B Supplementary Appendix: Not for Publication

B.1 Additional Proofs

Proof of Example 1

For notational simplicity, let $\pi^e_{x,s} \equiv f(x,s|e)$ denote the probability of state (x,s) conditional on effort e, $\bar{\pi}^e_x \equiv \int \pi^e_{x,s} ds$ denote the marginal probability of output x, and $\bar{\Pi}^e_x$ denote the associated cumulative distribution function ("CDF"). Suppose that $\pi^1_{x,s}$ and $\pi^0_{x,s}$ are both independent of s. As in Grossman and Hart (1983), it is convenient to write the principal's program in terms of "utils". Ignoring intermediate effort levels, the program is:

$$\min_V \int h\left(V\left(x\right)\right)\bar{\pi}^1_x dx \text{ s.t.}$$

$$\int V\left(x\right)\bar{\pi}^1_x dx \geq \overline{U} \tag{30}$$

$$\int V\left(x\right)\left(\bar{\pi}^1_x - \bar{\pi}^0_x\right)dx \geq 1, \tag{31}$$

where $h = V^{-1}$.

We wish to study conditions under which the solution to this relaxed program also solves the original program – i.e. under which the following omitted ICs are satisfied:

$$\int_S \int_X V\left(x\right)\left(\pi^1_{x,s} - \pi^e_{x,s}\right)dxds \geq 1 - e, \ \forall e.$$

Using the marginal distributions, we can rewrite these constraints as

$$\xi(e) \equiv \int_X V\left(x\right)\left(\bar{\pi}^1_x - \bar{\pi}^e_x\right)dx - (1-e) \geq 0.$$

Note that $\xi(1) = 0$ and, by the binding IC (31), $\xi(0) = 0$. Thus, it suffices to show that ξ is concave.

For part (i), note that if $V(x) \geq 0 \ \forall \ x$, a sufficient condition for the concavity of ξ is that $\bar{\pi}^e_x$ is weakly convex in e.

Next, we consider part (ii). Applying integration by parts to the solution of the relaxed program, we obtain

$$\int V\left(x\right)\left(\bar{\pi}^1_x - \bar{\pi}^e_x\right)dx = \int \dot{V}\left(x\right)\left(\bar{\Pi}^e_x - \bar{\Pi}^1_x\right)dx,$$

where $\bar{\Pi}$ is the CDF associated with $\bar{\pi}$. Substituting back in the definition of ξ yields

$$\xi(e) = \int \dot{V}\left(x\right)\left(\bar{\Pi}^x_e - \bar{\Pi}^x_1\right)dx + e - 1.$$

Since the likelihood ratio $\bar{\pi}_x^1/\bar{\pi}_x^0$ is non-decreasing in x, the solution of the relaxed program is monotonic: $\dot{V} \geq 0$. Then, since $\bar{\Pi}_x^e$ is a concave function of e, ξ is concave.

Proof of Lemma 1

Suppose that exactly one IC binds in Program (9)-(11) and let e^* be an effort for which the first best is not feasible. The necessary Kuhn-Tucker conditions from the principal's program yield, $\forall$ (x, s) in the support,

$$-h'(u_{x,s}) + \mu \left(K(e^*) - K(e') \frac{p_{x,s}^{e'}}{p_{x,s}^{e^*}} \right) + \lambda K(e^*) = 0, \quad (32)$$

where $\mu \geq 0$ is the multiplier associated with the binding IC. Subtracting these conditions in states (x, s) and (x, s') gives

$$h'(u_{x,s}) - h'(u_{x,s'}) = \mu K(e') \left(\frac{p_{x,s'}^{e'}}{p_{x,s'}^{e^*}} - \frac{p_{x,s}^{e'}}{p_{x,s}^{e^*}} \right). \quad (33)$$

If $\mu = 0$, then (33) implies a constant wage, which contradicts our assumption that the first best is not feasible.[9] Therefore, $\mu > 0$ and, because $K(e) > 0$ $\forall$ e, it follows from (33) and the convexity of h that $u_{x,s} \neq u_{x,s'}$ whenever $\frac{p_{x,s'}^{e'}}{p_{x,s'}^{e}} \neq \frac{p_{x,s}^{e'}}{p_{x,s}^{e}}$.

