},$$

$$p(t) = \frac{\alpha_1^2 e^{-\rho_1 t}/y_1 + \alpha_2^2 e^{-\rho_2 t}/y_2}{\alpha_1^1 e^{-\rho_1 t}/y_1 + \alpha_2^1 e^{-\rho_2 t}/y_2} \frac{\delta^1(t)}{\delta^2(t)},$$

Making the standard identification with the weights in the Representative Agent in the economy $\eta = 1/y_1$, $1 - \eta = 1/y_2$, we obtain expressions (4.10) and (4.11). Applying Itô's lemma to the above to identify the dynamics of the relative price process $p(t)$, we derive the following for the volatility of $p(t)$:

$$\sigma_p(t) = \sigma_\delta^1(t) - \sigma_\delta^2(t). \tag{A.40}$$

The no-arbitrage condition for the stock prices yields

$$q_s^1(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s)\, \delta^1(s)\, ds \,|\mathcal{F}_t\right] \quad \text{and} \quad q_s^2(t) = \frac{1}{\xi(t)} E\left[\int_t^T \xi(s) p(s)\, \delta^2(s)\, ds \,|\mathcal{F}_t\right]. \tag{A.41}$$

Upon substitution of (4.10) and (4.11) in the above, we explicitly evaluate the conditional expectations to yield (4.15)-(4.16). Expressions for the volatilities of the stock prices are then obtained by applying Itô's lemma to (4.15)-(4.16):

$$\sigma_s^1(t) = \sigma_\delta^1(t), \quad \sigma_s^2(t) = \sigma_\delta^2(t) + \sigma_p(t).$$

This together with (A.40) implies

$$\Sigma(t) = \begin{pmatrix} \sigma_\delta^1(t) \\ \sigma_\delta^1(t) \end{pmatrix}.$$

The volatility matrix $\Sigma(t)$ is not invertible, yielding the desired contradiction.

Step 2. Since there are no equilibria in the model in which $\Sigma(t)$ is invertible, we concentrate on the only remaining possibility for an equilibium: the one in which the two stocks represent the same investment opportunity and hence financial markets are incomplete.

Since one of (q_s^1, q_s^2) is redundant, define a composite security (or a hybrid tree), q_S, paying out δ^1 and δ^2. Households' trading strategies for investing in individual securities are indeterminate, however the position in the composite security (consisting of one share of both stocks) would be uniquely identified. The composite security has dynamics

$$dq_S(t) + (\delta^1(t) + p(t)\delta^2(t))dt = q_S(t)[\mu_S(t)dt + \sigma_S(t)dw(t)].$$

In the remainder of the proof, consider an incomplete market (q_b^1, q_S).

The first-order conditions to the optimization problem (4.6) subject to (4.8) are

$$\alpha_h^1 e^{-\rho_h t}/c_h^1(t) = y_h \xi_h(t), \tag{A.42}$$

$$\alpha_h^2 e^{-\rho_h t}/c_h^2(t) = y_h \xi_h(t) p(t), \tag{A.43}$$

where the state-price deflators ξ_1 and ξ_2 of the households are no longer the same up to a multiplicative constant.

At the optimum,

$$W_h(t) = \frac{1}{\xi_h(t)} E\left[\int_t^T \left(\xi_h(s) c_h^1(s) + \xi_h(s) p(s) c_h^2(s)\right) ds \,|\, \mathcal{F}_t\right].$$

Substituting in the first-order conditions (A.42)–(A.43), we have

$$
\begin{aligned}
W_h(t) &= \frac{1}{\xi_h(t)} E\left[\int_t^T \left(\frac{\alpha_h^1 e^{-\rho_h s}}{y_h} + \frac{\alpha_h^2 e^{-\rho_h s}}{y_h}\right) ds \,|\, \mathcal{F}_t\right] \\
&= \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h\, \rho_h\, \xi_h(t)} \,. \qquad \text{(A.44)}
\end{aligned}
$$

Hence

$$c_h^1(t) = \frac{\rho_h \alpha_h^1 e^{-\rho_h t}}{e^{-\rho_h t} - e^{-\rho_h T}}\, W_h(t)\,, \qquad \text{(A.45)}$$

$$c_h^2(t) = \frac{\rho_h \alpha_h^2 e^{-\rho_h t}}{(e^{-\rho_h t} - e^{-\rho_h T})\, p(t)}\, W_h(t)\,. \qquad \text{(A.46)}$$