Proof of Theorem 1, non-binding IR

This appendix completes the proof of Theorem 1, by considering the case where the IR (17) does not bind. We can thus ignore the IR from the principal's program. The first-order condition with respect to $u_{x,s}$ is

$$-p_{x,s}^{e^*} h'(u_{x,s}) - \mu_1 \left(K(1) p_{x,s}^1 - K(e^*) p_{x,s}^{e^*} \right) - \mu_2 \left(K(2) p_{x,s}^2 - K(e^*) p_{x,s}^{e^*} \right) = 0 \ \forall x, s. \quad (34)$$

For the wage to be independent of the signal, the system of equations (18) and (34) must have as a solution $u_{x,s} = u_x$ $\forall$ x, s. We can write this system of equations using

[9] Since the agent's preferences over efforts are independent of income (Assumption (1iii)), effort e^* can be implemented with the minimum constant wage $\bar{w}_{e^*}$ if and only if it can be implemented with any other wage $w \geq \bar{w}_{e^*}$.

the function $F:\mathbb{R}^{X(1+3S)+5}\to\mathbb{R}^{XS+2}$, where

$$F\left(\underbrace{u_1,...,u_X}_{X},\underbrace{\mu_1,\mu_2}_{2};\underbrace{\Theta}_{3},\underbrace{p^e_{1,1},...,p^e_{X,S}}_{3XS}\right)$$
$$=\left[\begin{array}{c} p^3_{1,1}h'(u_1)+\mu_1(K(1)p^1_{1,1}-K(3)p^3_{1,1})+\mu_2(K(2)p^2_{1,1}-K(3)p^3_{1,1}) \\ \vdots \\ p^3_{1,S}h'(u_1)+\mu_1(K(1)p^1_{1,S}-K(3)p^3_{1,S})+\mu_2(K(2)p^2_{1,S}-K(3)p^3_{1,S}) \\ \vdots \\ p^3_{X,1}h'(u_X)+\mu_1(K(1)p^1_{X,1}-K(3)p^3_{X,1})+\mu_2(K(2)p^2_{X,1}-K(3)p^3_{X,1}) \\ \vdots \\ p^3_{X,S}h'(u_X)+\mu_1(K(1)p^1_{X,S}-K(3)p^3_{X,S})+\mu_2(K(2)p^2_{X,S}-K(3)p^3_{X,S}) \\ \sum_{x=1}^X u_x\left(K(2)\sum_s p^2_{x,s}-K(3)\sum_s p^3_{x,s}\right)+G(2)-G(3) \\ \sum_{x=1}^X u_x\left(K(1)\sum_s p^1_{x,s}-K(3)\sum_s p^3_{x,s}\right)+G(1)-G(3) \end{array}\right].$$

To apply Corollary 1, we need to show that DF has full row rank. It is given by:

$$DF=\left[\begin{array}{cccccc} A_{XS\times X} & C_{XS\times 2} & D_\Theta & H^3_{XS\times XS} & H^2_{XS\times XS} & H^1_{XS\times XS} \\ B_{2\times X} & \mathbf{0}_{2\times 2} & E_\Theta & J^3_{2\times XS} & J^2_{2\times XS} & J^1_{2\times XS} \end{array}\right].$$

Matrices $A_{XS\times X}$ and $B_{2\times X}$ are, respectively, the derivative of the first XS equations and the last 2 equations (ICs) with respect to $\mathbf{u}$:

$$A_{XS\times X} = \left[\begin{array}{cccc} h''(u_1)\mathbf{P}^3_1 & 0 & ... & 0 \\ 0 & h''(u_2)\mathbf{P}^3_2 & ... & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & ... & h''(u_X)\mathbf{P}^3_X \end{array}\right],$$
$$B_{2\times X} = \left[\begin{array}{ccc} K(2)\mathbf{P}^2_1\cdot\mathbf{1}_S-K(3)\mathbf{P}^3_1\cdot\mathbf{1}_S & ... & K(2)\mathbf{P}^2_S\cdot\mathbf{1}_S-K(3)\mathbf{P}^3_X\cdot\mathbf{1}_S \\ K(1)\mathbf{P}^1_1\cdot\mathbf{1}_S-K(3)\mathbf{P}^3_1\cdot\mathbf{1}_S & ... & K(1)\mathbf{P}^1_S\cdot\mathbf{1}_S-K(3)\mathbf{P}^3_X\cdot\mathbf{1}_S \end{array}\right].$$

The derivatives with respect to the multipliers μ_1 and μ_2 are, respectively,

$$C_{XS\times 2}=\left[\begin{array}{cc} K(1)p^1_{1,1}-K(3)p^3_{1,1} & K(2)p^2_{1,1}-K(3)p^3_{1,1} \\ \vdots & \\ K(1)p^1_{1,S}-K(3)p^3_{1,S} & K(2)p^2_{1,S}-K(3)p^3_{1,S} \\ \vdots & \\ K(1)p^1_{X,1}-K(3)p^3_{X,1} & K(2)p^2_{X,1}-K(3)p^3_{X,1} \\ \vdots & \\ K(1)p^1_{X,S}-K(3)p^3_{X,S} & K(2)p^2_{X,S}-K(3)p^3_{X,S} \end{array}\right] \qquad (35)$$

and the null matrix $\mathbf{0}_{2\times 2}$. The derivatives with respect to $\{G(3), G(2), G(1)\}$ are, respectively, $\mathbf{0}_{XS\times 3}$ and

$$E_{\mathbf{G}} = \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

Thus, if $\mathbf{K}$ is constant, $\Theta = \mathbf{G}$, and we have $D_\Theta = D_{\mathbf{G}} = \mathbf{0}_{XS\times 3}$ and $E_\Theta = E_{\mathbf{G}}$.

The derivatives with respect to $\{K(3), K(2), K(1)\}$ are, respectively:

$$\begin{aligned} D_{\mathbf{K}} &= \begin{bmatrix} -\mu_1 p^3_{1,1} - \mu_2 p^3_{1,1} & \mu_2 p^2_{1,1} & \mu_1 p^1_{1,1} \\ \vdots & & \\ -\mu_1 p^3_{1,S} - \mu_2 p^3_{1,S} & \mu_2 p^2_{1,S} & \mu_1 p^1_{1,S} \\ \vdots & & \\ -\mu_1 p^3_{X,1} - \mu_2 p^3_{X,1} & \mu_2 p^2_{X,1} & \mu_1 p^1_{X,1} \\ \vdots & & \\ -\mu_1 p^3_{X,S} - \mu_2 p^3_{X,S} & \mu_2 p^2_{X,S} & \mu_1 p^1_{X,S} \end{bmatrix}, \\ E_{\mathbf{K}} &= \begin{bmatrix} -\sum_{x=1}^{X} u_x \sum_s p^3_{x,s} & \sum_{x=1}^{X} u_x \sum_s p^2_{x,s} & 0 \\ -\sum_{x=1}^{X} u_x \sum_s p^3_{x,s} & 0 & \sum_{x=1}^{X} u_x \sum_s p^1_{x,s} \end{bmatrix}. \end{aligned}$$

Thus, if $\mathbf{G}$ is constant, $\Theta = \mathbf{K}$, and we have $D_\Theta = D_{\mathbf{K}}$, and $E_\Theta = E_{\mathbf{K}}$.