The dynamic budget constraint that household h is facing is similar to (4.7), except that now there is a single composite risky security available for investment

$$dW_h(t) = W_h(t) r^1(t)dt - (c_h^1(t) + p(t)c_h^2(t))dt + s_h(t)(\mu_S(t) - r^1(t))dt + s_h(t)\sigma_S(t)dw(t)\,.$$

This combined with (A.45)–(A.46) gives

$$dW_h(t) = W_h(t)[r^1(t) - \frac{\rho_h e^{-\rho_h t}}{e^{-\rho_h t} - e^{-\rho_h T}} + \phi_h(t)(\mu_S(t) - r^1(t))]dt + W_h(t)\phi_h(t)\sigma_S(t)dw(t)\,, \qquad \text{(A.47)}$$

where ϕ_h denotes the proportion of the household's wealth invested in the composite security. Solving the above stochastic differential equation for $\log W_h(t)$, we obtain

$$
\begin{aligned}
\log W_h(t) = \log W_h(0) &+ \int_0^t [r^1(s) - \frac{\rho_h e^{-\rho_h s}}{e^{-\rho_h s} - e^{-\rho_h T}} + \phi_h(s)(\mu_S(s) - r^1(s)) - \frac{1}{2}|\phi_h(s)\sigma_S(s)|^2]\, ds \\
&+ \int_0^t \phi_h(s)\sigma_S(s)dw(s)\,. \qquad \text{(A.48)}
\end{aligned}
$$

Household h is solving

$$
\begin{aligned}
&\max_{c,\,\phi} E\int_0^T e^{-\rho_h t}[\alpha_h^1 \log c_h^1(t) + \alpha_h^2 \log c_h^2(t)]dt \\
&\quad = E\int_0^T e^{-\rho_h t}\Bigg[\log \frac{\rho_h e^{-\rho_h t}}{e^{-\rho_h t} - e^{-\rho_h T}} + \alpha_h^1 \log \alpha_h^1 + \alpha_h^2 \log \alpha_h^2 - \alpha_h^2\, p(t) + \log W_h(0) \\
&\qquad + \int_0^t \left(r^1(s) - \frac{\rho_h e^{-\rho_h s}}{e^{-\rho_h s} - e^{-\rho_h T}} + \phi_h(s)(\mu_S(s) - r^1(s)) - \frac{1}{2}|\phi_h(s)\sigma_S(s)|^2\right) ds\Bigg]\, dt\,,
\end{aligned}
$$

where we made use of (A.45)–(A.46) in the first equality and (A.48) in the second. Since $W_h(0)$, p and r^1 are taken as given by a household, the optimization problem of solving for the trading strategies becomes a pointwise problem

$$\max_{\phi_h(t)} \phi_h(t)(\mu_S(t) - r^1(t)) - \frac{1}{2}|\phi_h(t)\sigma_S(t)|^2$$

yielding at the optimum

$$\phi_h(t) = (\sigma_S(t)\sigma_S^\top(t))^{-1}(\mu_S(t) - r^1(t)). \tag{A.49}$$

Note that the proportion of wealth invested in the composite security is identical for both households.

We are now ready to show that the state-price deflators driving the investment opportunity sets of the two households are proportional. It is given by (A.44) that

$$\xi_h(t) = \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h \rho_h W_h(t)}.$$

Parameterizing the household-specific state-price deflator in the standard fashion by the interest rate r_h^1 and market price of risk θ_h^1 processes, applying Itô's lemma to both sides of this equality and simplifying we have

$$\begin{aligned}
\xi_h(t)[-r_h^1(t)dt - \theta_h^1(t)dw(t)] = &-\frac{e^{-\rho_h t}}{y_h W_h(t)} \\
&- \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h \rho_h W_h(t)}[r^1(t) - \frac{\rho_h e^{-\rho_h t}}{e^{-\rho_h t} - e^{-\rho_h T}} + \phi_h(t)(\mu_S(t) - r^1(t))]dt \\
&- \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h \rho_h W_h(t)}\phi_h(t)\sigma_S(t)dw(t) + \frac{e^{-\rho_h t} - e^{-\rho_h T}}{y_h \rho_h W_h(t)}|\phi_h(t)\sigma_S(t)|^2 dt \\
= &- \xi_h(t)[r^1(t)dt - \phi(t)\sigma_S(t)dw(t)]\,,
\end{aligned}$$

where we used (A.47) and (A.49). So, $r_h^1(t) = r^1(t)$ and $\theta_h^1(t) = \phi(t)\sigma_S(t)$, $\forall\, h = 1,2$. The two households face proportional state-prices deflators, hence markets are effectively complete, the weight η in the representative agent is constant, and a Pareto optimal allocation obtains.