The derivatives with respect to $\left(p^3_{x,s}\right)$, $\left(p^2_{x,s}\right)$, and $\left(p^1_{x,s}\right)$ are, respectively:

$$\begin{aligned} H^3_{XS\times XS} &= \begin{bmatrix} \left[h'\left(u_1\right) - K(3)(\mu_1+\mu_2)\right]\mathbf{I}_S & \mathbf{0}_{S\times S} & \dots & \mathbf{0}_{S\times S} \\ \mathbf{0}_{S\times S} & \ddots & \dots & \mathbf{0}_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{S\times S} & \mathbf{0}_{S\times S} & \dots & \left[h'\left(u_X\right) - K(3)(\mu_1+\mu_2)\right]\mathbf{I}_S \end{bmatrix} \\ J^3_{2\times XS} &= \begin{bmatrix} -u_1 K(3)\mathbf{1}_S & \dots & -u_X K(3)\mathbf{1}_S \\ -u_1 K(3)\mathbf{1}_S & \dots & -u_X K(3)\mathbf{1}_S \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} H^2_{XS\times XS} &= \begin{bmatrix} \mu_2 K(2)\mathbf{I}_S & \mathbf{0}_{S\times S} & \dots & \mathbf{0}_{S\times S} \\ \mathbf{0}_{S\times S} & \mu_2 K(2)\mathbf{I}_S & \dots & \mathbf{0}_{S\times S} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{S\times S} & \mathbf{0}_{S\times S} & \dots & \mu_2 K(2)\mathbf{I}_S \end{bmatrix} = \mu_2 \mathbf{I}_{XS}, \\ J^2_{2\times XS} &= \begin{bmatrix} u_1 K(2)\mathbf{1}_S & \dots & u_X K(2)\mathbf{1}_S \\ \mathbf{0}_S & \dots & \mathbf{0}_S \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} H^1_{XS\times XS} &= \mu_1 K(1)\mathbf{I}_{XS} \\ J^1_{2\times XS} &= \begin{bmatrix} \mathbf{0}_S & ... & \mathbf{0}_S \\ u_1 K(1)\mathbf{1}_S & ... & u_X K(1)\mathbf{1}_S \end{bmatrix}. \end{aligned}$$

Note that $DF_{\mathbf{P}} = \begin{bmatrix} H^3_{XS\times XS} & H^2_{XS\times XS} & H^1_{XS\times XS} \\ J^3_{2\times XS} & J^2_{2\times XS} & J^1_{2\times XS} \end{bmatrix}$ has $XS{+}2$ rows and $3XS$ columns. Since $XS+2 < 3XS$, it suffices to show that $DF_{\mathbf{P}}$ has full row rank to establish that DF has full row rank. We thus need to show that for any vector $\mathbf{y} \in \mathbb{R}^{XS+2}$,

$$\underbrace{\mathbf{y}}_{1\times(XS+2)} \times \underbrace{DF_{\mathbf{P}}}_{(XS+2)\times 3XS} = \underbrace{\mathbf{0}}_{1\times 3XS} \implies \mathbf{y} = \underbrace{\mathbf{0}}_{1\times(XS+2)}.$$

Let $DF_{\mathbf{P}_i} = \begin{bmatrix} H^i_{XS\times XS} \\ J^i_{2\times XS} \end{bmatrix}$. First, expanding $\mathbf{y} \times DF_{\mathbf{P}_2} = \mathbf{0}$ gives:

$$\begin{aligned} &\mu_2 K(2) y_1 + u_1 K(2) y_{XS+1} = ... = \mu_2 K(2) y_S + u_1 K(2) y_{XS+1} = 0 \\ &\mu_2 K(2) y_{S+1} + u_2 K(2) y_{XS+1} = ... = \mu_2 K(2) y_{2S} + u_2 K(2) y_{XS+1} = 0 \\ &\vdots \\ &\mu_2 K(2) y_{S(X-1)+1} + u_X K(2) y_{XS+1} = ... = \mu_2 K(2) y_{XS} + u_X K(2) y_{XS+1} = 0. \end{aligned}$$

Dividing through by $K(2) > 0$ and rearranging gives:

$$\begin{aligned} \mu_2 y_1 &= ... = \mu_2 y_S = -u_1 y_{XS+1} \\ \mu_2 y_{S+1} &= ... = \mu_2 y_{2S} = -u_2 y_{XS+1} \\ &\vdots \\ \mu_2 y_{S(X-1)+1} &= ... = \mu_2 y_{XS} = -u_X y_{XS+1}. \end{aligned} \tag{36}$$