We can then proceed with the same derivations as in Step 1 of this proof to derive (4.10)–(4.11) and then the no-arbitrage prices of redundant securities (4.15)–(4.16) from (A.41). (4.12)–(4.13) follow from (A.38)–(A.39) combined with (4.10)–(4.11). The constant η reflects an initial allocation of wealth and is determined from either household's static budget constraint with equilibrium quantities substituted in. Finally, to determine the interest rate, we apply Itô's lemma to (4.10) yielding

$$d\xi(t) = \xi(t)[(-\mu_\delta^1(t) - \frac{\alpha_1^1\,\eta\,\rho_1 e^{-\rho_1 t} + \alpha_2^1(1-\eta)\rho_2 e^{-\rho_2 t}}{\alpha_1^1\,\eta e^{-\rho_1 t} + \alpha_2^1(1-\eta)e^{-\rho_2 t}} + |\sigma_\delta^1(t)|^2)dt - \sigma_\delta^1(t)dw(t)]$$

and identify the negative of the drift term with the interest rate in the economy. ■

A.7.2 Proof of Proposition 6

The weight η is determined from either household's budget constraint with optimal quantities substituted in, e.g., household 1's:

$$E\left[\int_0^T \xi(t)[c_1^1(t) + p(t)c_1^2(t)]dt\right] = E\left[\int_0^T \xi(t)[e_1^1(t) + p(t)e_1^2(t)]dt\right].$$

Substituting (A.38)–(A.39) and (4.10)–(4.11), we have

$$E\left[\int_0^T e^{-\rho_1 t}\left(\frac{\alpha_1^1}{y_1}+\frac{\alpha_1^2}{y_1}\right)dt\right] = E\left[\int_0^T \frac{\alpha_1^1\,\eta\,e^{-\rho_1 t}+\alpha_2^1(1-\eta)e^{-\rho_2 t}}{\delta_1(t)}e_1^1(t)dt\right] + E\left[\int_0^T \frac{\alpha_1^2\,\eta\,e^{-\rho_1 t}+\alpha_2^2(1-\eta)e^{-\rho_2 t}}{\delta_2(t)}e_1^2(t)dt\right].$$

Rearranging and using $\eta = 1/y_1$, we arrive at

$$\frac{1-e^{-\rho_1 T}}{\rho_1} = E\left[\int_0^T e^{-\rho_1 t}\alpha_1^1\,\frac{e_1^1(t)}{\delta^1(t)}dt\right] + \frac{1-\eta}{\eta}E\left[\int_0^T e^{-\rho_2 t}\alpha_2^1\,\frac{e_1^1(t)}{\delta^1(t)}dt\right] + E\left[\int_0^T e^{-\rho_1 t}\alpha_1^2\,\frac{e_1^2(t)}{\delta^2(t)}dt\right] + \frac{1-\eta}{\eta}E\left[\int_0^T e^{-\rho_2 t}\alpha_2^1\,\frac{e_1^2(t)}{\delta^2(t)}dt\right]. \tag{A.50}$$

Due to (4.17) the sum of the first and third terms on the right-hand side of the last expression is $\frac{1-e^{-\rho_1 T}}{\rho_1}$, while the sum of the second and fourth is zero due to (4.18). Hence (A.50) is satisfied $\forall\,\eta \in (0,1)$. ■

A.7.3 Proof of Proposition 7

Obvious. ■

Appendix B: An Example of an OFE with a Portfolio Constraint

B.1 Further Simplification of TTE

B.1.1 For H>1

$(b_H^1,\, s_H^1)$ only appear in (A.27), so the latter can be used to define the former.