Similarly, expanding $\mathbf{y} \times DF_{\mathbf{P}_1} = \mathbf{0}$ yields

$$\begin{aligned} \mu_1 K(1) y_1 &= ... = \mu_1 K(1) y_S = -u_1 K(1) y_{XS+2} \\ \mu_1 K(1) y_{S+1} &= ... = \mu_1 K(1) y_{2S} = -u_2 K(1) y_{XS+2} \\ &\vdots \\ \mu_1 K(1) y_{S(X-1)+1} &= ... = \mu_1 K(1) y_{XS} = -u_X K(1) y_{XS+2} \end{aligned} \tag{37}$$

with $K(1) > 0$. Recall that $\mu_1 \geq 0$ and $\mu_2 \geq 0$ and at least one of them is strict. Thus,

$$\begin{aligned} y_1 &= ... = y_S = \bar{y}^1 \\ y_{S+1} &= ... = y_{2S} = \bar{y}^2 \\ &\vdots \\ y_{S(X-1)+1} &= ... = y_{XS} = \bar{y}^X. \end{aligned}$$

From equation (36), we have:

$$\begin{aligned} \mu_2 \bar{y}^1 &= -u_1 y_{XS+1} \\ &\vdots \\ \mu_2 \bar{y}^X &= -u_X y_{XS+1} \end{aligned} \tag{38}$$

Second, recall that $DF_{(\mu_1,\mu_2)} = \begin{bmatrix} C_{XS\times 2} \\ \mathbf{0}_{2\times 2} \end{bmatrix}$. Thus, $\mathbf{y} \times DF_{(\mu_1,\mu_2)} = \mathbf{0}$ gives

$$\sum_{x,s} \bar{y}^x \left[K(1)p^1_{x,s} - K(3)p^3_{x,s}\right] = 0, \quad \sum_{x,s} \bar{y}^x \left[K(2)p^2_{x,s} - K(3)p^3_{x,s}\right] = 0, \quad \forall x. \tag{39}$$

Multiplying both sides of the first equation in (39) by $\mu_2 \geq 0$:

$$\mu_2 \sum_{x,s} \bar{y}^x \left[K(1)p^1_{x,s} - K(3)p^3_{x,s}\right] = K(1) \sum_{x,s} (\mu_2 \bar{y}^x)\, p^1_{x,s} - K(3) \sum_{x,s} (\mu_2 \bar{y}^x)\, p^3_{x,s} = 0. \tag{40}$$

However, from equation (38), we have

$$\begin{aligned} &K(1) \textstyle\sum_{x,s} (\mu_2 \bar{y}^x)\, p^1_{x,s} - K(3) \sum_{x,s} (\mu_2 \bar{y}^x)\, p^3_{x,s} \\ &= -y_{XS+1} \left[K(1) \textstyle\sum_{x,s} u_x p^1_{x,s} - K(3) \sum_{x,s} u_x p^3_{x,s}\right] = -y_{XS+1}(G(3) - G(1)), \end{aligned} \tag{41}$$

where the last equality follows from the binding IC for $e = 1$. Let $G(3) \neq G(1)$ (the set of parameters for which $G(3) = G(1)$ have zero Lebesgue measure). Then, (40) and (41) imply $y_{XS+1} = 0$. Applying this logic to the second equation in (39) yields $y_{XS+2} = 0$.

Third, recall from equations (36) and (37) that, $\forall\, x$,

$$\mu_2 \bar{y}^x = -u_x y_{XS+1} \text{ and } \mu_1 \bar{y}^x = -u_x y_{XS+2}.$$

Moreover, $\mu_1 \geq 0$ and $\mu_2 \geq 0$ with at least one of them strict. Since $y_{XS+1} = y_{XS+2} = 0$, we have $\mu_1 \bar{y}^x = \mu_2 \bar{y}^x = 0$. Since either $\mu_1 \neq 0$ or $\mu_2 \neq 0$, this implies $\bar{y}^x = 0\ \forall\, x$. Thus, $\mathbf{y} \times DF_{\mathbf{P}} = \mathbf{0} \implies \mathbf{y} = \mathbf{0}$, i.e., $DF_{\mathbf{P}}$ has full row rank.