B.1.2 For H=2

We can use (A.28) to substitute

$$\eta_1(\omega) = \eta(\omega) \text{ and } \eta_2(\omega) = 1-\eta_1(\omega) = 1-\eta(\omega), \quad \text{all } \omega$$

into (A.21)-(A.26). This permits rewriting

$$\begin{aligned}&\eta_1(0)/\eta_1(\omega) - \eta_2(0)/\eta_2(\omega) = \\ &\eta(0)/\eta(\omega) - (1-\eta(0))/(1-\eta(\omega)) = \\ &\frac{\eta(0)(1-\eta(\omega)) - (1-\eta(0))\eta(\omega)}{\eta(\omega)(1-\eta(\omega))} = \\ &\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\,\frac{1}{\eta(\omega)}\end{aligned}$$

and thereby simplifying (A.23)-(A.24), which in turn permits "renormalizing" $p(\omega)/\eta(\omega) \rightarrow p(\omega)$

in (A.21)-(A.26), resulting finally in the system of equations

$$\alpha_1^0/\beta_1 + \frac{1-\eta(0)}{\eta(0)}\alpha_2^0/\beta_2 - p(0) = 0, \tag{B.1}$$

$$\pi(\omega)\alpha_1^1 + \frac{1-\eta(\omega)}{\eta(\omega)}\pi(\omega)\alpha_2^1 - p(\omega) = 0, \quad \omega > 0, \tag{B.2}$$

$$\sum_{\omega>0}\left(\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\right)p^1(\omega)/\delta^1(\omega) = 0 \tag{B.3}$$

$$\sum_{\omega>0}\left(\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\right)p^g(\omega) = 0, \quad g = 1, 2, \tag{B.4$'$}$$

$$(1 + 1/\beta_1) - \sum_{\omega}(p^1(\omega)/\delta^1(\omega),\, p(\omega))(b_1^0,\, s_1^0) = 0\,, \quad \text{and} \tag{B.5$'$}$$

$$1 - ((p^1(\omega)/\delta^1(\omega),\, p(\omega))/\pi(\omega))(b_1^1,\, s_1^1) = 0\,, \quad \omega > 0. \tag{B.6}$$

Note: The yield matrix has become

$$Y = [\eta(\omega)(p^1(\omega)/\delta^1(\omega),\, p(\omega)),\, \omega > 0],$$

which, since $\pi(\omega) > 0$, $\eta(\omega) > 0$, $\omega > 0$, necessarily has the same rank as the matrix which now defines Ms. 1's second period budget constraints

$$[(p^1(\omega)\delta^1(\omega),\, p(\omega))/\pi(\omega),\, \omega > 0].$$

B.2 The Leading Example When Ms. 1 Faces an Arbitrary Constraint on Transacting in Stock 2

Let $\phi : \mathbb{R} \to \mathbb{R}$ be differentiable and strictly quasi-concave, and add a constraint of the form $\phi(s_1^{12}) \geq 0$ to Ms. 1's optimization problem.

Note: There is an issue as to whether this requires – for logical consistency in describing the economy – adding the requirement for initial portfolios $\phi(s_1^{02}) \geq 0$. For simplicity, we sidestep this issue here, and permit $s_1^{02} \in \mathbb{R}$. The following construction can be elaborated to cover the case where we require, however, that $\phi(s_1^{02}) = 0$.

Note: It will be established that such a constraint can only be (effectively) binding at s_1^{12} if $s_1^{12} \notin [0, 1]$.

Let $\theta \geq 0$ be the multiplier associated with this constraint. If it is binding in a FE, then two changes are required of TTE.

Note: Bear in mind that, in terms of the original extended system of equations,

$$\eta_h(\omega) = \beta_h/\lambda_h(\omega),$$

and that, for present purposes, we have substituted $p(\omega)$ for $p(\omega)/\eta(\omega)$.

(i) The NAC's (B.4′) become

$$\sum_{\omega>0}\left(\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\right)p^g(\omega)=\begin{cases}0, & \text{if } g=1,\\ -\mu\eta(0), & \text{if } g=2, \quad \text{and}\end{cases} \tag{B.4}$$

(ii) Ms. 1's first period BC (B.5′) becomes

$$(1+1/\beta_1)-\sum_{\omega}(p^1(\omega)/\delta^1(\omega),\, p(\omega))(b_1^0,\, s_1^0)-\mu(s_1^{12}-s_1^{02})=0\,, \tag{B.5}$$

where μ is defined by

$$\mu=\theta D\phi(s_1^{12})/\beta_1.$$

This suggests the following approach to constructing an OFE. First, observing that neither μ nor $p(0)$ appears in the system consisting of (B.2), (B.3), (B.4) for $g=1$, and (B.6) (say, *System **I***, use System **I**, first, to construct values of the variables $p(\omega)$ and $\eta(\omega)\neq\eta(0),\ \omega>0$ such that

$$\text{rank}[(p^1(\omega)\delta^1(\omega),\, p(\omega))/\pi(\omega),\ \omega>0]=3,$$

and then construct $(b_1^1,\, s_1^1)$. Second, given values for these variables, use (B.1), (B.4) for $g=2$, and (B.5) (say, *System* **II**) to construct values of the remaining variables $p(0)$ and μ (which, by Proposition 4, is necessarily nonzero). Finally, simply specify any ϕ such that $\phi(s_1^{12})=0$ and $\text{sign}\, D\phi(s_1^{12})=\text{sign}\,\mu$.

Notice that, assuming that s_1^{12} has been fixed by $\phi(s_1^{12})=0$, the system (B.1)-(B.6) consists of 15 equations in the 15 variables

$$p(\omega),\ \text{all } \omega,\ \eta(\omega),\ \text{all } \omega,\ \mu,\ b_1^1 \text{ and } s_1^{11},$$

given the 12 independent parameters

$$0<\alpha_h^{t0},\ \alpha_h^{t1}=1-\alpha_h^{t0}<1,\ t=0,\,1,\ \beta_h>0,\ h=1,2,\ \delta^1(\omega)>0,\ \text{all } \omega,$$
$$\text{and } 0<\pi(1),\ \pi(2),\ \pi(3)=1-\pi(1)-\pi(2)<1.$$

B.3 Results

Claim 1. *If (B.2), (B.3), (B.4) and (B.6) has a solution in which $\eta(\omega)\neq\eta(0)$, (some) $\omega>0$, then $s_1^{12}\notin[0,1]$.*

Claim 2. *There is an economy for which (B.1)-(B.6) has a solution satisfying $\eta(\omega)\neq\eta(0),\ \omega>0$ (i.e., and OFE for an appropriately specified portfolio constraint function ϕ), as well as a unique solution satisfying $\eta(\omega)=\eta(0),\ \omega>0$ (i.e., a unique PFE).*

Proof of Claim 1. Let

$$\psi_h=\begin{cases}\sum_{\omega>0}\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\,\pi(\omega), & \text{if } h=1,\\ \sum_{\omega>0}\frac{\eta(0)-\eta(\omega)}{\eta(\omega)}\,\pi(\omega), & \text{if } h=2.\end{cases}$$

Then substituting from (B.2) into (B.4) yields the system

$$\alpha_1^{11}\psi_1 + \alpha_2^{11}\psi_2 = 0,$$
$$(1-\alpha_1^{11})\psi_1 + (1-\alpha_2^{11})\psi_2 = -\mu\,\eta(0),$$

which can only have a solution with $\mu \neq 0$ if $\alpha_2^{11} \neq \alpha_1^{11}$. This solution is

$$\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} = \frac{1}{\alpha_1^{11} - \alpha_2^{11}} \begin{pmatrix} \alpha_2^{11}\mu\eta(0) \\ -\alpha_1^{11}\mu\eta(0) \end{pmatrix}.$$

Hence,

$$\mu \neq 0 \;\Rightarrow\; \operatorname{sign}\psi_2 = -\operatorname{sign}\psi_1 \neq 0. \tag{B.7}$$

Note: Here it is useful to recall that the analogue of (B.6) also holds for $h = 2$:

$$\frac{1-\eta(\omega)}{\eta(\omega)} - ((p^1(\omega)\delta^1(\omega),\, p(\omega))/\pi(\omega))(b_2^1,\, s_2^1) = 0,\; \omega > 0. \tag{B.6$'$}$$