Proof of Proposition 2

The proof follows similar steps to Holmstrom (1982) and uses a trick introduced by Grossman and Hart (1983) to rewrite the principal's program as a minimization subject to linear constraints. Let the strictly convex function $h \equiv V^{-1}$ denote the inverse utility function and let F denote the cumulative distribution function ("CDF") associated with f. The principal's program can be written in terms of "utils" as:

$$\min_{u_{x,s}} \int h\left(u_{x,s}\right) dF(x,s|e^{*})$$

subject to the IR and IC:

$$\begin{aligned} G\left(e^{*}\right)+K\left(e^{*}\right)\int u_{x,s}dF(x,s|e^{*}) &\geq \bar{U}, \\ G\left(e^{*}\right)+K\left(e^{*}\right)\int u_{x,s}dF(x,s|e^{*}) &\geq G\left(e\right)+K\left(e\right)\int u_{x,s}dF(x,s|e)\ \forall e. \end{aligned}$$

We will present the discrete case here. The continuous case is analogous. Suppose that $\frac{f(x,s|e)}{f(x,s|e^{*})}=\phi_{e^{*}}(x,e)\ \forall\ x$. Then, the IC can be written as:

$$\sum_{x}\left(K(e^{*})-K(e)\phi_{e}(x)\right)\left[\sum_{s} f(x,s|e^{*})u_{x,s}\right]\geq G(e)-G(e^{*})\quad \forall e.$$

Suppose $(u_{x,s})$ satisfies IR and IC and, $\forall\ x$, substitute each entry of the vector $(u_{x,1},...,u_{x,S})$ by the expected value: $\bar{U}_{x}\equiv\sum_{s} f(x,s|e^{*})u_{x,s}$. This new vector also satisfies IC and IR. Since h is strictly convex, the principal's payoff rises if $u_{x,s}$ is not constant in s.

B.2 Multiple Binding ICs

This appendix shows that the case in which multiple ICs simultaneously bind is not knife-edge. The problem of implementing effort e at minimum cost is:

$$\min_{\{u_{x,s}\}} \sum_{x=x_1}^{x_X}\sum_{s=1}^{S} p_{x,s}^{e} b\left(u_{x,s}\right)$$

subject to

$$\begin{aligned} \sum_{x=x_1}^{x_X}\sum_{s=1}^{S} p_{x,s}^{e} u_{x,s}-c_{e} &\geq \bar{U} \\ \sum_{x=x_1}^{x_X}\sum_{s=1}^{S}\left(p_{x,s}^{e}-p_{x,s}^{\tilde{e}}\right)u_{x,s} &\geq c_{e}-c_{\tilde{e}}\quad \forall\tilde{e}. \end{aligned}$$

We study the case of three effort levels and three states. This is the simplest environment to study multiple binding ICs. With two effort levels, there is only one IC; with two states, wages are two-dimensional and, since the IR and at least one IC must bind for any effort except the least costly one, we generically can only have one binding IC.

Let $\mathcal{S} = \{1, 2, 3\}$ and $\mathcal{E} = \{1, 2, 3\}$, and take the utility function $u(c) = \sqrt{c + K}$, where $K > 0$ allows for negative wages. The inverse utility function is then

$$h(u) = u^2 - K.$$

Without loss of generality, let $e = 2$ denote the implemented effort. The program is:

$$\min_{\{u_s\}} \sum_{s=1,2,3} p_s^2 u_s^2$$

subject to

$$\begin{aligned} \sum_{s=1,2,3} p_s^2 u_s &\geq c_2 \\ \sum_{s=1,2,3} \left(p_s^2 - p_s^1\right) u_s &\geq c_2 - c_1 \\ \sum_{s=1,2,3} \left(p_s^2 - p_s^3\right) u_s &\geq c_2 - c_3 \end{aligned}$$