(This can be inferred from (B.2) and (B.6) together with the identity $(b_2^1,\, s_2^1) = (-b_1^1,\, 1-s_1^1)$, bearing in mind our having "renormalized" $p(\omega)/\eta(\omega) \to p(\omega)$ earlier.)
Now multiplying each of the equations in (B.6) and (B.6$'$) by

$$\frac{\eta(0)-\eta(\omega)}{1-\eta(\omega)}\pi(\omega),$$

summing over $\omega > 0$, and using (B.3)-(B.4) yields the property

$$\psi_h = \begin{cases} \mu\eta(0)s_1^{12}, & \text{if } h = 1, \\ \mu\eta(0)s_2^{12}, & \text{if } h = 2. \end{cases} \tag{B.8}$$

Since we have the identity $s_2^{12} = 1 - s_1^{12}$, both (B.7) and (B.8) can obtain with $\eta_1(\omega) \neq \eta(0)$, (some) $\omega > 0$, ($\Rightarrow\; \mu \neq 0$) only if $s_1^{12} \notin [0,1]$. ■

Proof of Claim 2. This merely requires displaying an example. See the following calculation. ■

B.4 Calculation

Step 1. Existence of a solution to System I in which $\eta(\omega) \neq \eta(0),\ \omega > 0$.

- **Parameter Values**

  $\alpha_1^{11} = 1/3,\ \alpha_2^{11} = 2/3,$
  $\delta^1(1) = 1,\ \delta^1(2) = 2,\ \delta^1(3) = 3/5,$
  $\pi(1) = 5/12,\ \pi(2) = 4/12,\ \pi(3) = 3/12.$

- **Proposed Solution**

For simplicity we set $\eta(2) = \eta(3) \Rightarrow p(2)/\pi(2) = p(3)/\pi(3)$, i.e., we impose identical spot market equilibrium for $\omega = 2, 3$.

$$p(1)/\pi(1) = (7/3,\, 5/3),\ p(2)/\pi(2) = p(3)/\pi(3) = (5/9,\, 5/9).$$

$$\eta(0) = 1/2,\ \eta(1) = 1/4,\ \eta(2) = \eta(3) = 3/4.$$

- **Checking the solution for (B.2), (B.3), and (B.4),** $g = 1$

  For (B.2):

$$p(1)/\pi(1) = (\alpha_1^{11},\, 1 - \alpha_1^{11}) + \frac{1 - \eta(1)}{\eta(1)}(\alpha_2^{11},\, 1 - \alpha_2^{11}) = (1/3,\, 2/3) + \frac{3/4}{1/4}(2/3,\, 1/3) = (7/3,\, 5/3)$$

$$p(2)/\pi(2) = p(3)/\pi(3) = (\alpha_1^{11},\, 1 - \alpha_1^{11}) + \frac{1 - \eta(2)}{\eta(2)}(\alpha_2^{11},\, 1 - \alpha_2^{11}) = (5/9,\, 7/9)$$

  For (B.3):

$$\sum_{\omega>0} \frac{\eta(0) - \eta(\omega)}{1 - \eta(\omega)} p^1(\omega)/\delta^1(\omega) = \frac{1/4}{3/4}(5/12)(7/3) - \frac{1/4}{1/4}(4/12)(5/9)(1/2) - \frac{1/4}{1/4}(3/12)(5/9)(5/3)$$
$$= 0$$

  For (B.4), $g = 1$:

$$\sum_{\omega>0} \frac{\eta(0) - \eta(\omega)}{1 - \eta(\omega)} p^1(\omega) = \frac{1/4}{3/4}(5/12)(7/3) - \frac{1/4}{1/4}(4/12)(5/9) - \frac{1/4}{1/4}(3/12)(5/9)$$
$$= 0$$

- **Verifying full rank of the yield matrix, i.e., the solution for (B.6)**

$$[(p^1(\omega)/\delta^1(\omega),\, p^1(\omega),\, p^2(\omega))/\pi(\omega),\, \omega > 0]$$
$$= \begin{bmatrix} 7/3 & 7/3 & 5/3 \\ 5/18 & 5/9 & 7/9 \\ 25/3 & 5/9 & 7/9 \end{bmatrix}$$
$$= \begin{bmatrix} 1/3 & 0 & 0 \\ 0 & 1/9 & 0 \\ 0 & 0 & 1/9 \end{bmatrix} \begin{bmatrix} 7 & 7 & 5 \\ 5/2 & 5 & 7 \\ 25/3 & 5 & 7 \end{bmatrix}$$