We know that IR binds. Substituting the binding IR into the two ICs, the IR and two ICs now become:

$$\sum_{s=1,2,3} p_s^2 u_s = c_2$$

$$\sum_{s=1,2,3} p_s^1 u_s \leq c_1 \tag{IC1}$$

$$\sum_{s=1,2,3} p_s^3 u_s \leq c_3 \tag{IC3}$$

An economy is parametrized by conditional distributions and costs: $\{p_1^e, p_2^e, c_e\}_{e=1,2,3}$ (p_3^e is given by $p_3^e = 1 - p_2^e - p_1^e$). We claim that there exists an open neighborhood of parameters in which both IC_1 and IC_3 bind. To show this, we will study the maximization program where we ignore one of the ICs. If the ignored IC is satisfied at the solution of this "relaxed program," this solution solves the principal's program. We will show that, for some open set of parameter values, IC_1 fails to hold when we ignore it and IC_3 fails to hold when we ignore it. Thus, both constraints bind.

First, consider the relaxed program where we omit IC_3. The Lagrangian is

$$L = -p_1^2 u_1^2 - p_2^2 u_2^2 - p_3^2 u_3^2 + \lambda \left(p_1^2 u_1 + p_2^2 u_2 + p_3^2 u_3 - c_2 \right) + \mu \left(p_1^1 u_1 + p_2^1 u_2 + p_3^1 u_3 - c_1 \right),$$

which has as first-order conditions the following linear system:

$$\begin{aligned} 2u_1 &= \lambda + \mu \frac{p_1^1}{p_1^2}, \ 2u_2 = \lambda + \mu \frac{p_2^1}{p_2^2}, \ 2u_3 = \lambda + \mu \frac{p_3^1}{p_3^2}, \\ p_1^2 u_1 + p_2^2 u_2 + p_3^2 u_3 &= c_2, \\ p_1^1 u_1 + p_2^1 u_2 + p_3^1 u_3 &= c_1. \end{aligned}$$

We will now combine the first three equations into one by eliminating λ. From the first equation, we have $2u_1 - \mu \frac{p_1^1}{p_1^2} = \lambda$. Substituting into the second and third and combining yields the following linear system with three equations and three unknowns:

$$\begin{bmatrix} \left(\frac{p_2^1}{p_2^2} - \frac{p_1^1}{p_1^2} \right) & \left(\frac{p_1^1}{p_1^2} - \frac{p_3^1}{p_3^2} \right) & \left(\frac{p_3^1}{p_3^2} - \frac{p_2^1}{p_2^2} \right) \\ p_3^2 & p_2^2 & p_1^2 \\ p_3^1 & p_2^1 & p_1^1 \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \\ c_1 \end{bmatrix},$$

which characterizes the solution of the relaxed program where we ignore IC_3.

Similarly, the solution of the relaxed program where we ignore IC_1 is given by:

$$\begin{bmatrix} \left(\frac{p_2^3}{p_2^2} - \frac{p_1^3}{p_1^2} \right) & \left(\frac{p_1^3}{p_1^2} - \frac{p_3^3}{p_3^2} \right) & \left(\frac{p_3^3}{p_3^2} - \frac{p_2^3}{p_2^2} \right) \\ p_3^2 & p_2^2 & p_1^2 \\ p_3^3 & p_2^3 & p_1^3 \end{bmatrix} \begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix} = \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix}.$$

It is easy to apply Cramer's rule to obtain a closed-form solution.

Use the following vector notation: $\mathbf{p}^e \equiv (p_1^e, p_2^e, p_3^e)$. Consider $\mathbf{p}^1 = (0.1,\ 0.28,\ 0.62)$, $\mathbf{p}^2 = (0.2,\ 0.15,\ 0.65)$, $\mathbf{p}^3 = (0.3,\ 0.1,\ 0.6)$, $c_1 = 0.75$, $c_2 = 1$, $c_3 = 0.5$.