  Consider the system

$$7v_1 + 7v_2 + 5v_3 = 0, \tag{B.9}$$
$$(5/2)v_1 + 5v_2 + 7v_3 = 0, \text{ and} \tag{B.10}$$
$$(25/3)v_1 + 5v_2 + 7v_3 = 0. \tag{B.11}$$

  We know that Y will have full rank if and only if the only solution to (B.9)-(B.11) is $v_1 = v_2 = v_3 = 0$. However

$$\text{(B.10) and (B.11)} \Rightarrow v_1 = 0,\ v_3 = -(5/7)v_2$$

while

$$v_1 = 0 \text{ and (B.9) } \Rightarrow \; v_3 = -(7/5)v_2,$$

which are consistent only if $v_2 = v_3 = 0$ as well.

- **Checking that the solution to**

$$[(p(\omega)/\delta^1(\omega),\, p(\omega))/\pi(\omega),\; \omega > 0](b_1^1,\, s_1^1) = 1 \tag{B.12}$$

satisfies $s_1^{12} \notin [0,1]$

(B.12) becomes (using $(b_1^1,\, s_1^{11},\, s_1^{12}) \to (b,\, s^1,\, s^2)$)

$$\begin{bmatrix} 7 & 7 & 5 \\ 5/2 & 5 & 7 \\ 25/3 & 5 & 7 \end{bmatrix} \begin{pmatrix} b \\ s^1 \\ s^2 \end{pmatrix} = \begin{pmatrix} 3 \\ 9 \\ 9 \end{pmatrix}$$

or

$$7b + 7s^1 + 5s^2 = 3 \tag{B.13}$$

$$(5/2)b + 5s^1 + 7s^2 = 9, \text{ and} \tag{B.14}$$

$$(25/3)b + 5s^1 + 7s^2 = 9. \tag{B.15}$$

From (B.13)-(B.15) it follows that

$$\text{(B.14) and (B.15) } \Rightarrow b = 0 \text{ (sic)},\; s^2 = 9/7 - (5/7)s^1.$$

The latter and (B.13) yield $7s^1 + 5(9/7 - (5/7)s^1) = 3$, so that

$$s^1 = -1, \text{ and } s^2 = 2 > 1.$$

Step 2. Given the solution to System I in which $\eta(\omega) \neq \eta(0),\ \omega > 0$, existence of a solution to System II for which $(b_1^0,\, s_1^0)$ is positive but small (which implies a unique PFE).

This is a "laydown." Use (B.4) for $g = 2$ to calculate μ. It is then easily verified that, for $b_1^0 = 0$ and $s_1^{02} = 0$, and for any choice of $0 < \alpha_1^{01} < 1$, $0 < \alpha_2^{01} < 1$, and $\delta^1(0) > 0$, when β_2 is sufficiently small, the solution to (B.5) satisfies $0 < s_1^{01} < 1$ (so that $0 < \bar{e}_1 = ((s_1^{01},\, 0),\, (s_1^{01},\, 0)) < \mathbf{1}$, and there is a unique CE, hence a unique PFE).

Note: Given the sign of μ (in the example, $\mu\eta(0) = 5/27$) and $s_1^{12} \notin [0,1]$, choose any ϕ such that

$$\phi(s_1^{12}) = 0 \text{ and } \operatorname{sign} D\phi(s_1^{12}) = \operatorname{sign}\mu.$$