The matrix in the relaxed program where we omit IC_3 is:

$$A_1 \equiv \begin{bmatrix} \left(\frac{p_2^1}{p_2^2} - \frac{p_1^1}{p_1^2} \right) & \left(\frac{p_1^1}{p_1^2} - \frac{p_3^1}{p_3^2} \right) & \left(\frac{p_3^1}{p_3^2} - \frac{p_2^1}{p_2^2} \right) \\ p_3^2 & p_2^2 & p_1^2 \\ p_3^1 & p_2^1 & p_1^1 \end{bmatrix} = \begin{bmatrix} 1.3667 & -0.4538 & -0.9128 \\ 0.65 & 0.15 & 0.2 \\ 0.62 & 0.28 & 0.1 \end{bmatrix}.$$

The solution is

$$\begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix} = (A_1)^{-1} \begin{bmatrix} 0 \\ c_2 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1.0703 \\ -0.3207 \\ 1.7620 \end{bmatrix},$$

where we used the fact that

$$(A_1)^{-1} = \begin{bmatrix} 0.2499 & 1.2813 & -0.2813 \\ -0.3596 & -4.2829 & 5.2829 \\ -0.5425 & 4.0478 & -3.0478 \end{bmatrix}.$$

Since A_1 has full rank, the solution is continuous in its parameters (conditional probabilities and costs) around these parameter values. Substituting in IC_3 gives

$$p_3^3 u_3 + p_2^3 u_2 + p_1^3 u_1 - c_3 = 0.6 \times 1.0703 + 0.1 \times (-0.3207) + 0.3 \times 1.7629 - 0.5 = 0.6387 > 0.$$

Thus, IC_3 fails to hold. Since the expression $p_3^3 u_3 + p_2^3 u_2 + p_1^3 u_1 - c_3$ is a continuous function of conditional probabilities, utilities, and costs, and utility is itself a continuous function of costs and probabilities, it follows that this expression is a continuous function of probabilities and costs. Thus, for parameter values in a neighborhood of the ones considered here, it is also the case that IC_3 fails to hold.

The matrix in the relaxed program where we omit IC_1 is:

$$A_3 = \begin{bmatrix} \left(\frac{p_2^3}{p_2^2} - \frac{p_1^3}{p_1^2}\right) & \left(\frac{p_1^3}{p_1^2} - \frac{p_3^3}{p_3^2}\right) & \left(\frac{p_3^3}{p_3^2} - \frac{p_2^3}{p_2^2}\right) \\ p_3^2 & p_2^2 & p_1^2 \\ p_3^3 & p_2^3 & p_1^3 \end{bmatrix} = \begin{bmatrix} -0.8333 & 0.5769 & 0.2564 \\ 0.65 & 0.15 & 0.2 \\ 0.6 & 0.1 & 0.3 \end{bmatrix},$$

which has inverse

$$(A_3)^{-1} = \begin{bmatrix} -0.3545 & 2.0909 & -1.0909 \\ 1.0626 & 5.7273 & -4.7273 \\ 0.3545 & -6.0909 & 7.0909 \end{bmatrix}.$$

The solution of the relaxed program is then

$$\begin{bmatrix} u_3 \\ u_2 \\ u_1 \end{bmatrix} = (A_3)^{-1} \begin{bmatrix} 0 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 1.5455 \\ 3.3636 \\ -2.5455 \end{bmatrix}.$$

Again, the solution is continuous in the parameters in a neighborhood of the parameters selected here. Substituting in the omitted IC gives:

$$p_3^1 u_3 + p_2^1 u_2 + p_1^1 u_1 - c_1 = 0.62 \times 1.5455 + 0.28 \times 3.3636 + 0.1 \times (-2.5455) - 0.75 = 0.8955 > 0.$$

Thus, IC_1 fails to hold. As before, by continuity, this is true for all parameter values in a neighborhood of the ones chosen here.

To summarize, for all parameter values in a neighborhood of the ones chosen here, both ICs simultaneously hold. Thus it is not true that generically only one IC binds.

References

[1] Holmstrom, Bengt (1982): "Moral hazard in teams." *Bell Journal of Economics* 13, 326–340.