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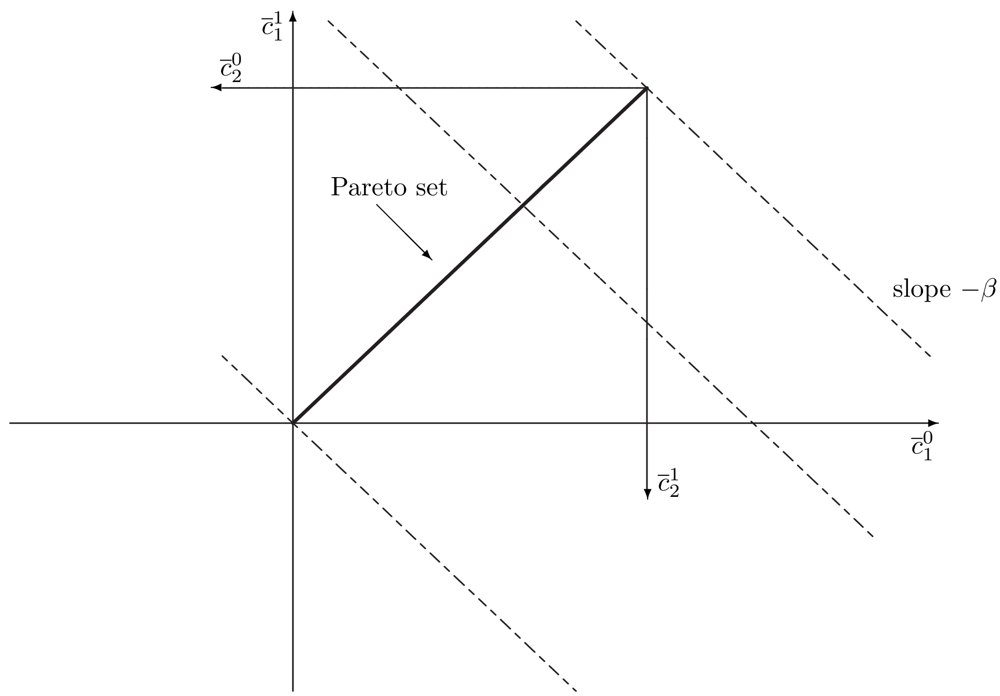
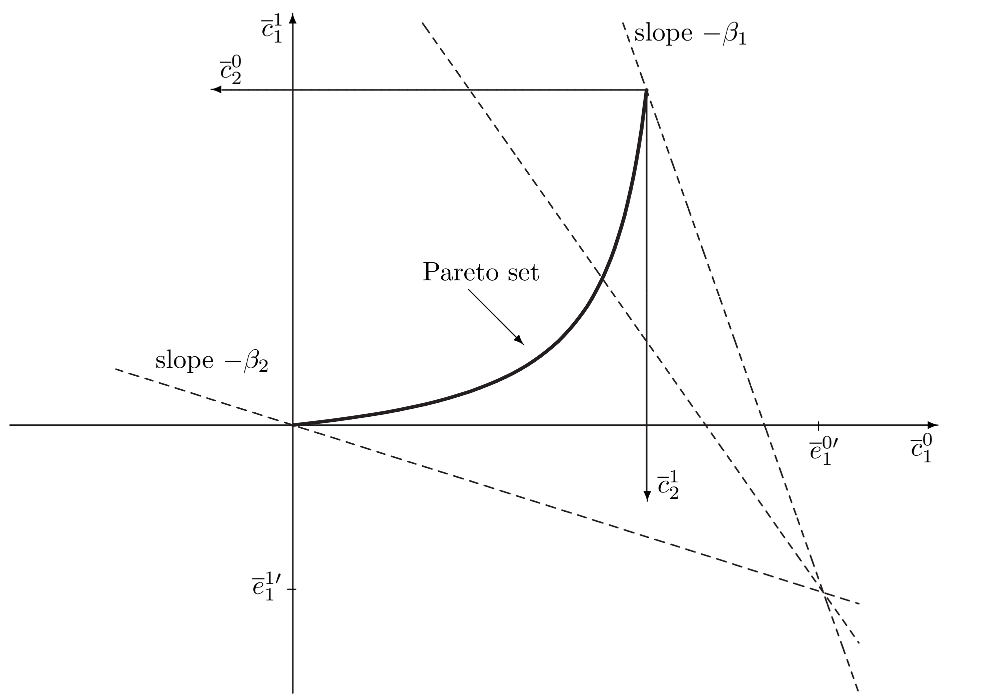


(a) Unique Equilibrium. $\beta_1 = \beta_2 = \beta$.



(b) Continuum of Equilibria. $\beta_1 > \beta_2$.

Figure 1. Equilibria in the TL-Model. The Edgeworth-Bowley box is presented for the (certainty) case of $G = \Omega = 1$, $\tilde{G} = 1$, $H = 2$. The thick solid line depicts the Pareto set, the dotted lines correspond to the prices which support allocations in the Pareto set. $(\bar{e}_1^{0\prime}, \bar{e}_1^{1\prime})$ is the endowment point for which a continuum of equilibria obtains.